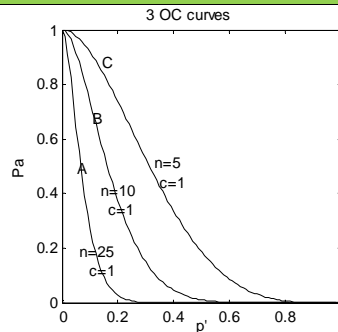
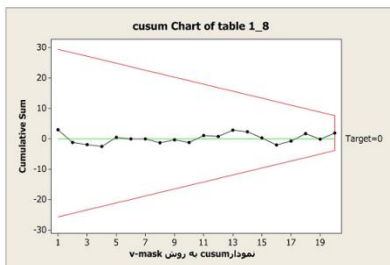


Statistical methods in

# Quality Control



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# Statistical Methods in Quality Control

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August 2020

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## FOREWORD

This book is the outcome of teaching, for more than 30 years, a course titled "statistical quality control" to B.S. students using many books especially the book written by Dr Grant & Dr. Leavenworth and the one written by Dr Kaoru Ishikawa on this subject.

The book is divided into 14 chapters that cover the materials on statistical quality control needed for a course of approximately 48 hours per semester (two times or 3 hours a week). Due to the importance of control charts and Acceptance Sampling Standards, most chapters of the book deal with the control charts and Standard Sampling Tables.

**Chapter 1** introduces basic concepts of quality control.

**Chapter 2** deal with Dr Shewhart's control charts for monitoring the variable characteristics of a product i.e.  $\bar{X}$ , R, S,  $S^2$  control charts.

**Chapter 3** is concerned with the way subgroups or samples have to be arranged; and with the process capability indices.

An introduction to probability is presented in **Chapter 4**.

**Chapters 5 & 6** deal with Dr Shewhart's control charts for monitoring the attribute characteristics of a product i.e. p & np, c, u and D control charts.

Some adaptation of control charts are introduced in **Chapter 7**.

**Chapter 8** introduces the so called Cumulative-Sum control charts for monitoring the process mean.

The precision, accuracy and errors of measuring systems are briefly discussed in **Chapter 9**.

**Chapter 10 to 13** deal with standard sampling tables including Dodge-Romg's, Cameron's, ISO2859 and ISO 3951 Tables used for acceptance sampling.

**Chapter 14** is concerned with product or system reliability and the related concepts including acceptance sampling tables based on product lifetime.

It has been tried to mention the source of the examples, figures or the tables. The examples are mainly taken from the practical books including the above 2 books i.e. Grants' and Ishikawa's.

Thanks God for making me successful to present this work which I hope to be useful in both academic and in industrial environments.

The author would be pleased if the readers write him about any kind of deficiencies of the book.

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# Chapter 1

## Introduction & Basic Concepts

## Chapter 1 Introduction and Basic Concepts

### Aims

In this chapter a history of quality control is provided; online and offline quality control are described and in this regard one of the ideas of Japanese Engineer Taguchi, entitled Quality Loss Function, is introduced. The chapter also briefly introduces some quality manuals used in automotive industry(FMEA, MSA, PPAP, APQ) and well known statistical process control tools(cause and effect diagram, scatter diagram, control charts,···). Other materials such as central limit theorem and some basic concepts of descriptive statistics, are also dealt in this chapter.

### 1.1 Introduction

Quality has been known as an important influencing factor on the success and competitiveness of production and service institutions in the internal and world markets. The return of capital is the result of good quality plans. Appropriate quality strategies lead in fruitful results. Some of the objectives of controlling the quality are the followings:

Assuring that the product specification comply with the design specifications,  
Reducing the scrap and rework  
Increasing the customer satisfaction and the organization reputation  
Increasing the profit

#### 1.1.1 History of Quality Control

Feigenbaum(1991) summarizes the phases in the evolution of quality control as follows:

1. Operator quality control, up to the end of 19<sup>th</sup> century
2. Foreman quality control
3. Inspection quality control in 1920s and 1930s
4. Statistical quality control,
5. Total quality control era [begun from 1960]

The following phase could be added to the above phases

6. ISO 9000 standards era beginning in 1987.

For a brief description of the first 5 phases see Feigenbaum(1991) page 15; however a some sentences from the book on the phases is followed:

Dr. Feigenbaum writes

" The first step in the development of the quality field i.e. operators quality control, was inherent in the manufacturing job up to the end of nineteenth century. Under that system, one worker or at least a very small number of workers, was responsible for the manufacture of the entire product; and therefore each worker could totally control the quality of personal work.

In the early 1900s we progressed to foreman quality control. This period saw the large-scale advent of our modern factory concept, in which many individuals performing a similar task were grouped so that they could be directed by a foreman who then assumed responsibility for the quality of their work.

The manufacturing system became more complex during World War I, involving large numbers of workers reporting to each production foreman. As a result, the first full-time inspectors appeared on the scene, initialing the third step, which we can call inspection quality control.

This step peaked in the large inspection organization of the 1920s and 1930s, separately organized from production and big enough to be headed by superintendents. This program remained in vogue until the tremendous mass-production requirements of World War II necessitated the fourth step of quality control, which we now identify as statistical quality control. In effect, this made big inspection organizations more efficient. Inspectors were provided with a few statistical tools, such as sampling and control charts. The most significant contribution of statistical quality control was that it provided sampling inspection rather than 100 % inspection. The work of quality control, however, remained restricted to production areas and grew rather slowly.... Recommendations resulting from the statistical techniques often could not be handled by existing decision making structures. Certainly they were not effectively handled by the existing inspection group....This brought us to the fifth step, total quality control...."( Feigenbaum,1991)

The present book focuses on the statistical techniques used in quality control, necessitated from the tremendous mass production requirement, because statistical quality control utilizing sampling and control charts, provides sampling inspection instead of 100 percent inspection.

### **1.1.2 Classification of quality control activities**

The activities of quality control are classified into offline and online. Quality control activities at the product planning, design and production engineering phases is referred to as offline quality control or quality engineering, whereas the quality control activities during actual running of production process is referred to as online quality control (Taguchi et al, 1989).

### 1.1.2.1 Online Quality Control

Online quality control deals with those quality activities which are done during the actual the implementation of the production process. According to Taguchi, the daily activities to control process conditions by observing either quality characteristics of product or process parameters are online quality control activities (Taguchi et al, 1989 page8). Once the details of the design and production process are ready and the process is running, possible unfavorable events in the production line are necessary to be identified and remedied on time ; events such as the quality loss of the input of the line, fault in the machine setting or error in its softwares, carelessness of the operator and the manager. Two conventional statistical techniques used in online quality control are control charts and acceptance sampling. The use of control charts helps to identify whether the process is “ in control” or “out of control”. An application of acceptance sampling could be preventing inferior material to enter the production.

### 1.1.2.2 Off line quality control

Quality control activities at the product planning , design and production engineering phases is referred to as offline quality control (Taguchi et al , 1987 page 4). These activities deal with the activities required before the start of the actual production. It has become widely recognized that producing a quality product at a reasonable cost requires an emphasis on its design. A part of the design might include determining factors such as pressure, temperature and types of catalyst to employ. Determining the optimal choices for the factors can often be approximated by a statistical analysis of a designed experiment (Derman & Ross, 1997). One of the contribution of Taguchi is developing an important statistical approach for the design of experiments (DOE) used in off-line quality control. For a description of DOE one could refer to such references as Roy (2001).

Quality loss function is another concept introduced by Taguchi in the field of offline quality control. A description of this concept follows.

### 1.1.3 Quality loss function(QFL)

The quality loss is operationalized through the so-called quality loss function, a quadratic function which shows that a reduction in variability around the target leads in a decrease in loss and a subsequent increase in quality. Loss is minimum when the product is on  $\theta = \frac{U+L}{2}$  (Kelemen, 2003). Taguchi proposed the quadratic symmetric function  $L(X, \theta)$  as follows:

$$L(x, \theta) = C(x - \theta)^2$$

where



$X$	The actual value of a selected feature of the product
$C$	Constant of the proportionality
$\theta$	The nominal value of the product feature = $\frac{U+L}{2}$
$L(X, \theta)$	Quality loss
$\mu_x$	The mean value of the feature resulted from the production
$\sigma_x$	The standard deviation of the feature resulted from the production

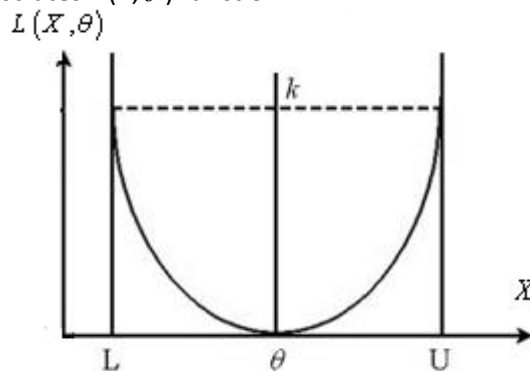
The mean of the loss values in the quality, denoted here by QLF, is measured in the following manner

$$L(X, \theta) = C(X - \theta)^2, QLF(\theta) = E[L(X, \theta)] =$$

$$CE(X - \theta)^2 = C[E(X - \mu_x)^2 + (\mu_x - \theta)^2] \Rightarrow$$

$$QLF(\theta) = C[\sigma_x^2 + (\mu_x - \theta)^2]$$

Figure 1-1 illustrates  $L(X, \theta)$  function.



**Fig. 1-1 Taguchi symmetric function,  $L(X, \theta)$ .**

The value of  $C$  determines the slope of the function, the larger the value of  $C$  the steeper the parabola. This is a symmetric function because it is assumed that there is a constant  $C$  for the whole loss function. The value of  $C$  is a major component in the loss function. This value which represents the intangible quality costs of a product could be calculated from (Kim and Liao, 1994):

$$C = k/d^2$$

where:

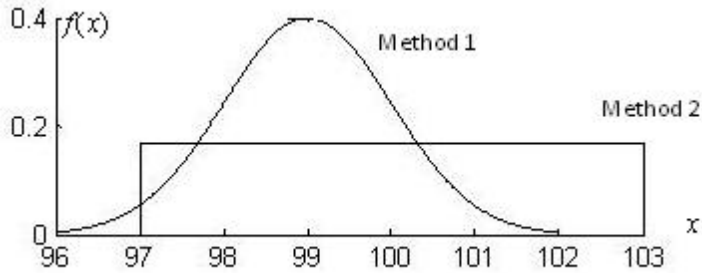
$k$  = loss associated with the feature specification limit ( $L$  or  $U$ ), and  
 $d$  = deviation of the specification from the target value ( $\theta$ ) i.e.

$$d = x - \theta$$

The applications of QLF and DOE are discussed in many books such as Taguchi et al (1987), Roy (2001).

**Example 1.1**

A product might be produced using 2 different methods. The weight of the product produced by Method 1 follows a normal distribution with parameters  $\mu = 99, \sigma = 1$  while the weight by Method 2 follows a uniform distribution in the interval [97 103]. If the target is 100, use QLF with the same constant to select a method.

**Solution**

Let  $X$  denotes

the weight of the product manufactured by Method 1 .

and  $Y$

the weight of the product manufactured by Method 1 .

To compare the quality functions, we write

$$QLF_X(\theta) = C[\sigma_X^2 + (\mu_X - \theta)^2] = C[1 + (99 - 100)^2] = 2C$$

$$QLF_Y(\theta) = C[\sigma_Y^2 + (\mu_Y - \theta)^2] = C\left[\frac{(103-97)^2}{12} + 0\right] = 3C$$

Since  $QLF_X < QLF_Y$ , it is concluded the method having the normal distribution is preferred despite the deviation of its mean from the target.

**Example 1.2**

When the diameter of a steel pipe exceeds its nominal value as much as 0.5, the product could be reworked with a cost of 2 dollars. Calculate the constant of the quality loss function for the production process.

**Solution**

$$L_X = C(x - \theta)^2 \quad x = \theta + 0.5 \quad L = 2$$

$$2 = C(0.5)^2 \Rightarrow C = 8$$

## 1.2 Introducing 5 quality manuals (FMEA, MSA, PPAP, APQP, SPC)<sup>1</sup>

Automotive Industry Action Group(AIAG) is an association where auto industry members collaborate to develop common global standards for quality. Among the publications of this association are AIAG core tools APQP, PPAP, FMEA, SPC, and MSA. These tools provide a framework for identifying areas that may need improvement and allow each work unit to analyze processes and work spaces to help an organizations in its journey through the never ending quest for continual improvement. The tools were primarily used in automotive industry but can be applied universally to identify defects at an early stage, sometimes much before actual start of production or service, and to qualify processes before bulk processing and to mitigate risks effectively.

There are many references which deal with these tools including Supplier Quality Assurance Manual posted on the VOLVO supplier portal

(<http://www.brunel.ac.uk/~emstaam/material/pcm/Volvo-Advanced-Product-Quality-Planning.pdf>).

A brief discussion of each of the tools follows.

### 1.2.1 FMEA

Failure mode and effects analysis (FMEA), also sometimes referred as FMECA (Failure mode effect and criticality analysis) is a prevention based early warning system used widely in Six Sigma, TS<sup>2</sup> 16949, Reliability Engineering, Product Development and Operations Management. FMEA is a team activity which identifies potential failure modes based on past experience with similar products or processes or based on common failures to prevent such failures in advance and saving potential losses, cost and time.

### 1.2.2 MSA

Measurement System Analysis (MSA) is a collection of statistical analysis methods of evaluating variability in measurement processes. Some popular methods of MSA include Gauge R&R analysis, Bias study, Linearity study, Measurement uncertainty etc. MSA is also widely used in TS 16949, Six Sigma and quality improvement projects. Just as processes that produce a product may vary, the process of obtaining measurements and data may have variation and produce defects. A Measurement Systems Analysis evaluates the test method, measuring instruments, and the entire process of obtaining measurements to ensure that variability of measurement system is within acceptable limits and to evaluate its impact on process acceptance indicators.

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<sup>1</sup>This section is based on a lecture at class by Mr Maysam Yousefi, a graduate of our university.

<sup>2</sup> ISO/TS 16949 is a quality management system designed exclusively for automotive sector. Any organizations in automotive supply chain can claim certification against this standard .See <http://www.qualicon.in/iatf-16949.php>

### 1.2.3 PPAP

Part Approval Process (PPAP) is the mechanism of qualifying a process for production during various stages e.g. Prototype development, Pre-launch and routine / bulk production. In TS 16949 system, a supplier obtains PPAP approval from customer based on fulfillment of process qualification requirements set by customer or as per PPAP manual where no specific requirements are set by customer. PPAP in TS 16949 has 5 levels and 19 requirements, each level having slightly different submission requirements. The 19 requirements include Part Submission Warrant (PSW), FMEA, Control Plan, Appearance approval reports, Dimensional reports etc.

### 1.2.4 APQP

Advanced product quality planning (APQP) is a framework of procedures and techniques used to develop products in industry, particularly the automotive industry. It is a product development methodology followed in TS 16949. The purpose of APQP is "to produce a product quality plan which will support development of a product or service that will satisfy the customer."

APQP covers all stages of Development, Production and Corrective action or Feedback in 5 phases:

- Plan and define program
- Product design and development verification
- Process design and development verification
- Product and process validation and production feedback
- Launch assessment and corrective action.

Major elements of APQP include:

- Understanding the needs of the customer
- Proactive feedback and corrective action
- Designing within the process capabilities
- Analyzing and mitigating failure modes
- Verification and validation
- Design reviews
- Control special / critical characteristics.

## 1.3 Statistical Quality Control

Statistical Quality Control is a number of statistical techniques used in all stages of industrial production (design of the product, laboratory testing of the resulting design actual implementation of the production process, testing the incoming supplies and material...) to ensure and measure the constant quality of the production process. With the objective of producing high - quality and reasonable -cost items, the statistical techniques are used in all steps of the production process including the design phase, laboratory testing of the prototype, performing the actual implementation of the

manufacturing operations and in the control of the quality of incoming materials.

The contributions of Dr Walter A. Shewhart, at the Bell Telephone Laboratories in 1920s by bringing together the disciplines of statistics, engineering and economics, could be regarded as the beginning of the field of statistical quality control. At that time two other researchers in the Bell company i.e. Harald Dodge and Harold Romig introduced **Acceptance Sampling**. In 1931 Dr Shewhart published a book titled “Economic Control of Quality of Manufactured Product.” It challenged the inspection-based approach to quality and introduced the modern era of quality management. Up until this time, statistical process control was largely a Bell Telephone quality tool. Shewhart’s book popularized statistical control and its use then spread throughout industry. In 1970s and 1980s the ideas of an international consultant in the field of quality control and assurance, Dr Genichi Taguchi, grew. Design of experiment s(DOE) and Quality loss function are two ideas developed by Dr Taguchi.

### 1.3.1 SPC tools (QC tools)

Statistical process control tool box(SPC tools) is a collection of statistical methods used to control a process within desired limits and to identify patterns of normal (random or chance cause) variation and special (assignable cause) variation in order to identify opportunities for intervention in the process and to take correct decision on process and product acceptance. They are also called quality control tools or simply quality tools. These quality control tools are very useful in Quality Management. Here follows a brief description of the tools i.e.

- Check sheets
- Flow Chart (Process Flow Diagram)
- Cause and Effect Diagram
- Scatter Diagrams
- Pareto Chart(Pareto Diagram)
- Histograms
- Control Charts

the detailed description could be found in many references.

#### 1.3.1.1 Check Sheets and Check lists

(summarized from <http://www.ifm.eng.cam.ac.uk/research/dstools/tqm-tools/>)

A Check sheet is a data recording form that has been designed to readily interpret results from the form itself. It needs to be designed for the specific data it is to gather.

A Check list contains items that are important or relevant to a specific

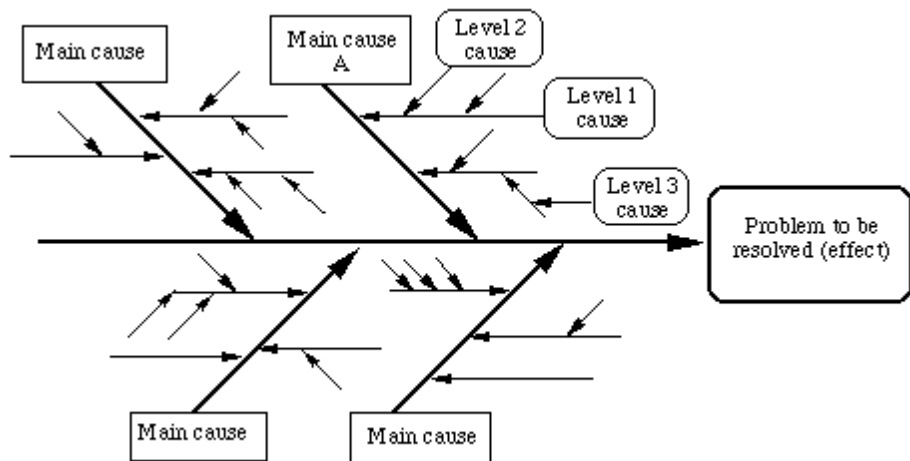
issue or situation. Checklists are used under operational conditions to ensure that all important steps or actions have been taken.

### 1.3.1.2 Process Flow Diagram

This is a schematic diagram that shows the flow of the product or service as it moves through the various processing stations or operations

### 1.3.1.3 Cause and Effect Diagram (Fishbone or Ishikawa Diagram)

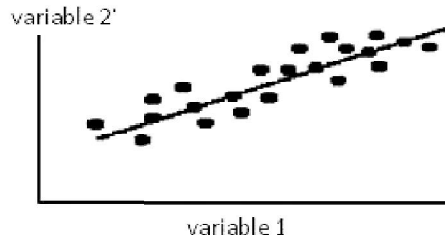
The *Cause and effect* or *fishbone diagrams*, introduced by Kaoru Ishikawa (1915-1989) is a method for analyzing process dispersion. The purpose of the diagram is to relate causes and effects. It used to analyze complex situations in a way that clarifies what the root issues are.



**Fig1- 2 A sample Cause and effect Diagram**  
(from <http://www.ifm.eng.cam.ac.uk/research/dstools/tqm-tools/>)

### 1.3.1.4 Scatter Diagram (Scatter plot)

The *Scatter Diagram* graphs pairs of numerical data to look for a relationship between them. In other words, the plot is a graphical representation of two variables taken from a data set. The Y axis is conventionally used for the characteristic whose behavior we would like to predict. Figure 1-3 is a typical scatter plot.



**Fig.1-3 A sample Scatter Plot**

(after <http://www.managementtutor.com/project-management/7-Quality-Control-Tools.html>)

A sample application of is to determine relationship between the production speed of an operation and the number of defective parts made.

#### 1.3.1.4Pareto Charts<sup>1</sup>

Pareto charts, Proposed by Walras Pareto while at the University of Lausanne in 1896, are used to identify and prioritize problems to be resolved immediately. These charts are based on the idea that large number of problems are caused by small number of causes,

The Pareto principle suggests that most effects come from relatively few causes. In quantitative terms: 80% of the problems come from 20% of the causes (machines, raw materials, operators etc.); 80% of the wealth is owned by 20% of the people etc. Therefore effort aimed at the right 20% can solve 80% of the problems. A production line may experience a range of defects. The manufacturer could concentrate on reducing or eliminating the defects causing a considerable percentage of monetary loss.

Pareto charts are used when products are suffering from different defects but the defects are occurring at a different frequency. In this situation, the company will concentrate reducing defects having major chunk or eliminate defects which cause financial loss.

##### 1.3.1.4.1 Constructing Pareto chart

Following are the steps for constructing a Pareto chart:

- Identify the defects which occur as a result of the completed process. Denote these defects by  $D_1, \dots, D_k$ .
- Take a sample of size  $n$  from the product.

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<sup>1</sup>(<http://www.managementtutor.com/project-management/7-Quality-Control-Tools.html>)

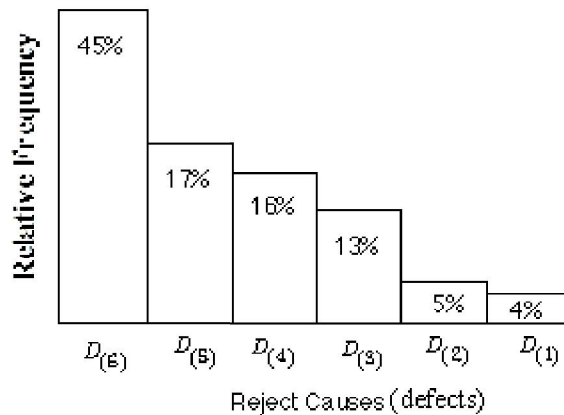
➤ check the  $n$  product individually for possible defects and fill a table like the following to find the frequencies of the defects  $D_1, \dots, D_k$

Defect	Product 1	Product 2	..	...	Product n	Frequency
$D_1$	✓	✓				$f_1$
$D_2$		✓				$f_2$
...	✓				✓	
$D_k$	✓				✓	$f_k$

Defect	$D_{(k)}$	.	.	.	$D_{(1)}$
Frequency	$f_{(k)} = f_{\max}$	.	.		$f_{(1)} = f_{\min}$
Relative Frequency	$\frac{f_{(k)}}{\sum_{i=1}^k f_i}$				$\frac{f_{(1)}}{\sum_{i=1}^k f_i}$

➤ Sort the defects from maximum frequency to the minimum:  
 $D_{(k)}$  is the defect in the sample with maximum frequency  $f_{(k)}$  and  
 $D_{(1)}$  is the defect in the sample with minimum frequency  $f_{(1)}$ .

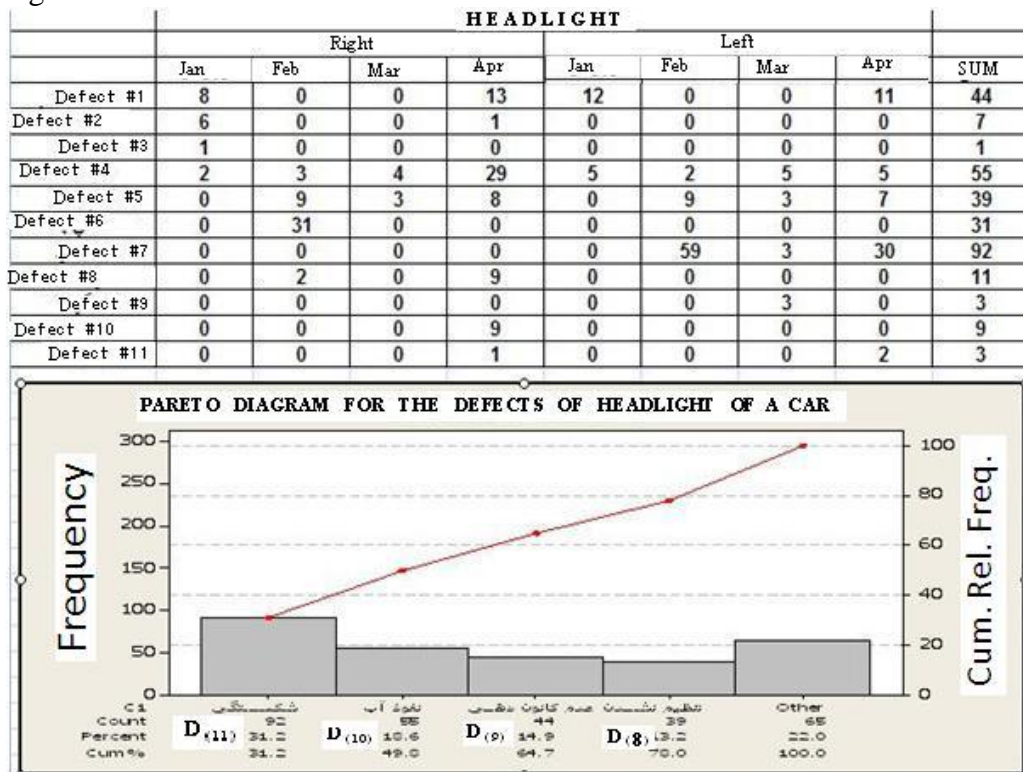
➤ Show  $D_{(k)}, \dots, D_{(1)}$  on the horizontal axis of an X-Y plot; and the corresponding frequencies ( or relative frequencies) of the defect on the left-hand vertical axis. The result could be a figure like the following.



**Fig.1-4-a An Example of a Pareto Diagram**



Usually a vertical axis is added to the right-hand side of the plot, showing the cumulative relative frequencies of the defects. Figure 1-4-b shows the result of a study in an automotive plant<sup>1</sup>. A sample of headlights were checked against 11 different defects on a 4-month horizon; the frequencies of the defects were recorded. The details of the study and the corresponding Pareto diagram are shown in the Figure.



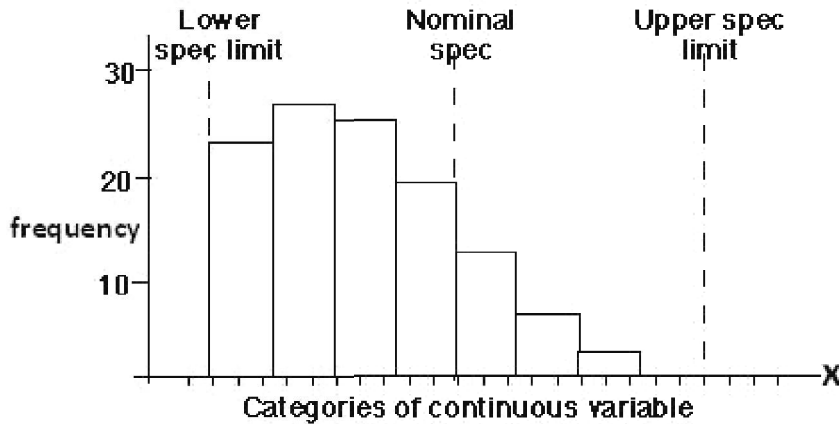
**Fig.1-4-b The result of study of defects of a car headlight and the Pareto Diagram**

### 1.3.1.5 Histogram

Without using some form of graphic tools it can be difficult to analyze, recognize or identify the pattern of the variations of data. A histogram is a graphic summary of variations in a set of data. It enables us to see patterns that are difficult to see in a simple table of numbers. In this graph the continuous variable is clustered into categories and the value of each cluster is plotted to give a series of bars as above. A histogram helps us to see the distribution of a set of product measurements is symmetric or skew. If we

<sup>1</sup> From a report by a former student of Shahid Bahonar University of Kerman, Iran (Mr Amin Yousephi )in 2009.

would like to know whether the distribution of a set of data is normal, exponential ...; an initial evaluation could be made by plotting the histogram. In summary, histograms can be used to draw conclusions about the data set. Fig. 1-5 reveals the skewed distribution of a set of measurements that remain nevertheless within specified limits



**Fig. 1-5 The histogram of a set of data**

#### 1.3.1.5.1 Constructing Histogram by hand

Student should know how to construct a histogram by hand; even if he/she uses software, to be sure he understands the process. The steps for drawing a histogram are:

1) Given a sample of data, determine the range of the sample by subtracting the largest value from the smallest value. This gives the interval between the smallest and the largest observations of the data set. So the range needed on the horizontal axis is known.

2) Choose a reasonable number of "classes" or "bins" to divide the range. There is no set rule, but as a rough guide the range should be divided by five for small sets of data and 20 for larger sets. Sturges (1926) has given the following formula for approximating the number of classes:

$$k = 1 + \log_2 n = 1 + 3.32193 \times \log_{10} n \cong 1 + 3.3 \log_{10} n$$

Where

k is the number of classes

n is the size of sample data.

If n is a power of 2, it is advised to use  $k = 1 + \log_2 n$ .

A simple alternative to Sturges rule is the Rice Rule presented as:

$$k = 2\sqrt[3]{n}.$$

Another alternative is  $k = \sqrt{n}$  utilized by MATLAB.

Decide on a reasonable bin size  $k$ . If necessary, round up  $k$  to the nearest integer.

3) Determine the class width ( $h$ ) by dividing the range ( $R$ ) to  $k$ .

$$h = \frac{R}{k}$$

Classes are usually equal-length segments; the first class will include the minimum data value and the final class includes the maximum data value. If the distribution is approximately normal, class width could be calculated from (Scott, 1979):

$$h = 3.5 \times s \sqrt[3]{n}$$

where

$s$  is the standard deviation of the data sample,

$n$  is the sample size (number of observations).

Always modify  $h$  calculated from the above formulas to reach a reasonable choice and make the resulting graph easy to read. An example of unreasonable choices would be to use a bin size of 2.4 and start it at 14.7. Instead use a bin size of 2 starting at 15.

Each observation should fit into exactly one category. For observations exactly on an endpoint, make some reasonable decision about whether you will put it into the lower interval or the upper interval and then do it consistently for the entire histogram (<http://www.austincc.edu>).

4) The next step is to make a table of frequencies. Begin with a column that lists the classes in increasing order. The next column should have a tally for each of the classes. The third column is for the count or frequency of data in each class. Another column might be added for the relative frequency of each class. This indicates what proportion of the data lies in that particular class.

5) Start through the data, making a tally mark for each observation in the correct class. When finished, count the tally marks to know the frequency for each class.

6) Draw the histogram, with the bins along the horizontal axis and the frequencies up the vertical axis. The steps are illustrated in the following example.

### Example 1.3

A sample of size 98 containing canned fish was randomly selected from a production line and weighed; the results are:

$X_1=200.0$ ,  $X_2=183.5$ ,  $X_3=164.0$ ,  $X_4=201.5$ ,  $X_5=194.0$ ,  $X_6=170.0$ ,  $X_7=195.0$ , ...,  $X_{92}=158.5$ ,  $X_{93}=174.0$ ,  $X_{98}=196.0$ . Draw a histogram for the weights.

### Solution:

Step 1) suppose the smallest and largest weight are 158.5 and 201.5 then  $R=201.5-158.5=43.0$

Step 2)  $k = \sqrt[3]{98} \rightarrow 9$

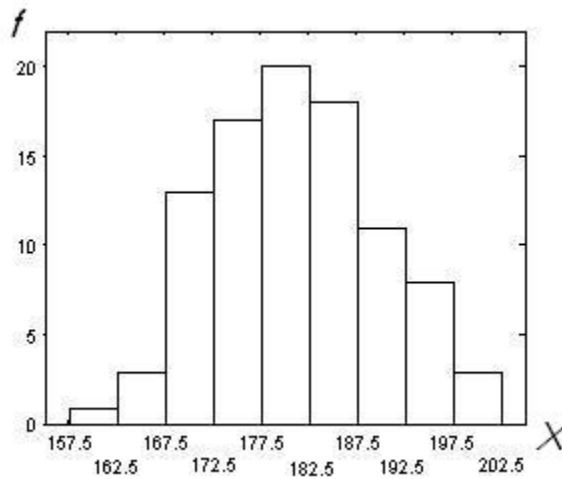
Step 3)  $h = \frac{R}{k} = \frac{43}{9} = 4.78 \rightarrow 5$

We choose 9 bins of length 5 starting at 157.5.

Steps 4 & 5 ) A tally mark for each weight is put in the corresponding subgroup. Tally is only partially shown, in order to show how it should be done. After finishing putting a tally mark for each weight, the frequency column is filled using the result of counting them. The result of step 4 is shown in the table below:

Class No.(i)	Class limits	Tally	Frequency ( $f_i$ )
1	$157.5 \leq X < 162.5$		1
2	$162.5 \leq X < 167.5$		3
3	$167.5 \leq X < 172.5$	etc	13
4	$172.5 \leq X < 177.5$	etc	17
5	$177.5 \leq X < 182.5$		20
6	$182.5 \leq X < 187.5$	etc	18
7	$187.5 \leq X < 192.5$		11
8	$192.5 \leq X < 197.5$	etc	8
9	$197.5 \leq X \leq 202.5$	etc	3

and the histogram, is shown in the following figure:



### 1.3.1.5.2 Making histogram in MATLAB

If all observations are available, to make a histogram in MATLAB, give all data in a vector as input; e.g.

X=

[ 200.0, 183.5, 164.0, 201.5, 194.0, 170.0, 195.0, ..., 158.5, 174.0, 196.0];

then use hist command in MATLAB:

hist(X).

If a frequency table is given instead of all observations, give the midpoints and frequencies of the subintervals in 2 separate vectors; e.g.

Y=[160 165 170 175 180 185 190 195 200];

nn=[ 1 3 13 17 20 18 11 8 3]';

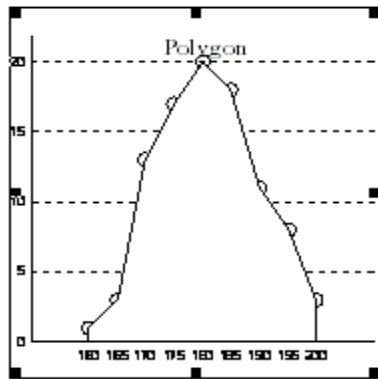
then Use bar command to make the histogram:

bar(Y,nn,'hist')

### 1.3.1.6 Frequency Polygon

A frequency polygon is a line graph that represents the shapes of the statistical distributions. To draw the graph, simply join the top middle points of the bins of the histogram with a straight line.

Figure 1.6 illustrates the polygon related to the previous example.

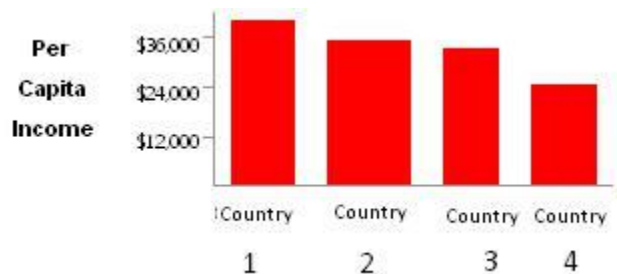


**Fig 1-6 Frequency Polygon for the data of Example 1.3**

Since **bar charts** as well as **histograms** are used to compare the sizes of different groups, a short description of the chart follows.

### 1.3.1.7 Bar chart

Bar charts are used to display and compare the number, frequency or other measures (e.g. mean) for different discrete categories or groups, such as the amount of rainfall that occurred during different months of a year, or the average salary in different states or countries. The graph is constructed such that the heights or lengths of the different rectangular bars are proportional to the size of the category they represent. Since the horizontal axis represents the different categories, it has no scale. The vertical axis does have a scale which indicates the units of measurement. The bars can be plotted vertically or horizontally; however, they are most commonly drawn vertically. A vertical bar chart is sometimes called a line graph. Here is the main difference between bar charts and histograms: With bar charts, each column represents a group defined by a category or group; and with histograms, each column represents a group defined by a quantitative variable. Figure 1-7 shows a sample bar chart.



**Fig 1-7 Sample bar- chart**

### 1.3.1.5.3 Histogram with unequal-length segments

Some times it is not necessary to use equal-length bins for constructing histograms. One such case is when the frequencies of some bins are small and they should be merged. When constructing a histogram with non-uniform (unequal) class widths, we must ensure that the areas of the rectangles are proportional to the class frequencies. This means that we would need to consider the widths in order to determine the height of each rectangle. The following example illustrates how to construct a histogram with non-uniform bin width.

#### Example 1.4

The following frequency distribution gives the range of the income of a sample of 2049,913 taxpayers. Draw a histogram of the relative frequency to illustrate the data.

Income (\$)	Less than 1000	1000-1999	2000-2999	3000-4999	5000-10000	10000-20000
Frequency	12299	209091	274688	600625	746169	207041

#### Solution:

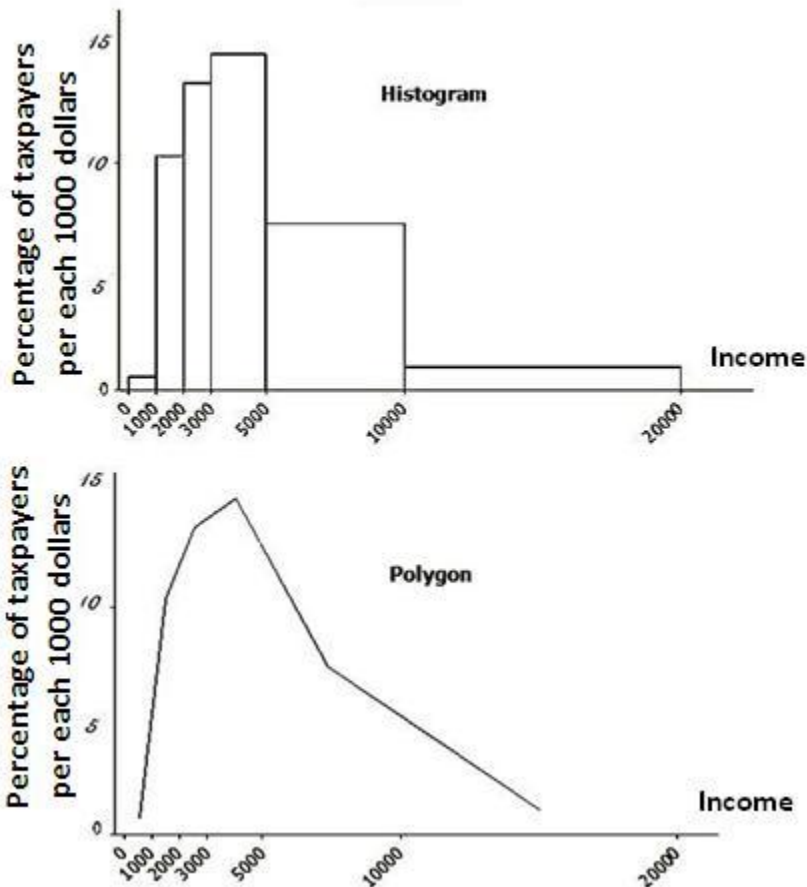
Determining each class widths:

Income (\$)	Less than 1000	1000- Less than 2000	2000- Less than 3000	3000- Less than 5000	5000- Less than 10000	10000- 20000
Frequency	12299	209091	274688	600625	746169	207041
Class width	1000	1000	1000	1000	5000	10000

Since the class widths are not equal, we choose a convenient width such as 1000, which is the smallest width, as a standard and adjust the heights of the rectangles accordingly. The other widths are then multiples of the standard width. The following table shows the calculations of the heights of the rectangles.

Income (\$)	Less than 1000	1000- Less than 2000	2000- Less than 3000	3000- Less than 5000	5000- Less than 10000	10000-20000
Frequency	12299	209091	274688	600625	746169	207041
Relative Frequency%	0.6	10.1	13.4	29.3	36.3	10.1
Class width	1000	1000	1000	2000	5000	10000
	standard	standard	standard	2× standard	5× standard	10× standard
Rectangle's height in histogram	0.6	10.1	13.4	$29.3/2=14.25$	$36.3/5=7.28$	$10.1/10=1.01$

The following figures show the histogram and the polygon



## 1.4 Control Charts

### Definition

The control chart may be defined as

"A graphical method for evaluating whether a process is or is not in a "state of statistical control ..."(Feigenbaum, 1991).

Statistical control charts could be regarded a tool for continuous testing a hypothesis regarding the parameters of a process.

The concept of statistical in-control will be defined later. Dr Walter Shewhart was the first researcher who worked in the field of control charts (in 1920's at Bell Telephone Company).



Forms other than forms of Shewhart's control charts such as CuSum control charts (for mean, range and defects) have come in use in recent years. A control chart <sup>1</sup>

- is a proven technique for improving productivity;
- is effective in defect prevention;
- prevents unnecessary process adjustment;
- provides diagnostic information;
- provides information about process capability.

### 1.4.1 Attributes and variables

Basically, there are two types of **data** to collect as a part of a problem-solving process: Attribute data and variable data.

Attribute data, or pass/fail or yes/no type or go/no-go information are features, characteristics or qualities given to an entity to describe it. Attributes are discrete in nature, for example, the number of nonfunctioning light bulbs, the proportion of broken eggs in a carton, the number of scratches on a tile. Attributes of instant coffee, for example, may include its aroma, flavor, color, packaging and presentation, etc. Attributes have only two possible ratings (negative or positive) expressed as acceptable or unacceptable, desirable or undesirable, good or bad, etc.

([www.businessdictionary.com/definition/attribute.html](http://www.businessdictionary.com/definition/attribute.html))

Variable data, is acquired through measurements, such as length, time, diameter, strength, height, temperature, density, thickness, pressure, and height. A variable, as the name implies, is "something that varies". It may be weight, density, hardness, internal diameter and so on.

Sometimes variables are treated as attributes. For example when you can check the internal diameter of a bushing with a go-not-go measuring device, you are treating the variable an attribute.

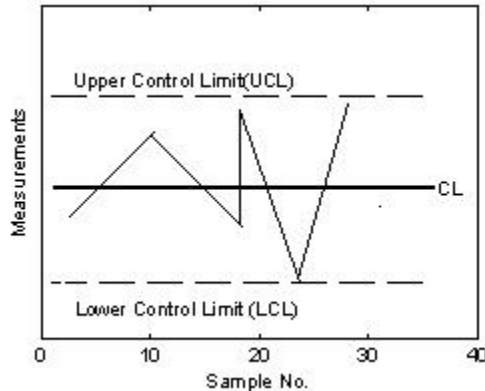
In statistical quality control, some of the control charts and standards ( such as np-control chart and ISO 2859 )deal with the quality attributes of products and some deal with variable characteristics of products ( such as  $\bar{X}$  chart and ISO 3951).

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<sup>1</sup> ([https://ay14-15.moodle.wisc.edu/prod/pluginfile.php/81841/mod\\_resource/content/0/ISyE512\\_Chapter%205%20modified.pdf](https://ay14-15.moodle.wisc.edu/prod/pluginfile.php/81841/mod_resource/content/0/ISyE512_Chapter%205%20modified.pdf))

### 1.4.2 General View of Dr. Shewhart's control charts

Dr. Shewhart's control charts look like the following figure:



**Fig 1-8 A sample Shewhart's control chart**

The control charts he introduced have horizontal and vertical axes, central line, upper and lower control limits calculated for a process. Sampled measures are regularly plotted about a central line between the limits.

The plotted lines corresponds to the stability/trend of the process. Action can be taken based on trend rather than on individual variation. This prevents over-correction/compensation for random variation, which would lead to many rejects.

#### 1.4.2.1 Basic Model of Shewhart's Control Charts

To calculate the upper limit, the lower limit and the central line of Dr Shewhart's control charts (Fig1.9), let us suppose we would like to determine the so-called 3- sigma limits of a hypothetical control chart entitled Y chart; where Y is a characteristic of a product. The limits are calculated as follows:

$$\text{Upper Control Limit } UCL_Y = E(Y) + 3\sigma_Y$$

$$\text{Center Line} = CL = E(Y)$$

$$\text{Lower Control Limit } LCL_Y = E(Y) - 3\sigma_Y$$

where

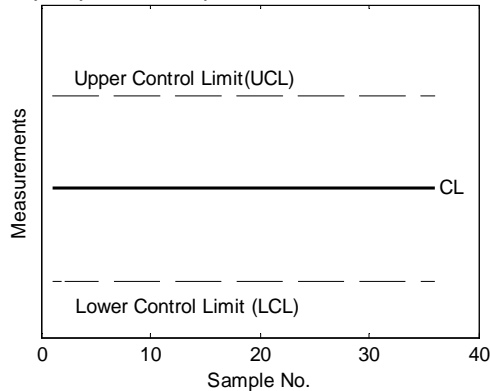
$UCL_Y$  = upper control limit of the control chart for a parameter, say Y, of the process

$LCL_Y$  = Lower control limit of Y control chart

$E(Y)$  = the mean of parameter Y

$\sigma_Y$  = The standard deviation of parameter Y

Note that only in  $\bar{X}$  chart, we can say that 0.27% of the samples are expected to lie outside the 3-sigma limits; because the statistical distribution of sample mean ( $\bar{X}$ ) is approximately normal; while in other charts we could only say that this percent is low.



**Fig 1-9 General view of the limits of Dr Shewhart's charts**

### 1.4.3 Types of control charts

The control charts used in statistical quality control are classified into two general categories: control charts for *variables* and control charts for *attributes*.

#### 1.4.3.1 Control charts for variables

Control charts for variables monitor characteristics that can be measured and have a continuous scale, such as height, weight, volume, or width. When an item is inspected, the variable being monitored is measured and recorded. In this class of charts lies charts such as

- $\bar{X}$ , R, S,  $S^2$  control charts,
- CuSum control charts

-Control charts based on the quality loss function (Derman&Ross,1997page 98)  
To construct these kind of control charts, variables data is used.

#### 1.4.3.2 Control charts for attributes

Control charts for attributes are used for quality characteristics that are counted rather than measured. C, u, np and p control charts lie in this class. These charts present the quality characteristic of a sample versus the sample number or time.

Control charts are easily plotted and widely used. With one *glance* they give you an indication as to how close you are to your target. The following steps should be followed for plotting and using a control chart.

(extracted from Persian translation of Ishikawa, 1983 with minor changes)

Specify the product and its difficulties.

Determine which product should be controlled and specify its difficulties. Based on this decision the type of information which is need will be clarified.

Choosing the appropriate control chart

Depending on the difficulty, specify which of the  $\bar{X}$ ,  $R$ ,  $s$ ,  $s^2$  or  $c$ ,  $u$ ,  $np$  &  $p$  charts is more appropriate for your purpose.

Use the past history

To prepare the control chat for using in future, the history of the past periods could be used. If some unusual points were observed in the control chart among the future samples, look for possible assignable causes. If the causes of the bad unusual quality were found, do appropriate actions to eliminate the causes, if possible.

Suppose appropriate actions were done in the production phases regarding the bad quality and the production phases are controlled. Now it should be tested whether the production complies with standards or not. The work methods are then modified, if needed. Recording the daily data is continued.

Control of production phases

If unusual condition was observed, the cause is readily searched for and the proper action is done.

Recalculate the control limits

If the work methods or machines are changed, the control limits are to be recalculated. If the quality is getting improved the control limits are revised unless we have used standards for the calculation of the limits. The following points should be considered during recalculation:

a) The outlier points (lying outside the control limits) whose causes are not random and have been fixed, must not be considered for the recalculation of control limits.

b )The outlier points whose causes cannot be found or no action could be taken to eliminate them have to be considered in the recalculation of control limits. **End of citation from Ishikawa(1983).**

The following section deals with the calculation of the portion of a set of numbers that fall within a given limits using the average and standard deviation of the data.

## 1.5 What the Average and Standard Deviation of a Set of numbers Really Tell

(Based on Grant and Leavenworth, 1988 page 65)

The combination of sample average and standard deviation could tell us what proportion of a set of data fall within a specified limits. To illustrate, we distinguish two cases based on whether the distribution of the variable is known or not.

### 1.5.1 The statistical distribution is known

First of all suppose we know that a given distribution with mean  $\mu$ , standard deviation  $\sigma$  and density function  $f(x)$  fits the set of data. The portion that lies between  $[a, b]$  is calculated from  $\Pr(a \leq X \leq b) = \int_a^b f(x)dx$ . If  $a$  and  $b$  are given as the closed range  $\mu \pm k\sigma$  and the distribution is normal, it is very easy to calculate the proportion of the products falling within this range.

#### Example 1.5

The life of a certain product is exponentially distributed with mean  $\mu$ . A sample of this product have a mean life of  $\bar{X} = 100$ . If the standard deviation of the life of the product is denoted by  $\sigma$ ; what proportion data fall within the range  $\mu \pm \sigma$ .

#### Solution

The life denoted by  $X$  has an exponential distribution with mean and standard deviation equal to 100, then

$$\Pr(\mu - \sigma \leq X \leq \mu + \sigma) = \Pr(0 \leq X \leq 2\mu)$$

$$= \Pr(0 \leq X \leq 2\mu) = \int_0^{2\mu} \frac{1}{\mu} e^{-\frac{1}{\mu}t} dt = 1 - e^{-2} = 0.865 \text{ or } 86.5\%$$

With Matlab: the proportion =  $\text{expcdf}(200, 100) - \text{expcdf}(0, 100) = 0.8647$ .

#### Example 1.6

If the weight of a product is normally distributed with mean  $\mu$  and the standard deviation  $\sigma$ , what proportion of the weight fall inside

$$\mu \pm t\sigma \quad t = 1, 2, 3, 4, 6 \quad \& \quad t = 0.6745.$$

#### Solution

$$\Pr(\mu - \sigma \leq X \leq \mu + \sigma) = \Pr(-1 \leq Z \leq 1)$$

Using statistical toolbox of MATLAB:

$$\Pr(-1 \leq Z \leq 1) = \text{normcdf}(1) - \text{normcdf}(-1) = 0.6827$$

The answer for the ranges are as follows:

$$\begin{aligned}
\mu \pm 4\sigma &\rightarrow 99.994\% \\
\mu \pm 3\sigma &\rightarrow 99.73\% \\
\mu \pm 2\sigma &\rightarrow 95.45\% \\
\mu \pm \sigma &\rightarrow 68.27\% \\
\mu \pm 0.6745\sigma &\rightarrow 50\%
\end{aligned}$$

**Example 1.7**

Determine what proportion of a product having a normally distributed inner diameter ( $X$ ) ( $X \sim N(\mu, \sigma)$ ) fall within  $\mu - 2\sigma, \mu + 4\sigma$ .

**Solution**

Since  $X \sim N(\mu, \sigma)$  then using MATLAB normcdf command we have:

$$\Pr(\mu - 2\sigma < X < \mu + 4\sigma) = \Pr(-2 < Z < 4) =$$

$$\Pr(Z < 4) - \Pr(Z < -2) = 0.999968 - 0.022750 \cong 0.9772$$

**1.5.2 The statistical distribution is unknown**

If nothing whatever is known about the pattern of variation, what do sample mean and standard deviation ( $\bar{X}$  and  $s$ ) tell?

One answer is given by Tchebycheff's inequality<sup>1</sup> (Grant and Leavenworth, 1988 page 65). According to this theorem more than  $1 - \frac{1}{t^2}$  of any set of finite numbers must fall within the closed range  $\mu \pm t\sigma$   $t \geq 1$ , where  $\mu$  is estimated by  $\bar{X}$  and  $\sigma$  is estimated by  $s\sqrt{\frac{n-1}{n}}$ .

Therefore if nothing is known about the pattern of variation of the characteristic expect the mean and variance, and we would like to know the percentage falling in the range  $\mu \pm t\sigma$   $t \geq 1$ ;

According to the inequality more than  $1 - \frac{1}{t^2}$  of any distribution fall within the range  $\mu \pm t\sigma$  ( $t \geq 1$ ) and at most  $\frac{1}{t^2}$  fall outside these limits.

if  $\mu$  is unknown,  $\bar{X}$  is a good estimate for it.

If  $\sigma$  is not known, estimate it with  $s\sqrt{\frac{n-1}{n}}$  (Grant -leavenworth, 1988 page 66).

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<sup>1</sup>Tchebycheff's inequality could be stated as follows: If  $X$  is a random variable with mean  $\mu$  and finite standard deviation  $\sigma$ ; for  $t > 1$

$$\Pr(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2} \quad 0 < \sigma < \infty. \text{ Then Tchebycheff' (1867) states}$$

$$\Pr(|X - \mu| \geq t\sigma) \leq \frac{1}{t^2}, \quad \Pr(-t\sigma < X - \mu < t\sigma) > 1 - \frac{1}{t^2} \quad t \geq 1 \quad 0 < \sigma < \infty.$$

### Example 1.8

The mean and standard deviation of the statistical distribution of one characteristic of a product are  $\hat{\mu}=\bar{X}=40$  and  $\hat{\sigma} = 2$ . What percentage of the distribution fall in the interval (34, 40) in each of the following cases :

- The characteristic is normally distributed with  $\hat{\mu} = 40$  and  $\hat{\sigma} = 2$ .
- The characteristic is exponentially distributed with  $\hat{\mu} = 40$ .
- Nothing is known about the distribution except  $\hat{\mu}=\bar{X}=40$  and  $\hat{\sigma} = 2$ .

### Solution

$$a) \Pr(34 < X < 46) = \Pr\left(\frac{34-40}{2} < Z < \frac{46-40}{2}\right) = \Pr(-3 < Z < 3)$$

$$\Pr(Z < -3) = 0.00135, \quad \Pr(Z < 3) = 0.99865$$

$$\Rightarrow \Pr(-3 < Z < 3) = 0.99865 - 0.00135 = 0.9973$$

$$b) \Pr(34 < X < 46) = \text{expcdf}(46, 40) - \text{expcdf}(34, 40) = 0.11$$

$$c) \bar{X} = 40 \quad \sigma = 2 \quad \bar{X} - t\sigma = 34 \quad \bar{X} + t\sigma = 40$$

$$\Pr(\bar{X} - t\sigma < X < \bar{X} + t\sigma) > 1 - \frac{1}{t^2}$$

$$\Pr(34 < X < 46) = \Pr(40 - 3 \times 2 < X < 40 + 3 \times 2) > 1 - \frac{1}{t^2} = 1 - \frac{1}{3^2} = \frac{8}{9} = \%88.9$$

Note that for part b  $\sigma = \mu$ , therefore  $34 \neq \mu - t\sigma$  &  $40 \neq \mu + t\sigma$ .

$$\Pr(34 < X < 46) = \Pr(40 - 3 \times 2 < X < 40 + 3 \times 2) > 1 - \frac{1}{t^2} = 1 - \frac{1}{3^2} = \frac{8}{9} = \%88.9$$

### 1.5.2.1 The statistical distribution is unknown but Camp-Meidell conditions hold

Camp-Meidell (1922) states if

- the distribution of X has one mode and
- this mode coincides the mean of the distribution and
- the frequencies in the frequency table decline continuously on both sides of

the mode, then more than  $1 - \frac{1}{2.25t^2} = 1 - \frac{4}{9t^2} = 1 - \left(\frac{2}{3t}\right)^2$  of any

distribution satisfying the above conditions fall within the closed range

$\{\mu \pm t\sigma, t \geq 1, 0 < \sigma < \infty\}$  and at most  $\frac{1}{2.25t^2} = \frac{4}{9t^2} = \left(\frac{2}{3t}\right)^2$  fall outside

the limits  $\mu \pm t\sigma$  ( $t \geq 1$ ).

Camp-Meidell's inequality provides a tighter bound than Tchebychev's.

**Example 1.9**

Assume the Camp-Meidell conditions hold in a distribution having  $\mu = 40$  and  $\sigma = 2$ , what proportion of the distribution fall within the range (34,46)?

**Solution**

$$\mu = 40 \text{ and } \sigma = 2$$

$$(\mu - t\sigma, \mu + t\sigma) = (34, 46) \Rightarrow \mu = 40, \sigma = 2, t = 3$$

by Camp-Meidell's inequality:

$$\Pr(\mu - t\sigma < X < \mu + t\sigma) > 1 - \frac{1}{2.25t^2}$$

$$\Pr(34 < X < 46) > 1 - \frac{1}{2.25 \times 9} = 1 - \frac{1}{2.25 \times 9} = \%95.06$$

The proportion indicated by the normal curve and by the 2 inequalities are compared as follows (Grant and Leavenworth, 1988 page 66)

The percent falling outside the range			
Range	Normal Distribution	Unknown Distribution	
		Under all circumstances Less than the following percentage (Tcheby- chev inequality)	Under Camp-Meidell conditions Less than the following
$\mu \pm 2\sigma$	4.55%	25%	11.1%
$\mu \pm 3\sigma$	0.27%	11.1%	4.94%
$E(\alpha)$	0.006%	6.25%	2.78%
$\mu \pm 6\sigma$	1.97ppm	2.78%	1.23%
If $\mu, \sigma$ are unknown, having a sample of size n with mean $\bar{X}$ and standard deviation s; replace $\mu, \sigma$ with $\bar{X}$ and $\sigma_{rms} = s\sqrt{\frac{n-1}{n}}$ (Grant and Leavenworth, 1988 page 66).			

**1.6 The central limit theorem**

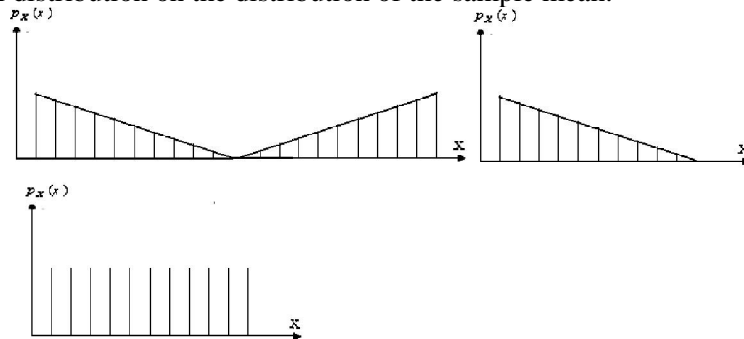
In this section a version of central limit theorem, which is frequently used in statistical quality control, is stated. According to the **central limit theorem** the mean of random samples of sufficiently large size n from a population with mean  $\mu$  and finite variance  $\sigma^2$ , tends towards a normal distribution with mean  $\mu$  and finite variance  $\frac{\sigma^2}{\sqrt{n}}$ ; even if the original distribution is not normally distributed. The theorem also states that sum of the sample elements ( $\sum X_i$ ) tends towards a normal distribution with mean  $n\mu$  and finite variance  $n\sigma^2$ ;

The following figures illustrate some aspects of the theorem. Figure 1.10 shows three possible original distribution of which samples are drawn. Figure 1-11 shows a case where as the size of a sample taken from an

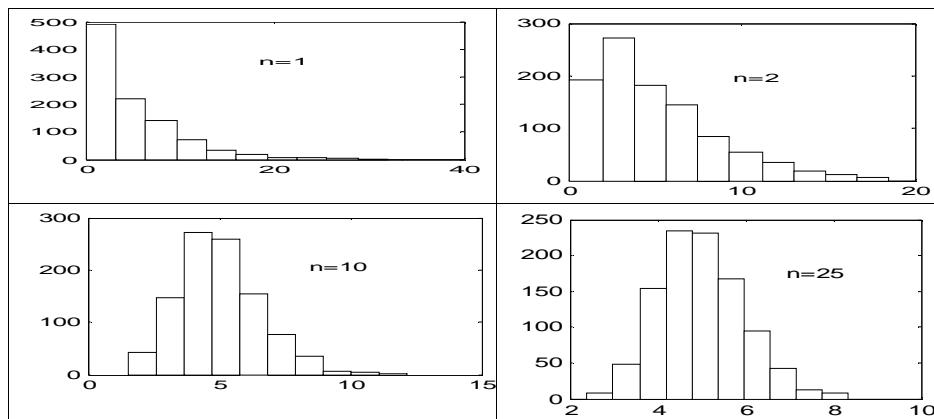


exponential distribution increases, the pattern of variation of the mean of a sample tends to symmetry which is inherent in normal distribution.

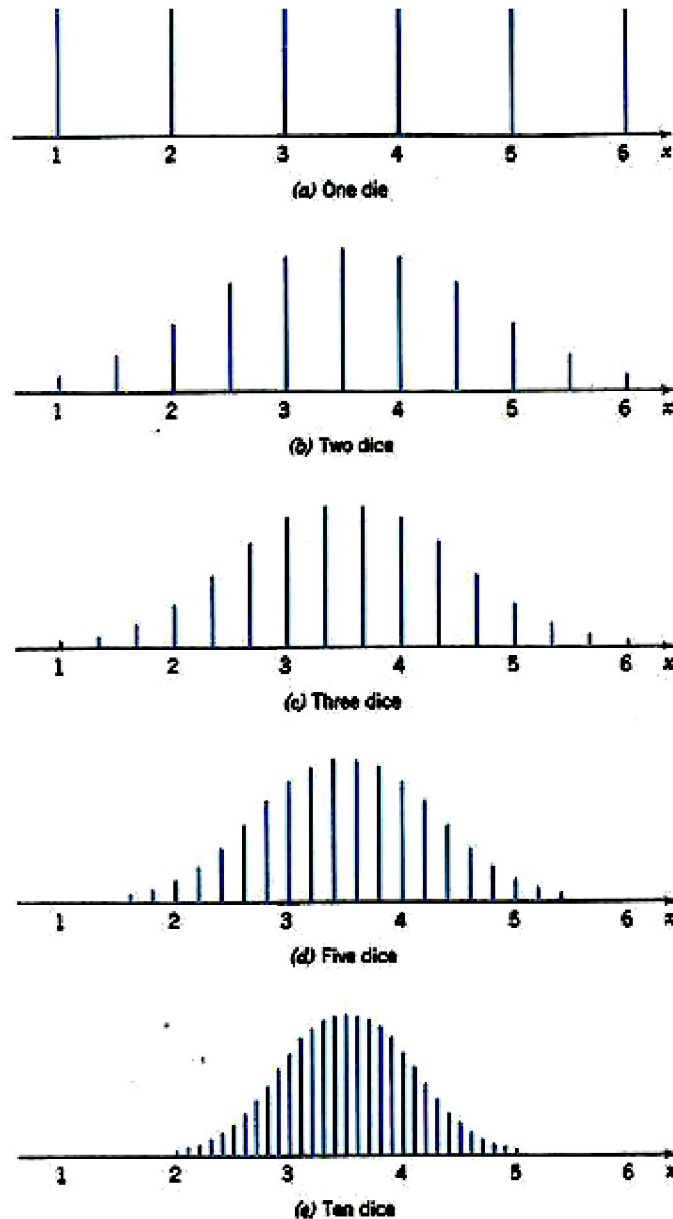
Figure 1-12 shows the effect of the size(n) of samples from a discrete original distribution on the distribution of the sample mean.



**Fig 1-10 3 different original distribution**



**Fig 1-11 The effect of the size of samples from a continuous original distribution, on the distribution of sample mean**



**Figure 6-5** Distributions of average scores from throwing dice. [Adapted with permission from Box, Hunter, and Hunter (1978).]

**Fig 1-12** The effect of the size of samples from a discrete distribution, on the distribution of sample mean  
(Montgomery & Runger, 1994)

## Exercises

**1.1** A small electronic device is designed to emit a special signal. The time of emitting the signal is a characteristic of the device. In the production of this device, 25 subgroups of 5 units were taken at periodic review intervals and tested. The results are given in the following table.

Subgroup No.	Duration of signal					Subgroup No.	Duration of signal				
	Sample Letter						Sample Letter				
	a	b	c	d	e		a	b	c	d	e
1	390	393	395	405	420	14	387	390	398	400	408
2	376	381	381	383	401	15	370	395	395	397	403
3	380	387	395	397	407	16	382	399	401	406	406
4	377	383	387	390	493	17	390	395	395	400	410
5	393	395	403	405	414	18	381	390	394	497	399
6	379	388	395	397	400	19	387	389	398	401	415
7	387	400	400	403	410	20	372	379	396	400	405
8	391	392	394	497	405	21	387	389	391	391	406
9	390	391	395	401	405	22	376	380	391	406	412
10	379	391	395	494	410	23	395	396	397	400	400
11	390	397	400	406	428	24	392	394	397	399	400
12	380	392	389	391	399	25	385	390	390	392	392
13	375	393	392	395	404						

Use enough number of subintervals with mid-point 370, 375,380,... prepare a frequency distribution for all of these 125 data. Then calculate

a) The mean and standard deviation of the all 125 measurements.

b) Plot a histogram for the 125 measurements, a histogram for 25 sample means and a histogram for 25 sample standard deviations.

### 1.2( Grant & Leavenworth , 1988 page 70)

In the production of an electrical device operated by a thermostatic control, five control switches were tested each hour to determine the "on" temperature at which the thermostat actually operated under a given setting. Results of the test over a 3-day production period were as follows

Date	Subgroup No.	"on" temperature at which a thermostatic switch operates (temperature units not specified)				
		a	b	c	d	e
April 24	1	54	56	56	56	55
	2	51	52	54	56	49
	3	54	52	50	57	55
	4	56	55	56	53	50
April 25	5	53	54	57	56	52
	6	53	57	58	55	54
	7	52	55	54	55	56
	8	56	53	53	54	55
April 26	9	55	52	53	56	55
	10	50	54	53	55	55
	11	57	54	53	52	53
	12	52	52	54	53	55
April 27	13	54	53	55	52	52
	14	54	55	54	53	55
	15	56	53	57	56	54
	16	58	57	56	54	54
April 28	17	55	55	55	56	53
	18	54	57	54	55	54
	19	54	53	56	53	53
	20	53	53	57	54	53
April 28	21	53	55	57	56	55
	22	51	54	53	54	55
	23	54	55	50	55	54
	24	56	53	51	55	51
	25	56	55	55	55	53

Make a tally of these 125 measurements and arrange them in a frequency distribution.

Calculate the mean of each sample and plot a histogram for the 25 sample means.

### 1.3( Grant & Leavenworth , 1988 page 72)

A manufacture of electrical products purchases many parts from outside suppliers. A lot of 20,000 of certain small component is received from a new supplier. The receiving in inspection department for the manufacturer has taken a random sample of 200 components from this lot and measured the resistance of each component. These resistance in ohms have been arranged the following frequency distribution.

Cell boundaries (Ohms)	Frequency $f_i$	Midpoint $M_i$	Cell boundaries (Ohms)	Frequency $f_i$	Midpoint $M_i$
66.5-68.5	2	67.5	78.5-80.5	40	79.5
68.5 -70.5	7	69.5	80.5-82.5	24	81.5
70.5-72.5	13	71.5	82.5-84.5	16	83.5
72.5-74.5	22	73.5	84.5-86.5	5	85.5
74.5-76.5	25	75.5	86.5-88.5	2	87.5
76.5-78.5	44	77.5	sum	200	

- a) Compute the average and sample standard deviation of this frequency distribution
- b) What percentage of a normal distribution having your computed estimates of  $\mu$  and  $\sigma$  would fall outside the specification limits  $75 \pm 10$  ohms.
- c) If you make the arbitrary assumption that resistances are distributed uniformly throughout each cell, what percentage of the actual distribution fell outside these limits?

Ans. a)  $\bar{X} = \frac{\sum f_i M_i}{\sum f_i} = 77.7$      $s = 4$ , b) 3.5%    c) 2.9%

**1.4**( Grant & Leavenworth , 1988 page 72)

What proportion of a frequency distribution would you expect to fall outside  $\bar{X} \pm 2.2\sigma$  limit if:

- a) it is known to be approximately normal.
- b) it is known only that it satisfies the conditions of the Camp-Meidell inequality.
- c) nothing is known about the form of distribution.

Ans. a) 2.78%    b) 9.2%    c) 21%

**It is our duty to act in such a manner that the action can be universalized i.e. we would want everyone else to act in a similar manner**

**(based on Immanuel Kant's philosophy)**



# Chapter 2

## Control Charts for Variables ( $\bar{X}$ , R, S & $S^2$ charts)

## Chapter 2:

### Control Charts for Variables( $\bar{X}$ , R, S and $S^2$ Charts)

#### Aims

This chapter, after defining "being in control" and "being out of control" concepts, teaches how to plot  $\bar{X}$  control chart in order to check whether the mean of a parameter related to the product population is under control or not. The chapter also deals with R, s and  $s^2$  control charts to investigate the "in" and "out" of control status of the process standard deviation. It is worth mentioning that the statistical distribution of the observations in the samples of this chapter is assumed to be approximately normal.

#### 2.1 $\bar{X}$ control chart

An  $\bar{X}$  control chart is a type of statistical process control chart for use with continuous data collected in samples of size usually between 3 to 5 pieces. This chart is used to monitor the process mean. Dr. Schewart has proposed  $E(\bar{X}) \pm 3\sigma_{\bar{X}}$  for the 3-sigma limits of X-Bar control chart where

$E(\bar{X}) = \mu$  is the expected value of the sample mean ( $\bar{X}$ ),  
 $\mu$  is the process mean,

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}},$$

$\sigma$  is the process standard deviation,  
 $n$  is the sample size.

The reason for using the 3-sigma limits could be described as follows: Based on central limit theorem, the distribution of sample mean ( $\bar{X}$ ) is approximately normal; therefore 99.73% and 0.27% of the sample means ( $\bar{X}$ s) are expected to fall inside and outside of the 3-sigma limits  $\mu \pm 3\sigma_{\bar{X}}$  respectively; i.e. from each 10,000  $\bar{X}$ s only 27, from each 100  $\bar{X}$ s no  $\bar{X}$  is expected to fall outside limits. That is why  $E(\bar{X}) \pm 3\sigma_{\bar{X}}$  is chosen for the upper and lower control limits. To calculate the limits, two cases are distinguished:  $\mu$  and  $\sigma$  known or unknown.

##### 2.1.1 $\bar{X}$ chart limits - $\mu$ and $\sigma$ known

If specific or standard values, say  $\mu$  and  $\sigma$ , are given (or could be obtained from the specification limits) for the mean and standard deviation of the process, the lower control limit for X-bar chart



$(LCL_{\bar{X}})$ , the upper control limit for the chart  $(LCL_{\bar{X}})$  and its central line (C.L.) are calculated as follows:

$$UCL_{\bar{X}} = \mu + \frac{3}{\sqrt{n}}\sigma \quad \text{or} \quad UCL_{\bar{X}} = \mu + A\sigma$$

$$\text{C.L.} = \mu$$

$$LCL_{\bar{X}} = \mu - \frac{3}{\sqrt{n}}\sigma \quad \text{or} \quad LCL_{\bar{X}} = \mu - A\sigma$$

where

$$A = \frac{3}{\sqrt{n}} \text{ is given in Table U and}$$

$n$  = sample size.

**It is worth noting** that using  $\bar{X}$  chart with limits  $\mu_0 \pm A\sigma$  is in fact testing the hypothesis  $H_0: \mu = \mu_0$  versus  $H_1: \mu \neq \mu_0$  using several samples, with level of significance  $\alpha = 0.27\%$ .

### 2.1.2 $\bar{X}$ chart limits - $\mu$ and $\sigma$ unknown

If specific or standard values, say  $\mu$  and  $\sigma$ , are not available for the mean and standard deviation of the process, supposing the distribution of the measurements (X population) is normal, we know from statistics that the estimate for the standard deviation of the population (process) is given by:

$$\hat{\sigma} = \frac{\bar{s}}{c_4} \quad \text{or} \quad \hat{\sigma} = \frac{\bar{R}}{d_2}$$

where

$$\bar{s} = \frac{\sum S_i}{K}$$

$S_i$  is the standard deviation of  $i^{\text{th}}$  sample  $i=1,2,\dots,k$

$$\bar{R} = \frac{\sum R_i}{K}$$

$R_i$  is the range (the difference between maximum & minimum) of  $i^{\text{th}}$  sample  $i=1,2,\dots,k$

$c_4$  and  $d_2$  are 2 ratios whose values, which depend on sample size ( $n$ ), are given in Table U. The values could be calculated using the following phrases in MATLAB:

`c4 = sqrt(2/(n-1)).*gamma(n/2)./gamma((n-1)/2).`

`pd = makedist('normal',0,1); fun = @(x) (1-(1-cdf(pd,x)).^n-(cdf(pd,x)).^n); d2 = integral(fun,-inf,inf)`

Then estimating the mean of the process with  $\bar{\bar{X}}$  and the standard deviation with  $\hat{\sigma} = \frac{\bar{s}}{c_4}$  we have

$$3\sigma_{\bar{X}} = \frac{3}{\sqrt{n}}\sigma \cong \frac{3\bar{s}}{c_4\sqrt{n}} = A_3\bar{s}$$

$$UCL_{\bar{X}} = \mu + \frac{3}{\sqrt{n}}\sigma \Rightarrow UCL_{\bar{X}} = \bar{\bar{X}} + A_3\bar{s}$$

$$\text{central line} = \bar{\bar{X}}$$

$$LCL_{\bar{X}} = \mu - \frac{3}{\sqrt{n}}\sigma \Rightarrow LCL_{\bar{X}} = \bar{\bar{X}} - A_3\bar{s}$$

Factor  $A_3$  is given in Table U.

Similarly if we let  $\hat{\sigma} = \frac{\bar{R}}{d_2}$ , then

$$3\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{3}{\sqrt{n}} \times \frac{\bar{R}}{d_2} = A_2\bar{R} \text{ and}$$

$LCL_{\bar{X}}$ ,  $UCL_{\bar{X}}$  and the central line are calculated as follows:

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_2\bar{R}$$

$$\text{Central line} = \bar{\bar{X}}$$

$$LCL_{\bar{X}} = \bar{\bar{X}} - A_2\bar{R}$$

Factor  $A_2$  is given in Table U.

### 2.1.3 Plotting $\bar{X}$ chart in MATLAB

Given  $k$  ( $k \geq 5$ ) observed samples of size  $n$ , give these data in MATLAB as a matrix of  $k \times n$  and then run `xbarplot` command or `controlchart` command; these commands have several options.

#### Example 2.1

Given the following table data, calculate the 3-sigma limits for  $\bar{X}$  chart:

Sample No.	Observations				$\bar{X}$	$s = \sqrt{\frac{\sum x_i^2 - n\bar{X}^2}{n-1}}$	R
1	20	15	18	24	19.25	3.77	9
2	32	19	20	14	21.25	7.63	18
3	35	17	45	34	32.75	11.62	28
4	34	19	25	16	23.5	7.94	18
5	30	10	11	21	18	9.42	20
6	42	9	18	36	26.25	15.37	32
7	45	32	44	18	34.75	12.63	27

**Solution**

$\sigma, \mu$  are not given . we use the following formulae:

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_3 \bar{s}$$

$$LCL_{\bar{X}} = \bar{\bar{X}} - A_3 \bar{s}$$

$$\text{Central line} = \bar{\bar{X}}$$

$$\bar{s} = \frac{\sum s}{7} = 9.77, \bar{\bar{X}} = \frac{\sum \bar{X}}{7} = 25.11$$

From Table U for  $n=7$  :  $A_3=1.628$ , then

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_3 \bar{s} = 25.11 + 1.628 \times 9.77 = 41.02$$

$$LCL_{\bar{X}} = \bar{\bar{X}} - A_3 \bar{s} = 25.11 - 1.628 \times 9.77 = 9.20$$

$$\text{Central line} = 25.11$$

Using the following commands we could plot the chart as shown below:

```
>>A=[...
```

```
20    15    18    24
32    19    20    14
35    17    45    34
34    19    25    16
30    10    11    21
42    9     18    36
45    32    44    18];
```

```
>> controlchart(A,'chart','xbar' )
```

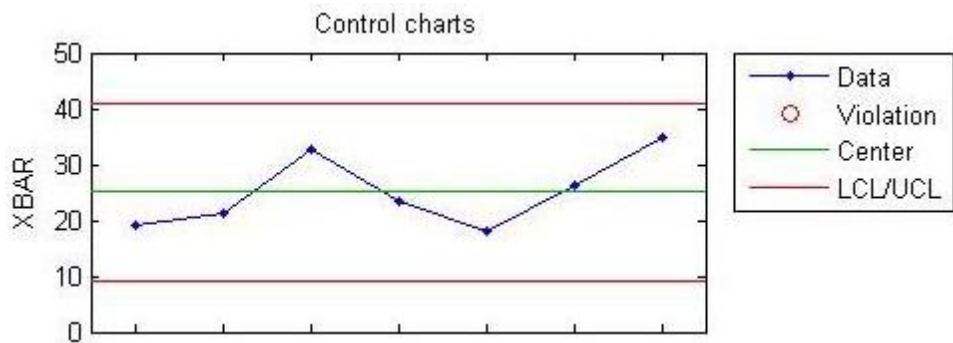


Fig. 2-1 X-bar chart for Example 2.1

**Example 2.2**

Plot X-bar chart for the 3 samples of size 4 given in the following Table .

Sample No.	measurements				$\bar{X}$	R	S	$S^2$	$\sigma_{rms} = \sqrt{\frac{\sum x_i^2 - n\bar{X}^2}{n}}$
1	100	101	99	98	99.5	3	1.291	1.667	1.118
2	98	106	100	94	99.5	12	5.000	25.000	4.330
3	104	101	100	103	102.0	4	1.826	3.334	1.581
					$\bar{\bar{X}} =$ 100.3	$\bar{R} =$ 6.33	$\bar{s} =$ 2.706	$\bar{S}^2 =$ 10.00	

$\mu$  and  $\sigma$  are not given . We use the following formulae:

$$LCL_{\bar{X}} = \bar{\bar{X}} - A_2 \bar{R}$$

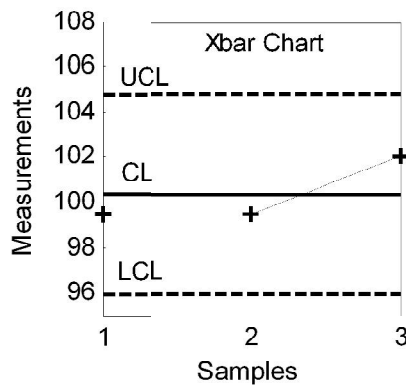
$$\text{Central line} = \bar{\bar{X}}$$

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_2 \bar{R}$$

$$UCL_{\bar{X}} = 100.3 + 0.729 \times 6.33 = 104.92,$$

$$LCL_{\bar{X}} = 100.3 - 0.729 \times 6.33 = 95.67, \bar{\bar{X}} = 100.3$$

Figure 2-1 shows X-bar chart.



**Fig.2-2 X-bar chart for Example 2.**

Another way of calculating the limits is :

( $A_3$  is read 1.128 for  $n=4$  from Table U)

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_3 \bar{S} = 100.3 + 1.128 \times 2.71 = 104.71$$

$$\bar{\bar{X}} = 100.3.$$

$$LCL_{\bar{X}} = \bar{\bar{X}} - A_3 \bar{S} = 100.3 - 1.128 \times 2.71 = 95.89$$

**Example 2.3**

A small electronic device is designed to emit a special signal. The time of emitting the signal is a characteristic of the device. In the production of this device, 3 subgroups of 4 units were taken and tested. The results are given in the following table. The standard given for the mean is 398.00 and for the variance is 8; plot the  $\bar{X}$  control chart.

Sample No.	Duration of automatic signal (Xi)			
1	390	393	393	405
2	393	395	403	405
3	382	399	401	406

**Solution**

Sample No.	$\bar{X}$	$s^2 = \frac{\sum (x_i - \bar{X})^2}{n-1}$	S	R	$\sigma_{rms} = \sqrt{\frac{\sum (x_i - \bar{X})^2}{n}}$
1	395.25	44.25	6.6521	15	5.761
2	399.00	34.66	5.8878	12	5.0990
3	397.00	108.67	10.4243	24	9.027
	$\bar{\bar{X}} = 397.08$	$\bar{S}^2 = 62.52$	$\bar{S} = 7.65$	$\bar{R} = 17$	$\bar{\sigma}_{rms} = 6.63$

$$UCL_{\bar{X}} = \mu + A\sigma$$

$$\text{Central line} = \mu$$

$$LCL_{\bar{X}} = \mu - A\sigma$$

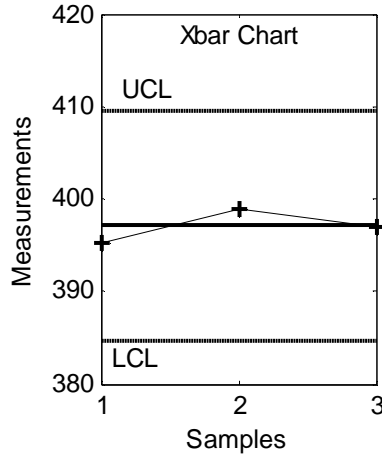
$$A = \frac{3}{\sqrt{n}} = 1.5$$

$$UCL_{\bar{X}} = 398.00 + 1.5 \times 8 = 410.00$$

$$C.L = \mu = 398.00$$

$$LCL_{\bar{X}} = 398.00 - 1.5 \times 8 = 386.00.$$

Figure 2-2 shows the  $\bar{X}$ -bar chart.



**Fig. 2-3  $\bar{X}$  chart for Example 2.3**

It is worth noting that LSL and USL(or L and U ) are determined from specification limits and should not be mistaken from LCL &UCL for X-bar control chart( $UCL_{\bar{x}}$  and  $LCL_{\bar{x}}$ ). LSL and USL are specification limit and pertain to individual measurements , however  $UCL_{\bar{x}}$  and  $LCL_{\bar{x}}$  are the allowed limits for the means of the samples of the measurements.

## 2.2 Determining of standard $\mu$ and $\sigma$ from product specification limits i.e. L and U

Suppose design limits U and L have been specified for a dimension of a product say for the external diameter of a bushing to be produced by a machining process such as lathe working. In this section it is desired to determine the mean and standard deviation on which the lathe should be adjusted. This mean and standard deviation , driven from U and L, could be used for determining the control limits of  $\bar{X}$ , R&S charts.

Given a nominal value  $\mu$  and tolerance  $\pm t$  for a dimension of the product (or specification limits  $L = \mu - t$  &  $U = \mu + t$ ), it is reasonable to adjust the process on  $\mu = \frac{U+L}{2}$ . To calculate the standard deviation for the product process, the producer has to specify an acceptable percentage of waste i.e. the portion of the produced products accepted to fall outside L and U. Let this portion be denoted by  $\alpha$  and the distribution of the dimension (X) produced by the process (call it process distribution ) is normal.; then to determine the allowed standard deviation we have

$$\Pr(X < L) = \Pr(X > U) = \frac{\alpha}{2} \text{ or } \Pr\left(Z > \frac{U-\mu}{\sigma}\right) = \Pr\left(Z > \frac{t}{\sigma}\right) = \frac{\alpha}{2} \Rightarrow \frac{t}{\sigma} = Z_{\frac{\alpha}{2}} \Rightarrow \sigma = \frac{t}{Z_{\frac{\alpha}{2}}}.$$

It is obvious that standard deviation of the process could be less than this value, therefore given the tolerance  $\mu \pm t$  and an acceptable waste percent  $\alpha$ , the mean of the process should be adjusted on  $\mu$  and the allowable standard deviation( $\sigma_{all}$ ) for the process is

$$\sigma_{all} = \frac{t}{Z_{\frac{\alpha}{2}}}$$

where

$Z_{\frac{\alpha}{2}}$  is a value related to standard normal such that  $\Pr\left(Z > Z_{\frac{\alpha}{2}}\right) = \frac{\alpha}{2}$  is obtainable in MATLAB by `norminv(1-alpha/2)`.

#### Example 2.4

The tolerance for one of the dimensions of a product is  $23 \pm 0.1$ . If 0.27% of the product is allowed to fall outside the specification limits  $U=23.1$  and  $L=22.9$ , calculate the mean and the standard deviation on which the process should be adjusted.

$$\mu = \frac{23.1 + 22.9}{2} = 23$$

$$\Pr(X > 23.1) = \frac{0.0027}{2} \Rightarrow \Pr\left(Z > \frac{23.1 - 23}{\sigma}\right) = 0.00135 \Rightarrow \frac{23.1 - 23}{\sigma} = Z_{0.00135} = 3 \Rightarrow \sigma = \frac{0.1}{3}.$$

$$Z_{\frac{\alpha}{2}} = \text{norminv}\left(1 - \frac{0.0027}{2}\right) = 3.0000$$

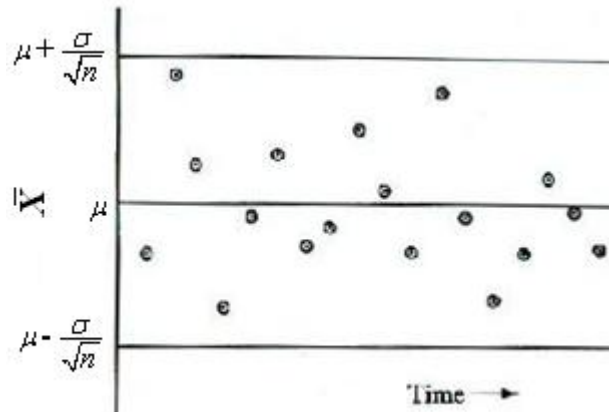
Therefore

$$\sigma_{all} = \frac{t}{Z_{\frac{\alpha}{2}}} = \frac{0.1}{Z_{0.0027/2}} = \frac{0.1}{Z_{0.00135}} = \frac{0.1}{3}.$$

### 2.3 Concept of "in control" status and "out of control" status

When the plotting of a control chart is done; if all sample points are distributed within the UCL and LCL randomly and no special pattern of variation is evident it is said that the parameter being monitored is under control<sup>1</sup>. Figure 2-4 is an example of this case.

<sup>1</sup> It is reminded that when we say that  $\mu$  is under control based on an  $\bar{X}$  chart with  $\mu_0 \pm A \sigma$  as limits; actually we are accepting the null hypothesis  $H_0: \mu = \mu_0$  versus  $H_1: \mu \neq \mu_0$  with using several samples  $\alpha = 0.27\%$



**Fig. 2-4 A sample X-bar chart indicating process mean is in control**

Lack of control of a process parameter is indicated by at least one point falling outside control limits of the corresponding control chart or the sample points forming a special pattern. Figures 2-5 and 2-6 show two control charts indicating the parameter of the process (in this case the process mean) is out of control. In Fig. 2-5 some points are outside the control limits and in Fig. 2-6 the points have formed a special pattern.

The author believes that those points falling on the UCL and LCL could be acceptable; because in statistical tests the equal signs generally belong to the acceptance region and control charts are continuous form of testing hypothesis.

Supplementary to the ordinary rule "One point exceeds the control limits" are the rules suggested by the Western Electric Co. (WECO). These WECO rules would conclude the process is out of control if (as referenced by Montgomery & Runger, 1994, page 846),:

One point plots outside 3-sigma control limits,

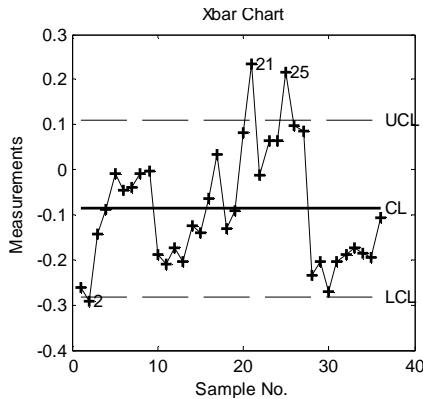
Two out of three consecutive points beyond one 2-sigma limit,

4 out of 5 consecutive points plot at a distance of 1-sigma or beyond, from the center line,

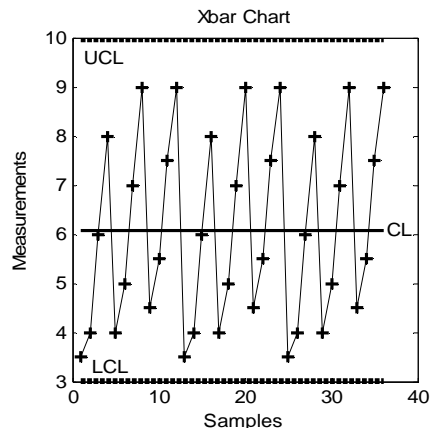
Eight consecutive points on one side of the center line.

There are also some tests based on the theory of runs indicating the process might be running out of control. These tests will follow shortly.





Fig, 2-5 A chart indicating out of control status



Fig, 2-6 A chart indicating out of control status

Reaching to the conclusion that the process is out of control, indicate the presence of assignable causes. The quality –man, then, has to initiate problem solving to improve the manufacturing process and to prevent defects and defective .

## Control charts for monitoring process variance (R-chart, s-chart, $s^2$ -chart )

R-chart, s-chart,  $s^2$ -chart , are used to monitor the variation of a quantitative characteristic of products such a height, width, hardness... measurable on a continuous scale. Along with  $\bar{X}$  – chart, usually R –chart or s-chart is used than  $s^2$ -chart.

### 2.4 R-chart( Range control chart)

R control chart is used to monitor process variations, when the variable of interest is a quantitative measure;(http://www.freequality.org/documents/knowledge/x-bar%20R%20Charts.pdf). It is used to track instantaneous variations, and to evaluate the stability of the variability, within a process.(http://www.businessdictionary.com/ definition/range-chart-R-Chart.html). This control chart is widely used to examine the stability of production processes in many industries.

**Theorem** (Kume, 1992, page 164)

If R is the range of a sample of size n extracted from a **normal** distribution  $N(\mu, \sigma^2)$ , Then

$$E(R) = d_2\sigma, \sigma_R = d_2\sigma$$

where

$\sigma$  is the standard deviation of the distribution or process,

$d_2$  and  $d_3$  are 2 ratios whose values depend on sample size ( $n$ ).  $d_2$  values are given in Table U. **End of Theorem.**

$d_2$  could also be calculated from:

$$d_2 = \int_{-\infty}^{+\infty} [1 - [1 - \varphi(z)]^n - \varphi(z)]^n dz,$$

Where  $\varphi(z)$  is the cumulative distribution function of standard normal.

Or from MATLAB for a given  $n$ :

```
n = ... ;pd = makedist('normal',0,1);fun = @(x) (1-(1- cdf(pd,x)).^n-(cdf(pd,x)).^n);
```

```
d2 = integral(fun,-inf,inf)
```

$d_3$  values are given in references such as Grant & Leavenworth(1988). Note that  $n$  is the sample size(volume) not the number of samples. For the calculation of the 3-sigma upper and lower control limits, 2 cases are distinguished below :  $\sigma$  known or unknown.

### 2.4.1 R control chart limits - $\sigma$ known

when a standard or specific value is given for the process standard deviation or is obtainable from the product specification limits L and U then

$$UCL_R = E(R) + 3\sigma_R = d_2\sigma + 3d_3\sigma = (d_2 + 3d_3)\sigma = D_2\sigma$$

Central line=  $E(R)$

$$LCL_R = E(R) - 3\sigma_R = d_2\sigma - 3d_3\sigma = (d_2 - 3d_3)\sigma = D_1\sigma$$

$D_1$  and  $D_2$ , given in Table U, are 2 factors whose values depend on sample size  $n$ .

When a negative value is obtained for  $LCL_R$ , set  $LCL_R = 0$ , because

$$R = X_{\max} - X_{\min} \geq 0.$$

### 2.4.2 R control chart limits - $\sigma$ unknown

When a standard or specific value is not given for the process standard deviation  $\sigma$  or is not obtainable from the product specification limit and the process follow a normal distribution, then

$$UCL_R = D_2\sigma \cong D_2 \frac{\bar{R}}{d_2} = D_4 \bar{R}$$

Central line=  $\bar{R}$

$$LCL_R = D_1\sigma \cong D_1 \frac{\bar{R}}{d_2} = D_3 \bar{R},$$

where

$\bar{R}$  is the mean of samples' ranges i.e.  $\bar{R} = \frac{\sum_{i=1}^k R_i}{k}$  given k successive subgroups,  
 $D_3$  and  $D_4$  are factors given in Table U.

### Example 2.5

Plot the R –chart for the information given the following table

Sample No.					R
1	20	15	18	24	9
2	32	19	20	14	18
3	35	17	45	34	28
4	34	19	25	16	18
5	30	10	11	21	20
6	42	9	18	36	32
7	45	32	44	18	27
					$\bar{R} = 21.71$

### Solution

Since  $\sigma$  is not given ,the limits are calculated as follows:

$$UCL_R = D_4 \bar{R} = 2.28 \times 21.71 = 49.5$$

$$\text{Central line} = \bar{R} = 21.71$$

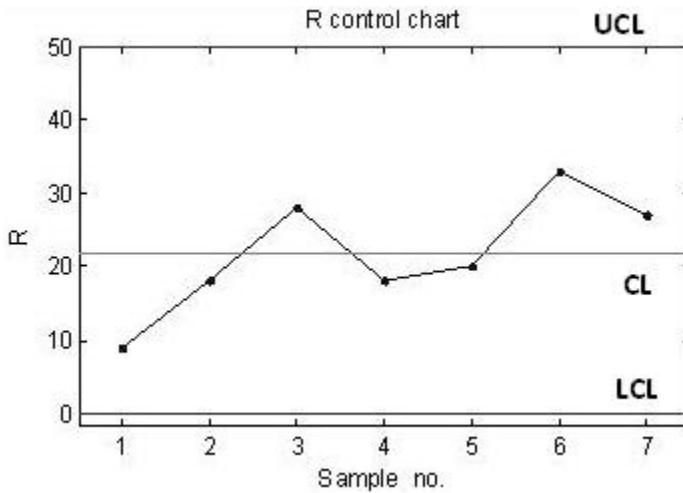
$$LCL_R = D_3 \bar{R} = 0 \times 21.71 = 0$$

The following commands could plot the chart as shown below in MATLAB( having statistical toolbox):

```
>>A=[...
    20    15    18    24
    32    19    20    14
    35    17    45    34
    34    19    25    16
    30    10    11    21
    42     9    18    36
    45    32    44    18];
```

Figure 2-7 shows the R control chart using the following MATLAB command

```
>> controlchart(A,'chart','R' )
```



**Fig. 2-7 R- chart for Example 2.5**

### Interpretation

Since the ranges of all the samples have fallen between the limits and no specific pattern is evident from the points, it is concluded the standard deviation of the process is under control. Once the R chart shows a state of statistical control, the process standard deviation can be estimated using

$$\hat{\sigma} = \frac{\bar{R}}{d_2}.$$

It should be added that this example is for teaching purposes; for practical use at least 20 ([www.freequality.org/documents/knowledge/x-bar R Charts.pdf](http://www.freequality.org/documents/knowledge/x-bar%20R%20Charts.pdf)) samples are needed to start constructing a control chart.

### 2.5 S-control chart

S chart is another control chart used to monitor the process variations when the characteristic of interest is quantitative. Note that

sample standard deviation ( $s = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{X})^2}{n-1}}$ ) is itself a random variable, and

therefore has its mean and standard deviation denoted by  $E(s)$  &  $\sigma_s$  respectively. The theory of statistics shows that if the process is normally distributed, the following relationship is available between  $\sigma_s$  and the process standard deviation ( $\sigma$ ):

$$\sigma_s = \sigma \sqrt{1 - c_4^2}.$$

And also the expected value of sample standard deviation satisfy the following relationship

$$E(s) = c_4 \sigma,$$

where

$$c_4 = \sqrt{\frac{2}{n-1}} \frac{\left(\frac{n-2}{2}\right)!}{\left(\frac{n-3}{2}\right)!} \text{ read from Table U}$$

$$\left(\frac{k}{2}\right)! = \Gamma\left(\frac{k}{2} + 1\right) = \text{Gamma function}\left(\frac{k}{2} + 1\right),$$

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad x > 0,$$

$$\left(\frac{n}{2} - 1\right)! = \Gamma\left(\frac{n}{2} - 1 + 1\right) = \Gamma\left(\frac{n}{2}\right),$$

$$\left(\frac{n-3}{2}\right)! = \Gamma\left(\frac{n-3}{2} + 1\right);$$

The following MATLAB command calculates  $c_4$  for a given sample size  $n$ :

$$c_4 = \text{sqrt}(2/(n-1)).*\text{gamma}(n/2)./ \text{gamma}((n-1)/2)$$

For the calculation of the upper- and lower 3-sigma control limits, 2 cases are distinguished:  $\sigma$  known or unknown.

### 2.5.1 S chart limits - $\sigma$ known

When a standard or specific value is given for the process standard deviation or it is obtainable from the product specification limits i.e. L and U then:

since

$$UCL_s = E(s) + 3\sigma_s$$

$$\text{Central line} = E(s)$$

$$LCL_s = E(s) - 3\sigma_s$$

Substituting for  $E(s)$  and  $\sigma_s$ :

$$UCL_s = c_4\sigma + 3\sigma\sqrt{1-c_4^2} = B_6\sigma$$

$$\text{Central line} = c_4\sigma$$

$$LCL_s = c_4\sigma - 3\sigma\sqrt{1-c_4^2} = B_5\sigma$$

If LCLs turned to be negative, let LCLs = 0.

Factors  $B_5$  and  $B_6$  whose values depend on sample size ( $n$ ) are given in Table U or by the following commands in MATLAB:

$$c4 = \text{sqrt}(2/(n-1)).*\text{gamma}(n/2)./ \text{gamma}((n-1)/2)$$

$$B6 = c4 + 3*\text{sqrt}(1-c4^2)$$

$$B5 = c4 - 3*\text{sqrt}(1-c4^2).$$

### 2.5.2 s chart limits - $\sigma$ unknown

When the standard deviation of the process is not given or could not be determined:

$$UCL_S = B_6\sigma = B_6 \frac{\bar{S}}{c_4} = B_4\bar{S}$$

Central line  $= \bar{S}$

$$LCL_S = B_3\bar{S}$$

Factors  $B_5$  and  $B_6$  are ratios whose values depend on sample size ( $n$ ) and given in Table U.

### Example 2.6

Plot the  $S$ -chart for the information given in the following table.

Sample No.					$s = \sqrt{\frac{\sum x_i^2 - n\bar{X}^2}{n-1}}$
1	20	15	18	24	3.77
2	32	19	20	14	7.63
3	35	17	45	34	11.62
4	34	19	25	16	7.94
5	30	10	11	21	9.42
6	42	9	18	36	15.37
7	45	32	44	18	12.63
					$\bar{S} = 9.77$

### Solution

$\sigma$  is not given; the following formulae are used:

$$UCL_S = B_4\bar{S} = 2.27 \times 9.77 = 22.17 =$$

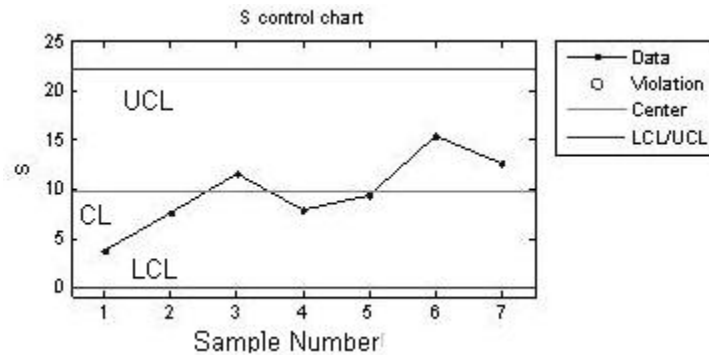
Central line  $= \bar{S}$

$$LCL_S = B_3\bar{S} = 0 \times 9.77 = 0.$$

The following commands could plot the chart as shown below in MATLAB (having statistical toolbox):

```
>>A=[...
    20      15      18      24
    32      19      20      14
    35      17      45      34
    34      19      25      16
    30      10      11      21
    42       9      18      36
    45      32      44      18];
```

```
>> controlchart(A,'chart','S' )
```



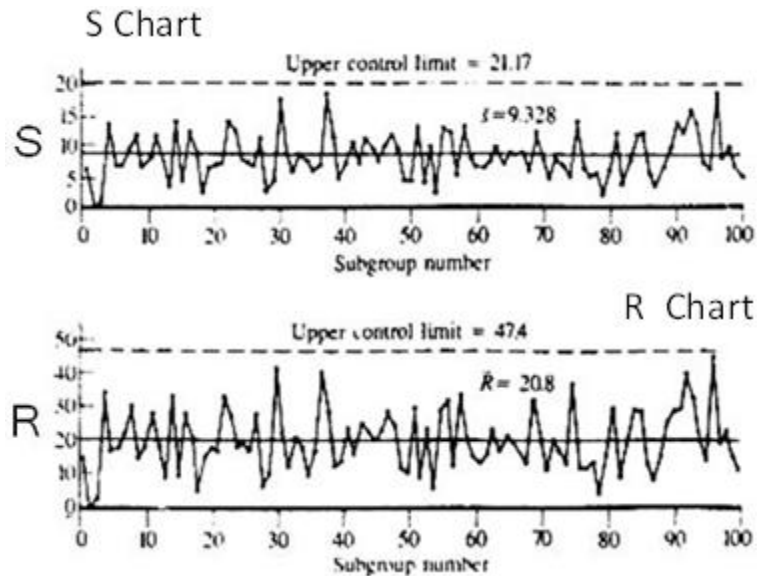
**Fig. 2-8 S- chart for Example 2.6**

Figures 2-7 & 2-8 show R-chart and s-chart for the same data. A quotation from Grant and Leavenworth(1988) is worth noting:

The similarity between the variations from subgroup to subgroup shown in these 2 charts(Fig2-9) is emphasized by the use of connecting lines<sup>1</sup>. It seems clear that these two charts tell practically the same story; there is no need to use both, either one may be used in any instance( Grant & Leavenworth, 1988 page 81).

Figure 2-9 illustrates this fact. It is worth remembering that the sample standard deviation ( $s$ ) uses all observations of the sample. However the calculation of  $R$  is easier when it is done manually. One must, however, be careful when using  $s$ -control chart. For even though one is using the limits  $E(s) \pm 3\sigma_s$ , it does not follow that the probability of having a value outside the control limits when the process is in control is 0.0027. Even if the data come from a normal distribution, the sample standard deviation ( $s$ ) will not be normally distributed( Derman & Ross, 1997 page 94). The same is true for all other charts when computing 3-sigma limits.

<sup>1</sup> In practical control –chart work in industry the points on the charts are sometimes connected and sometimes not.



**Fig 2-9 Control charts for sample standard deviation and range of 100 samples** (Grant & Leavenworth, 1988 page 80)

## 2.6 $S^2$ Control Chart

"There is no reason to first compute the subgroup  $S^2$  and then take its square root to obtain the sample standard deviation. One should just plot the successive values of  $S^2$  until one falls outside the lower and upper control limits of  $S^2$ -chart" (Derma & Ross, 1997 page 95). The upper and lower limits of this chart are given by the following relationships assuming the process is normally distributed.

### 2.6.1 $S^2$ Chart limits - $\sigma^2$ known

When a standard for the process variance is given or could be derived, the following relationships are applicable:

$$UCL_{S^2} = \sigma^2 \frac{\chi^2_{n-1, 0.00135}}{n-1}$$

$$\text{Central line} = E(S^2) = \sigma^2$$

$$LCL_{S^2} = \sigma^2 \frac{\chi^2_{n-1, 0.99865}}{n-1}$$

When a point does fall outside the chart, the process should temporarily cease and the system declared out of control.



### 2.6.2 $S^2$ Chart limits - $\sigma^2$ unknown

When a system is just starting out,  $\sigma^2$  is unknown but can be estimated from the average of the sample variances ( $\widehat{\sigma^2} = \overline{S^2} = \frac{\sum_i S_i^2}{k}$ ). The limits are as follows (Derman&Ross,1997 page 96):

$$UCL_{S^2} = \overline{S^2} \frac{\chi_{n-1,0.00135}^2}{n-1}$$

$$\text{Central line} = \overline{S^2}$$

$$LCL_{S^2} = \overline{S^2} \frac{\chi_{n-1,0.99865}^2}{n-1}.$$

As in the case of stating up an X-bar chart it then should be checked that the k subgroup variances used in estimating the process variance(  $\sigma^2$ ), all fall within these estimated control limits. If any of them fall outside , then a decision should be made as to whether the system was temporarily out of control or that no control has yet been reached. If the former decision is made then those subgroups whose variances fall outside the control limits should be discarded and the estimate of  $\sigma^2$  recomputed (Derman& Ross, 1997 p 96).

If we follow the convention of computing the three-sigma upper and lower limits for  $S^2$ -control chart we might get a more reasonable limits:

$$UCL_{S^2} = E(s^2) + 3\sigma_{s^2} =$$

$$\text{Central line} = E(s^2) = \sigma^2$$

$$LCL_{S^2} = E(s^2) - 3\sigma_{s^2}$$

assuming the process is normally distributed,  $\frac{n-1}{\sigma^2} s^2$  has a chi-squared distribution with n-1 degree of freedom; then

$$Var[\frac{(n-1)s^2}{\sigma^2}] = 2(n-1) \Rightarrow \sigma_{s^2} = \sigma^2 \sqrt{\frac{2}{n-1}} \text{ therefore:}$$

$$UCL_{S^2} = E(s^2) + 3\sigma_{s^2} = (1 + 3\sqrt{\frac{2}{n-1}})\sigma^2$$

$$\text{Central line} = \sigma^2$$

$$LCL_{S^2} = (1 - 3\sqrt{\frac{2}{n-1}})\sigma^2$$

If the lower limit turned to be negative let it be zero i.e.  $LCL_s=0$ .

When the process variance is not known, estimating it from

$$\widehat{\sigma^2} = \overline{S^2} = \frac{\sum_i S_i^2}{k},$$

we have:

$$UCL_{S^2} = (1 + 3\sqrt{\frac{2}{n-1}})\bar{S}^2$$

$$\text{Central line} = \bar{S}^2$$

$$LCL_{S^2} = (1 - 3\sqrt{\frac{2}{n-1}})\bar{S}^2$$

**Example 2.7**

Plot the  $S^2$ -chart for the information given in the following table.

Sample No.					$S^2 = \frac{\sum x_i^2 - n\bar{X}^2}{n-1}$
1	20	15	18	24	14.21
2	32	19	20	14	58.22
3	35	17	45	34	135.02
4	34	19	25	16	63.04
5	30	10	11	21	88.74
6	42	9	18	36	236.24
7	45	32	44	18	159.52
					$\bar{S}^2 = 107.86$

**Solution**

$\sigma^2$  is not known, therefore the following formulae are used

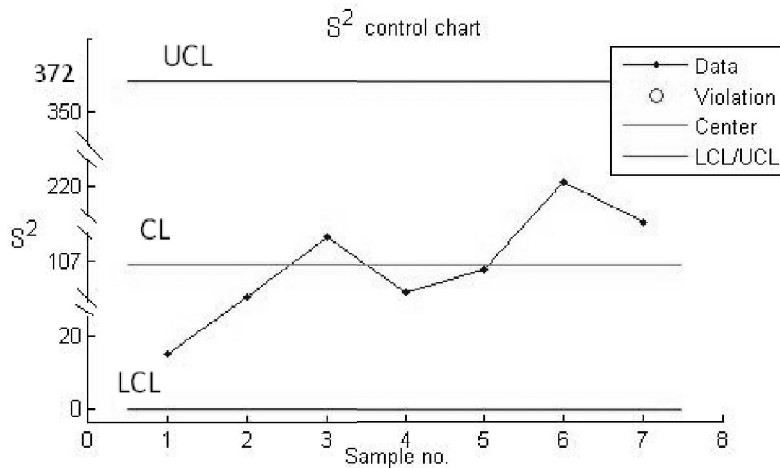
$$UCL_{S^2} = (1 + 3\sqrt{\frac{2}{n-1}})\bar{S}^2 = (1 + 3*\text{sqrt}(2/(4-1)))*107.86 = 372.06$$

$$= (1 + 3*\text{sqrt}(2/(4-1)))*107.86 = 372.06$$

$$\text{Central line} = \bar{S}^2 = 107.86$$

$$LCL_{S^2} = (1 - 3\sqrt{\frac{2}{n-1}})\bar{S}^2 = -156.342 \rightarrow 0$$

Figure 2-10 shows the control chart.



**Fig. 2-10 S² control chart for Example 2.7**

As the following calculations shows, if the formulae recommended by Derman & Ross (1997) were used, a wider interval between the limits would be gained:

$$USL_{S^2} = \bar{S^2} \frac{\chi_{4-1,0.00135}^2}{4-1}$$

$$\text{Central line} = \bar{S^2}$$

$$LSL_{S^2} = \bar{S^2} \frac{\chi_{4-1,0.99865}^2}{4-1}$$

$$\chi_{3,0.00135}^2 = 15.63 \quad \chi_{3,0.99865}^2 = 0.03 \quad \bar{S^2} = 107.86$$

Then:

$$USL_{S^2} = (107.86) \frac{15.63}{4-1} \cong 562$$

$$\text{Central line} = \bar{S^2} = 107.86$$

$$LSL_{S^2} = \bar{S^2} \frac{\chi_{4-1,0.99865}^2}{4-1} = (107.86) \frac{0.03}{4-1} \cong 10.8$$

## 2.8 Types of variations in product characteristics

Experienced quality men classify the variations in a characteristic of a product into two categories based on their causes: chance variations and special variations.

### Common or chance variations

Common variations are those whose sources could not be identified and are due to randomness; in other words the causes are un-assignable, and no special cause can be found for them. The chance causes are an inherent part of the process. When a control chart shows the process is in control, the variations are due to chance causes.

### Special or uncommon variations

Special variations are those variations that occur due to *assignable causes* which are identifiable outside natural variations. *Assignable* (or *special*) causes are those whose effect can be detected and controlled. Once detected, their removal is a relatively simple exercise. They occur due to outside elements such as people, material, machinery, measuring methods,...

When a control chart shows the status of in control, the process variations are due to chance causes and furthermore, if the limits have been calculated using a standard value given for a parameter, the in-control status indicates the parameter related to the chart is according to the standard.

When a control chart indicates the process parameter is out of control, the variations are due to special or assignable causes ; and the parameter of the process in practice is not according to the standard, if the limits have been calculated using a standard value.

It is worth mentioning that if the control limits have not been calculated using a standard value, the in-control status indicates only the production process continue to work under chance or common causes and the out-of-control status indicates that causes other than common causes called special or assignable causes govern the process.

#### 2.8.1 Some theory- of – run based tests for lack of control

For the study of research data, considerable work has been done on developing many tests based on the so-called theory of runs<sup>1</sup>, providing useful tools. To detect shifts in a manufacturing process parameter , the following rules are suggested(see Grant &Leavenwoth,1988, Page 89):  
There is suspicion that the process parameter has changed if

7 successive points on the control chart, all are on the same side of the central line(a *run of 7 points all above or all below the central line*).

In 11 successive points on the control chart, at least 10 are on the same side of the central line.

In 14 successive points on the control chart, at least 12 are on the same side of the central line.

In 17 successive points on the control chart, at least 14 are on the same side of the central line.

In 20 successive points on the control chart, at least 16 are on the same side of the central line.

---

<sup>1</sup> For studying the theoretical basis f the rules see references such as Grant & Leavenworth (1988) p 228

Therefore if any of the above happens in any Dr Shewhart 's chart, there is an indication that the parameter related to the control chart might have increased or decreased. For example If the control chart is  $\bar{X}$  Chart, the process mean; if it is R Chart, the process dispersion i.e. process standard deviation; if it is p chart, the defective fraction of the process has changed .

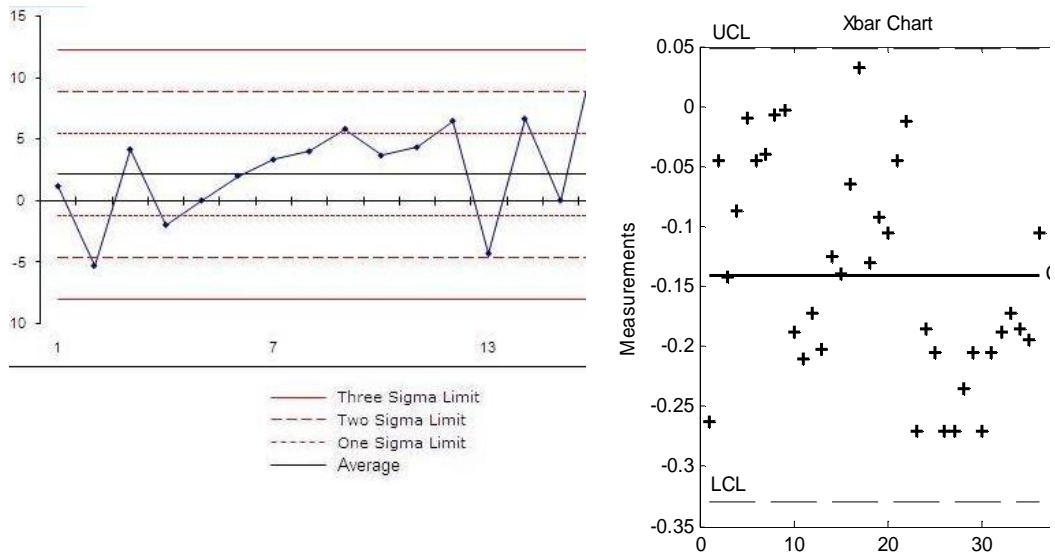


Fig. 2-12 An example for the run tests- case 1<sup>1</sup>

There are other tests; e.g. Western Electric co.( WECO) rules, AIAG rules; Boeing AQS Rules; Nelson rules. Nosekivicova(2012) lists more tests.

### 2.8.2 Application of $\bar{X}$ and (R or s or $s^2$ ) chart to make decision for the process

Two cases are distinguished when using  $\bar{X}$  chart together with R or s or  $s^2$  control charts for make decision on a production process :

The process is In-control

The process is out of control( whose causes have to be explored).

The in-control and out-control status of a process parameter were previously defined .

Out-of-control status in  $\bar{X}$  and R or s or  $s^2$  charts could indicate

The process mean ( $\mu$ )have changed

The process standard deviation( $\sigma$ ) has changed

Both  $\mu$  and  $\sigma$  have changed

<sup>1</sup> [http://www.brighthubpm.com/methods-strategies/74855-interpreting-control-charts-in-project-quality-management/#imgn\\_3](http://www.brighthubpm.com/methods-strategies/74855-interpreting-control-charts-in-project-quality-management/#imgn_3)

However, the control chart tells **when** to look for trouble but it cannot, by itself, tell **where** to look or **what** cause will be found( Grant& Leavenwoth , 1988 page 99)

Regarding the out of control status and the changes in  $\mu$  and  $\sigma$  it is pointed out that:

If the lack of control is due to the change in the process mean while the universe (process) standard deviation is unchanged, the lack of control and special variations will be indicated in the  $\bar{X}$  chart ; however, the change in the mean may cause lack of control to be indicated by a control chart. Since the control limits are located pretty far from the central line, a few points are expected to fall outside the chart; the above tests are helpful to see if the mean is in control or not.

A shift in process dispersion while the process mean is unchanged, may cause lack of control to be indicated on the  $\bar{X}$  chart as well as by a control chart for dispersion(Grant and Leavenworth,1988, pages 96-97) i.e. R or s or  $s^2$  chart.

In searching for the causes of an out of control condition, WECO Handbook advises the R-chart[or s or  $s^2$  chart ] pattern to be viewed first, making it possible to identify many causes directly; then to read the  $\bar{X}$  pattern first in the light of R-chart and then jointly with the R pattern [ or s or  $s^2$  pattern] to make it possible to obtain some other information( Grant& leavenwoth,1988 page97)

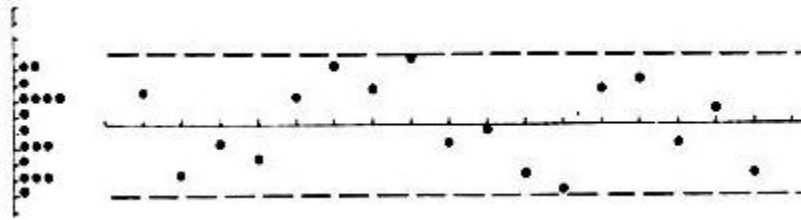
When the process(universe) dispersion as well as the process average is shifting, obviously lack of control will be indicated in both charts(Grant and Leavenworth,1988,pages 96-97), i.e. the R chart[ or s chart or  $S^2$  chart]as well as the  $\bar{X}$  chart.

It is reminded that assignable cause of variation may be due to measurement errors.

## 2.9 Interpreting pattern of variation on $\bar{X}$ and "R or s or $S^2$ chart"

It should be kept in mind that a control chart could tell us that there is a problem , but could not tell us by itself what or where the problem is. Studies including the one performed at Bell Telephone Labs in 1964 have categorized the most frequent assignable causes (Grant & Leavenwoth, 1988 page99). These researches could be used in teaching young experts, engineers and inspectors. As a sample , Fig. 2-13 illustrates a special pattern of variation and the corresponding frequent causes( Grant &Leavenworth,1988page100). Some other patterns are discussed in the same reference.

## Recurring cycles



Some causes affecting  $\bar{X}$  chart (stable variability):

Temperature or other recurring changes in physical environment  
Worker fatigue  
Differences in measuring or testing devices which are used in order  
Regular rotation of machines or operators  
Merging of subassemblies or other processes

Some causes affecting R chart:

Scheduled preventive maintenance  
Worker Fatigue  
Worn tools

**Fig. 2-13 A pattern of variations in the  $\bar{X}$  & R chart , and most frequent causes**(after Grant & Leavenworth,1988 page 100)

It is worth mentioning here that

if a point falls outside the  $\bar{X}$  chart, the cause could be disordering of the machine;

when rarely a control chart for dispersion( $R$ ,  $S$  or  $S^2$ ) indicates the status of out of control, using the chart could be stopped , even if the  $\bar{X}$  chart indicates lack of control frequently.

## 2.10 Type I and Type II errors in control chart

In statistical quality control, there are two types of errors:

*Type I error* : the **incorrect rejection** of a conforming production process

*Type II error* : the **incorrect accepting** (not rejecting) a nonconforming production process.

*The probability of Type I error is denoted by  $\alpha$  and*

*The probability of Type I error is denoted by  $\beta$ .*

*In using control charts*

$\alpha = \text{Pr}(\text{conclude statistically out of control although the process is truly in control})$ ;e.g.:

for x-bar control chart with  $\mu_0$  as the central line:

$$\alpha = 1 - \Pr(LCL_{\bar{X}} \leq \bar{X} \leq UCL_{\bar{X}} \mid \mu = \mu_0)$$

$\beta = \Pr(\text{conclude statistically in control} \mid \text{although the process is truly out of control})$ ; e.g. for  $\bar{x}$ -bar control chart with  $\mu_0$  as the central line:

$$\beta = \Pr(LCL_{\bar{X}} \leq \bar{X} \leq UCL_{\bar{X}} \mid \mu \neq \mu_0).$$

Using the theory –of-run –based tests might result in type I error.

The following table provides a representation of  $\alpha$  and  $\beta$ .

Representation of the probability of Type I & II errors		
	State of Nature	
	Process is conforming (in control )	Process is nonconforming (out of control)
Inspector accepts process (concludes it is in control )	OK (1 – $\alpha$ )	Error ( $\alpha$ )
Inspector rejects process (concludes it is out of control )	Error ( $\beta$ )	OK (1 – $\beta$ )

### 2.10.1 Effect of n on type II error in $\bar{X}$ control chart

Since  $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$ , if sample size n increases the distance between the central line and the limits in  $\bar{X}$  control chart decreases, the sensitivity of the chart increases and  $\beta$  decreases. This fact could be described as follows:

Suppose  $LCL_{\bar{X}} = \mu_0 - \frac{3\sigma}{\sqrt{n}}$  and  $UCL_{\bar{X}} = \mu_0 + \frac{3\sigma}{\sqrt{n}}$ ; and the process mean

has shifted to  $\mu = \mu_0 + k\sigma$ , then

$$\beta = \Pr(LCL_{\bar{X}} < \bar{X} < UCL_{\bar{X}} \mid \mu \neq \mu_0) =$$

$$\Pr\left(\frac{\mu_0 - 3\frac{\sigma}{\sqrt{n}} - (\mu_0 + k\sigma)}{\frac{\sigma}{\sqrt{n}}} < Z < \frac{\mu_0 + 3\frac{\sigma}{\sqrt{n}} - (\mu_0 + k\sigma)}{\frac{\sigma}{\sqrt{n}}}\right)$$



$$\Rightarrow \beta = \Pr(-3 - k\sqrt{n} < Z < 3 - k\sqrt{n}) = \int_{-3 - k\sqrt{n}}^{3 - k\sqrt{n}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

This integral is the area under the probability density function of standard normal distribution in an interval from  $-3 - k\sqrt{n}$  to  $3 - k\sqrt{n}$ . Although the length of the integral is 6 which does not depend on the  $k$  &  $n$ , but for a fixed constant  $k > 0$  as  $n$  increase the interval moves to the left and its area decreases. For a fixed constant  $k < 0$  as  $n$  increase the interval moves to the right and its area decreases; therefore in  $\bar{X}$  control chart when the process mean shifts to  $\mu = \mu_0 + k\sigma$ , and remains at this new mean ( $k$  is constant), as  $n$  increases  $\beta$  decreases. The following table shows some  $\beta$ 's computed from

$$\beta = \text{normcdf}(3 - k * \text{sqrt}(n)) - \text{normcdf}(-3 - k * \text{sqrt}(n)) \quad \text{or}$$

$$\beta = 0.5 * \text{erf}((3 - k * \text{sqrt}(n)) / \text{sqrt}(2)) - 0.5 * \text{erf}((-3 - k * \text{sqrt}(n)) / \text{sqrt}(2))$$

for  $k=3$  and several  $n$ ,  $\beta$  has the following values.

$n$	$\beta$
1	0.5000
2	0.1070
3	0.0140
4	0.0013
5	0.0001
6	0.000007

### 2.10.2 The effect of sample size ( $n$ ) on Type I error in 3-sigma $\bar{X}$ control chart

If  $UCL, LCL_{\bar{X}} = \mu_0 \pm 3\sigma_{\bar{X}}$ , the increase or decrease in sample size does not affect  $\alpha$ . The following calculations show in this situation if the process mean does not change and remains at  $\mu = \mu_0$ ,  $\alpha$  is constant and does not depend on sample size:

$$\begin{aligned} \alpha &= \Pr(\bar{X} > UCL_{\bar{X}} | \mu = \mu_0) + \Pr(\bar{X} < LCL_{\bar{X}} | \mu = \mu_0) \quad \Rightarrow \quad UCL, LCL_{\bar{X}} = \mu_0 \pm 3\sigma_{\bar{X}} \\ \alpha &= \Pr(Z > 3) + \Pr(Z < -3) \end{aligned}$$

### 2.11 Estimation of process mean and variance from $\bar{s}$ & $\bar{R}$ in an in-control process

When the control charts for process mean and dispersion i.e.  $\bar{X}$  control chart and  $R$  or  $S$  or  $S^2$  control charts indicate the process is in control, we

have accepted that if the means of the samples taken from the production process are kept between  $\bar{\bar{X}} \pm 3\sigma_{\bar{X}}$ , the population of the products will be between  $\mu \pm 3\sigma$ . If the process (and consequently the product population) is in control, the mean of the population or process is estimated as  $\hat{\mu} = \bar{\bar{X}}$ .

When R control chart indicates in-control status for a normally distributed process, the population or process dispersion  $\sigma$  could be estimated from  $\hat{\sigma} = \frac{\bar{R}}{d_2}$  (Grant & Leavenworth, 1988, page 130).

It should be pointed out that some books mention that estimator  $\hat{\sigma} = \frac{\bar{R}}{d_2}$

has a larger mean square error (MSE) than does  $\hat{\sigma} = \frac{\bar{s}}{c_4}$  and strongly

suggest using  $\hat{\sigma} = \frac{\bar{s}}{c_4}$  to estimate the standard deviation of in control

process (e.g. see Derman & Ross, 1997 page 92). Therefore when s-chart is used, to monitor the standard deviation of a normally distributed process

when the process is in control, estimate  $\sigma$  from  $\hat{\sigma} = \frac{\bar{s}}{c_4}$ .

If the  $S^2$  chart indicates the process is in control the process variance is estimated with the mean of the samples' variances i.e.  $\hat{\sigma}^2 = \bar{S}^2$ .

## 2.12 Estimation of process mean and variance from $\bar{s}$ , $\bar{R}$ in an out-of-control process

If we would like to estimate the process mean and standard deviation or to calculate the control limits of the charts for future use, when the process parameter(s) is(are) out of control based on the control charts whose control limits have been calculated as trial limits, act as follows:

### R-Chart:

If, for example, the range calculated for second sample falls outside the R-chart and its cause is assignable and could be eliminated or fixed, exclude the sample and calculate new values for  $\bar{R}$  and the control limits; if the new R-chart indicates in-control status,  $\sigma$  is estimated from  $\hat{\sigma} = \frac{\bar{R}}{d_2}$ , otherwise these steps are repeated until all out-of-limit values of R that have assignable and fixable cause has been excluded and R-chart indicates the process is in-control then estimate  $\hat{\sigma} = \frac{\bar{R}}{d_2}$ . In an algorithmic way:

- Try to judge from charts and all other pertinent information what may be the assignable causes and whether or it is likely they can be eliminated [or fixed]
- Discard the samples whose corresponding points fall outside the R or s or  $s^2$  chart and have assignable , fixable causes; then
- Calculate a revised  $\bar{R}$  or  $\bar{s}$  and new control limits for R-chart or s-chart or  $s^2$  chart, if again some points fall outside, those having assignable and fixable causes are eliminated;
- Repeat this procedure(calculate a revised  $\bar{R}$  or  $\bar{s}$  and control limits) until no point with assignable cause of variation fall outside the control.; then
- Estimate  $\sigma$  after all out-of-control values of R having assignable and fixable causes have been eliminated as follows using the final revised  $\bar{R}$  (Grant and Leavenwoth,1988 page 130)

$$\hat{\sigma} = \frac{\bar{R}}{d_2}.$$

This value is considered the process standard deviation as if the process were brought into control(Grant& Leavenworth,1988 page 130). Needless to say similar steps hold true, when using S or  $S^2$  chart for monitoring process dispersion.

If using S-chart similar steps, as described above in the case of using R-chart, have to be followed and repeated until the S-chart plotted with the remaining samples indicate the process is in control; then the process or population dispersion is estimated using  $\hat{\sigma} = \frac{\bar{s}}{c_4}$ .

### **$\bar{X}$ -Chart:**

Discard the points that fall outside the  $\bar{X}$  chart and have assignable , fixable causes of variation from the calculation of  $\bar{\bar{X}}$ ;

Calculate a revised  $\bar{\bar{X}}$  and new control limits for  $\bar{X}$  chart

Repeat the above steps if some points fall outside the new chart until no point having assignable and fixable causes fall outside.

Now your final  $\bar{\bar{X}}$  is as an estimate for the process mean.

The following quotation from Besterfield(1990) page 80 is worth noting:

There are 2 techniques used to discard data. If either the  $\bar{X}$  or the R value of a subgroup is out of control limits and has an assignable

cause, both are discarded, or only the out-of-control-limit value of a subgroup is discarded (Besterfield, 1990 page 80). In this book the latter is followed i.e. when an R or s value is discarded its corresponding  $\bar{X}$  value is not necessarily discarded and vice versa.

Note that Ishikawa (1983) advises:

The data of the samples falling outside control charts whose assignable cause has been identified and eliminated are not included in the recalculations.

The data of the samples falling outside charts whose cause cannot be identified or eliminated are included in the recalculations (Persian translation of Ishikawa, 1983).

### Example 2.8

Given the following data, calculate the trial upper and lower control limits for the R-chart and  $\bar{X}$  chart and estimate  $\mu$  &  $\sigma$ . The process is supposed to be normally distributed.

(Note the number of samples should be at least 20 to 25 at the beginning to calculate the trial limits, however, this is a class example;)

Sample No. (n=5)	$\bar{X}$	R
1	20	2
2	21	4.5
3	23	0.50
4	22	1.5

### Solution

Since no values have been given as standard  $\mu$  &  $\sigma$ , the following formulae are used:

### Trial Control limits for R-chart

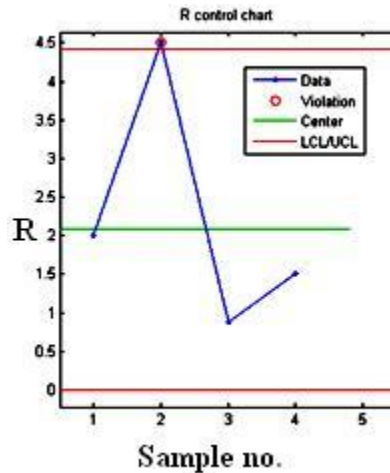
$$\text{Center line} = \bar{R} = \frac{2 + 4.5 + 0.5 + 1.5}{4} = 2.125$$

From Table U for  $n=5$   $D_4=2.11$   $D_3=0$

$$UCL_R = D_4 \bar{R} = 2.11 \times 2.125 = 4.48$$

$$LCL_R = D_3 \bar{R} = 0 \times 2.125 = 0$$

Plotting the R-chart will show that the range calculated for sample # 2 fall outside limits.



### Revised Control limits and estimation of $\sigma$

Suppose investigations reveals the cause of falling the range outside is assignable and could be eliminated; therefore we discard  $R=4.5$ ; the revised center line and new limits are:

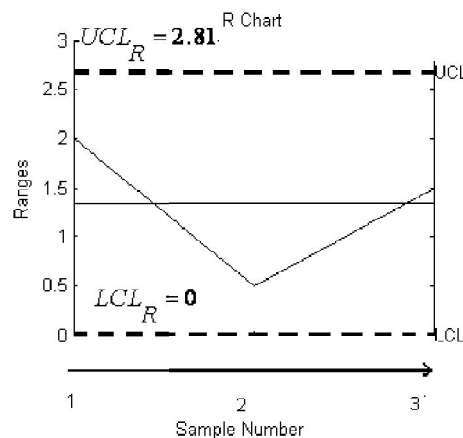
$$\bar{R} = \frac{2 + 0.9 + 1.5}{3} = 1.33$$

$$UCL_R = D_4 \bar{R} = 2.11 \times 1.33 = 2.81$$

$$\bar{R} = 1.33$$

$$LCL_R = D_3 \bar{R} = 0 \times 1.33 = 0$$

The following figure shows the R-chart with the revised control limits.



The new chart indicates the standard deviation is under control, there-fore

$$\hat{\sigma} = \frac{\bar{R}}{d_2} = \frac{1.33}{2.326} = 0.57 .$$

**Trial control limits for  $\bar{X}$ -chart**

Trial control limits for  $\bar{X}$ -chart are calculated and the chart is plotted:

$$\text{Central line } \bar{\bar{X}} = \frac{20+21+23+22}{4} = 21.50$$

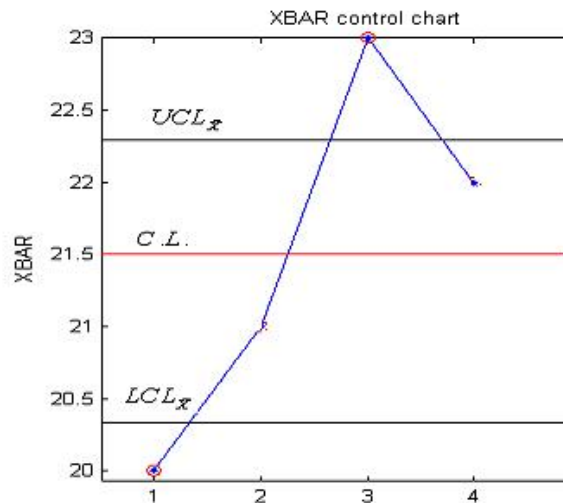
For  $n=5$  from Table U  $A_2=0.58$

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_2 \bar{R} = 21.5 + 0.58 * 1.33 = 22.27$$

$$CL = \bar{\bar{X}} = 21.5$$

$$LCL_{\bar{X}} = \bar{\bar{X}} - A_2 \bar{R} = 21.5 - 0.58 * 1.33 = 20.73$$

The following figure shows the  $\bar{X}$  control chart.



To estimate  $\bar{X}$ , suppose the cause of falling sample # 1 outside of control chart is un-assignable, but that of sample #3 was assignable therefore was eliminated.

The revised  $\bar{X}$  and the necessary calculations for limits is as follows:

$$\text{Central line } \bar{\bar{X}} = \frac{20+21+22}{3} = 21.00, R = \frac{2+4.5+1.5}{3} = 2.66$$

$n=5$  Table U  $A_2=0.58$

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_2 \bar{R} = 21.00 + 0.58 * 2.66 = 22.55$$

$$\bar{\bar{X}} = 21.00$$

$$LCL_{\bar{X}} = \bar{\bar{X}} - A_2 \bar{R} = 21.00 - 0.58 * 2.66 = 19.45$$

If we exclude sample #3 and plot the X-bar chart with the revised limits, we would see the chart shows the in-control status.

Since the new chart shows the mean under control, then  $\bar{\bar{X}} = 21.00$  is an estimate for the process mean.

### 2.13 Estimation of $\mu$ and $\sigma$ of a process- different sample sizes

We know that, given several samples, the mean and variance of the process is estimated as  $\hat{\mu} = \bar{\bar{X}}, \hat{\sigma}^2 = \bar{S}^2$ . If the samples have different sample sizes  $n_1, \dots, n_k$ , the following formulae are applicable:

$$\bar{X}_i = \frac{\sum_{j=1}^{n_i} x_{ij}}{n_i} \quad \bar{\bar{X}} = \frac{\sum_{i=1}^k n_i \bar{X}_i}{\sum_{i=1}^k n_i} \quad , \quad \bar{S}^2 = \frac{\sum_{i=1}^k (n_i - 1) s_i^2}{\sum_{i=1}^k (n_i - 1)} \quad \hat{\sigma}_i = \frac{s_i}{c_4}$$

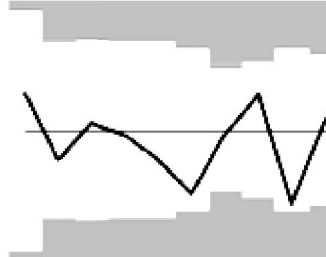
$$\hat{\sigma} = \frac{\sum \hat{\sigma}_i}{k}$$

$$UCL_{\bar{X}} = \bar{\bar{X}} + \frac{3\hat{\sigma}}{\sqrt{n_i}} \quad LCL_{\bar{X}} = \bar{\bar{X}} - \frac{3\hat{\sigma}}{\sqrt{n_i}}$$

where

$x_{i1}, x_{i2}, \dots, x_{ij}, \dots, x_{in_i} \quad i = 1, \dots, k$  are the observed values in  $i^{\text{th}}$  sample ( $i=1, \dots, k$ ).

The control chart with variable limits which look like the following figure will be discussed in chapter 7.



If the standard deviation of sample size is not large i.e. the sample sizes are not that different, an average sample size could be obtained from

$n = \bar{n} = \frac{\sum_{i=1}^k n_i}{k}$  and used in the calculation of control limits.  $\bar{\bar{X}}, \bar{s}$  are

calculated as usual way:  $\bar{\bar{X}} = \frac{\sum \bar{X}_i}{k}, \quad \bar{s} = \frac{\sum s_i}{k}.$

## 2.14 Continuing to use the chart- Revision of the central line and limits

Question:

To continue the use of charts for the coming period(week, month,...) can we use the trial control limits as calculated for the previous period?

Answer:

1)

If the process mean and standard deviation are in control and if the current ( trial )control limits have been calculated using the standard values for mean and standard deviation, and no point has fallen outside the control limits and the points are scattered between the limits randomly, continue using the current chart; if the standard has not changed.

2)

If the process mean and standard deviation are in control and if the current control limits have been calculated using  $\bar{\bar{X}}$  and  $\bar{S}$  or  $\bar{R}$  and the points are scattered between the limits randomly and no point has fallen outside the chart, the current control limits could be used in the future. They might be revised utilizing new information and accumulation of data.

3)

If the process mean and/or standard deviation are not in control and if the current control limits have been calculated using the standard values, even if points have fallen outside the control limits and the points are not scattered between the limits randomly, if the standard has not changed, continue using the current chart for future as well as trying to find the reason and fixing the cause(s).

4)

If the process mean and/or standard deviation are not in control and if the current control limits have been calculated using  $\bar{\bar{X}}$  and  $\bar{S}$  or  $\bar{R}$ , and some points have fallen outside the control limits, discard those points that fall outside and have assignable, fixable causes. Calculate new control limits using the remaining points and repeat the procedure until no point with assignable, fixable cause fall outside. Use the latest revised control chart for future. In the future every now and then revision might be necessary. More details are as follows:

If the process dispersion is in control but the mean is not i.e. S or R or  $S^2$  chart indicates in-control status but x-bar chart does not indicate the process mean is in control, to determine the revised mean and center line:

-if it is possible to determine a standard value  $\mu$ , use the standard value as the center line, otherwise a so-called aimed-at average  $\bar{X}'_0$  is calculated. A common procedure for obtaining an  $\bar{X}'_0$  is to modify the current process average with the information given by the process dispersion and specification limits; the modified value is taken as  $\bar{X}'_0$  ( refer to the extensive Example 2.13) .



- When with the calculated limits using this  $\bar{X}'_0$ , a considerable number of samples fall outside the limits, conclude the calculated  $\bar{X}'_0$  is poor and not a suitable one. Come to this conclusion if by a fairly simple and low-cost adjustment of manufacturing process, such as a machine adjustment influencing a dimension, the process average could not be altered. In such cases it is more sensible to use the past  $\bar{X}$  as the new central line or possibly a revised  $\bar{X}$  corrected by discarding the past out of control points (Grant and Leavenworth, 1988 page 132). A simple way to calculate a revised  $\bar{X}$ , is to eliminate the past out-of-limit points whose causes are assignable and can be eliminated.

5)

"When R[or s] control chart shows that the process dispersion is out of control, [and the X-bar chart shows the mean is in control] it is desirable to estimate the value of  $\sigma$  that might be attained if the dispersion were brought into control. This estimate is necessary even though it cannot be made with great assurance that it is correct. One possible method is to eliminate the values of R[or s] above the control limits and make a new calculation of  $\bar{R}$  [or  $\bar{s}$ ]. If new limits calculated from this  $\bar{R}$  throw more points above the control limits, the calculation of a revised  $\bar{R}$  [or  $\bar{s}$ ] may be repeated again. [This method is illustrated in the extensive example 2.13]. This revised  $\bar{R}$  [or  $\bar{s}$ ] may be used as the new central line on the chart as the basis for calculating the new limits on the R[or S] chart and for calculating  $A_2\bar{R}$  [or  $A_3\bar{s}$ ] to get the distance of the control limits from the central line on the  $\bar{X}$  chart"(Gran & Leavenworth, 1988 page 132).

6)

Where both X-bar chart and the chart for dispersion indicate out of control, the calculation of the revised  $\bar{R}$  or  $\bar{s}$  should be made before a decision on the  $\bar{X}'_0$ . When R [or s] control chart indicates lack of control, do not interpret  $\bar{X}$  chart. Frequently discarding the points that fall outside the R[or s] chart and have assignable causes which could be eliminated leads in the elimination of the special points of the  $\bar{X}$  chart"(Naghandarian, 1993 p163)

Regarding  $\bar{X}$  chart, calculate either aimed-at-mean (this will be shown in the extensive Example 2.13) or a revised  $\bar{X}$  and new control limits for the chart (discarding the points with assignable fixable causes falling outside the chart, repeat this procedure if necessary and use them for future ).

Needless to say, after the identification of special causes of the samples falling outside, sufficient efforts should be done to eliminate or fix the causes.

Some references study the relationship between an in-control process and one or two specification limits<sup>1</sup>; e.g. refer to Grant and Leavenworth(1988) page 127.

**Example 2.9** (Based on Problem 4-5 Grant & Leavenworth, 1988, page 145)

Automatic machinery is used to fill and seal 10-oz a cans of a certain liquid product. The process standard deviation is 0.20 oz. To ensure that every can meets or exceeds this 10-oz minimum, the company has set a target value for the process of 11.0 oz.

At this process average of 11.0 oz, what percent of cans will have less than 10.5 oz and greater than 10 oz. of product? Assume contained weight are normally distributed.

If the quality control section samples these cans in subgroups of four, what will 3-sigma control limits be for the  $\bar{X}$  chart,?

Assuming that  $\pm 3\sigma$  natural tolerance limits on the process cover virtually all the filled cans, what is the minimum value to which the process average may be lowered to ensure that virtually no cans are filled with less than a minimum of 10 oz?

**Solution**

a)

$$\Pr(10 < X < 10.5) = \text{normcdf}(10.5, 11, .2) - \text{normcdf}(10, 11, .2) = 0.0062 - 0.0000$$

or

$$= \Pr\left(\frac{10-11}{0.2} < Z < \frac{10.5-11}{0.2}\right) = \Pr\left(\frac{-1}{0.2} < Z < \frac{-0.5}{0.2}\right) = \Pr(-5 < Z < -2.5) = 0.0062 - 0 = 0.0062 = 0.62\%$$

b)

$$UCL_{\bar{X}} = \mu + 3\sigma_{\bar{X}} = \mu + \frac{3}{\sqrt{n}}\sigma = 11 + \left(\frac{3}{2} \times 0.2\right) = 11.3$$

$$CL = \mu = 11$$

$$LCL_{\bar{X}} = \mu - 3\sigma_{\bar{X}} = 11 - 0.3 = 10.7$$

c) since  $\Pr(Z < -3) = 0.00135 \approx 0.00$  then

$$\Pr(X < 10) = 0 \approx \Pr(Z < -3) \rightarrow \Pr\left(Z < \frac{10 - \mu}{\sigma}\right) = \Pr(Z < -3) \rightarrow \mu = 10.6$$

or

$$\mu - 3\sigma \geq 10 \Rightarrow \mu - 3(0.2) \geq 10 \Rightarrow \mu \geq 10 + 0.6 = 10.6$$

<sup>1</sup> The specification limits of a dimension of a product are usually denoted by USL or U for upper limit and by LSL or L for lower limit. For example if the tolerance is  $140.0 \pm 0.1$ , then  $USL = U = 139.9$  and  $LSL = L = 140.1$ .

**Example 2.10**

a) Find the control limits for the  $\bar{X}$  & R chart, given the data in the following table.

b) The specification is  $395 \pm 30$ . Does the process satisfy the specification limits?

c) What control limits do you propose for future?

$i$	$\sum_{j=1}^5 x_{ij}$	$\bar{X}_i$	$R_i$	Calculations for part (a)
1	2003	400.6	30	$n = 5, k = 25, \sum_{i=1}^{25} \sum_{j=1}^5 x_{ij} = 49323, \sum \bar{X}_i = 9864.6, \sum R_i = 544$ $\bar{X}_i = \frac{\sum_{j=1}^5 x_{ij}}{5}, \bar{\bar{X}} = \frac{9864.6}{25} = 394.58$ $\bar{R} = \frac{\sum R}{k} = \frac{544}{25} = 21.76, \hat{\sigma} = \frac{\bar{R}}{d_2} = \frac{21.76}{2.326} = 9.36$ $UCL_{\bar{X}} = \bar{\bar{X}} + \frac{3}{\sqrt{n}} \hat{\sigma} = 394.58 + \frac{3}{\sqrt{5}} \times 9.36 = 407.14$ $CL = \bar{\bar{X}} = 394.58$ $LCL_{\bar{X}} = \bar{\bar{X}} - \frac{3}{\sqrt{n}} \hat{\sigma} = 394.58 - \frac{3}{\sqrt{5}} \times 9.36 = 382.04$ $UCL_R = D_4 \bar{R} = 2.11 \times 21.76 = 45.91$ $CL = \bar{R} = 21.76$ $LCL_R = D_3 \bar{R} = 0 \times 21.76 = 0$
2	1922	84.4	25	
3	1966	93.2	27	
4	1930	386	16	
5	2010	402	21	
6	1956	91.2	24	
7	2000	400.	23	
8	1979	395.8	14	
9	1982	396.4	15	
10	1967	393.4	31	
11	2021	404.2	38	
12	1941	388.20	19	
13	1949	389.8	29	
14	1983	396.6	21	
15	1980	396	13	
16	1994	398.8	24	
17	1990	398	20	
18	1961	392.2	18	
19	1990	398.	28	
20	1951	390.2	33	
21	1964	392.8	19	
22	1965	393.0	36	
23	1988	397.6	5	
24	1982	396.4	8	
25	1949	389.8	7	
sum	49323	9864.6	544	

**Solution**

a) calculations are done inside the table.

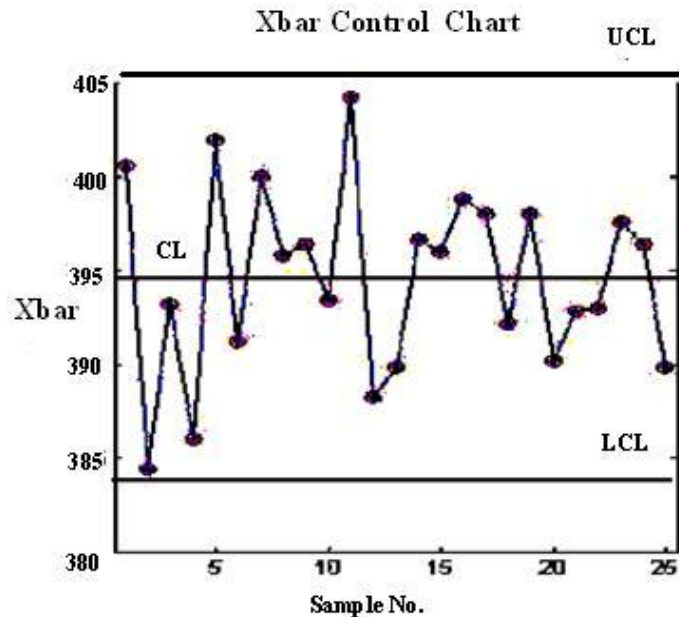
b) All 125 measurements fall within these limits and

$$U = 425, L = 365 \Rightarrow U - L = 60, 6\sigma = 6 \times 9.36 = 56.16 < U - L$$

$$\bar{\bar{X}} = 394.58 \cong \frac{U + L}{2}, U - L < 6\sigma;$$

Therefore it may be economical to discontinue the use of control chart.

c)  $\bar{X}$  chart is shown in the following figure. Plotting R chart is left an exercise for the students. No sample falls outside the limits of  $\bar{X}$  & R chart; it is not that necessary to use these control charts as far as the same condition prevail; if we want to use control charts, continue to use the current limits.



### Example 2.11

A production process is to be operated for a product with specification  $48 \pm 3$ .

a) Assuming a normal distribution, determine a mean and standard deviation on which the process should be adjusted so that the specification coincides  $\mu \pm 3\sigma$ .

b) Suppose sample size is 4. Find the upper and lower 3-sigma control limits for  $R, \bar{X}$  charts

### Solution

a)

$$U = 48 + 3 = 51 \quad \Rightarrow \quad U - L = 51 - 45 = 6$$

$$L = 48 - 3 = 45$$

$$U - L = 6\sigma \Rightarrow 6 = 6\sigma \Rightarrow \sigma = 1$$

$$\mu + 3\sigma = U \Rightarrow \mu + 3(1) = 51 \Rightarrow \mu = 51 - 3 = 48$$

$$b) UCL_{\bar{X}} = \mu + \frac{3}{\sqrt{n}}\sigma = 48 + \frac{3}{\sqrt{4}}(1) = 49.5$$

Central line:  $\mu = 48$

$$LCL_{\bar{X}} = \mu - \frac{3}{\sqrt{n}} \sigma = 48 - \frac{3}{\sqrt{4}} (1) = 46.5$$

$$UCL_R = D_2 \sigma = 4.70(1) = 4.70$$

$$\text{Central line: } d_2 \sigma = 2.059(1) = 2.06$$

$$LCL_R = D_1 \sigma = 0(1) = 0$$

### Example 2.12

(Problem 4-20 Grant & Leavenworth, 1988 page 148)

a) A fair percentage of a certain product requires costly work operations to change a certain quality characteristic. Rework is possible whenever the quality characteristic falls above the upper specification limit. If the value falls below the lower specification limit, the product must be scrapped. X-bar and R charts have been maintained for 50 subgroups of size 5 each with samples taken every two hours. The specification limits are  $119 \pm 10$ . The process appears to be in statistical control with mean of 124 and standard deviation of 5. On the assumption that the quality characteristic is normally distributed, approximately what percentage of defective articles is being produced? How much of this can be reworked?

b) Calculate the control limits for s-chart.

### Solution

a) The percent of conforming =

$$\Pr(109 < X < 129) \Rightarrow \Pr\left(\frac{109 - 124}{5} < Z < \frac{129 - 124}{5}\right) = \Pr(-3 < Z < 1) =$$

$$= 0.8413 - 0.00135 = 0.83995 \approx 84\%$$

normcdf(1) - normcdf(-3) = 0.84: MATLAB

The percent of nonconforming =  $1 - 0.84 = 0.16$

Rework:

$$\Pr(X > 129) = \Pr\left(Z > \frac{129 - 124}{5}\right) = \Pr(Z > 1) = 1 - 0.8413 = 0.1587 = 15.87\%$$

$$\text{b) } UCL_s = B_6 \sigma = 1.96 \times 5 = 9.8$$

$$\text{Central line} = c_4 \sigma = 0.94 \times 5 = 4.7$$

$$LCL_s = B_5 \sigma = 0 \times \sigma = 0$$

Factors  $C_4$ ,  $B_5$  &  $B_6$  are available from Table U or using the following MATLAB commands:

$$c_4 = \text{sqrt}(2/(n-1)) \cdot \text{gamma}(n/2) / \text{gamma}((n-1)/2)$$

$$B_6 = c_4 + 3 \cdot \text{sqrt}(1 - c_4^2)$$

$$B_5 = c_4 - 3 \cdot \text{sqrt}(1 - c_4^2)$$

**Example 2.13**

An extensive example - (Example 4-3 Grant and Leavenworth, 1988, page 136)

**Milling a slot in an aircraft terminal block .An Example to Illustrate the Steps in the Use of  $\bar{X}$  and R Charts on a Manufacturing Operation Decisions Preparatory to the Control Chart**

High percentages of rejections for many of the parts made in the machine shop of an aircraft company indicated the need for examination of the reasons for trouble. As most of the rejections were for failure to meet dimensional tolerances, it was decided to try to find the causes of trouble by the use of  $\bar{X}$  and R charts.

These charts, which of course required actual measurement of dimensions, were to be used only for those dimensions which were causing numerous rejections. Among many such dimensions, the ones selected for control charts were those having high costs of spoilage and rework, and those on which rejections were responsible for delays in assembly operations. Although the initial purpose of all the  $\bar{X}$  and R charts was to diagnose causes of trouble, it was anticipated that some of the charts would be continued for routine process control and possibly for acceptance inspection.

This example deals with one of these dimensions, the width of a slot on a duralumin forging used as a terminal block at the end of an airplane wing spar. The final machining of this slot width was a milling operation. The width of the slot was specified as  $0.8750 \begin{Bmatrix} +0.0050 \\ -0.0000 \end{Bmatrix}$  inch. The designing engineers had specified this dimension with a unilateral tolerance because of the fit requirements of the terminal block; it was essential that the slot width be at least 0.8750 in and desirable that it be as close to 0.8750 as possible.

Most of the aircraft parts produced in this machine shop were large parts fabricated in lots the size of which varied from a few hundred to several thousand. It was felt that practical considerations called for a single decision as to the method of subgrouping and the size and frequency of sample to apply to all the  $\bar{X}$  and R charts to be used. One limiting factor was the small number of available personnel for the control chart inspection in relation to the number of control charts it was desired to keep. On this basis it was decided that for each chart the sample inspected would be approximately 5% of the total production of the part in question. Because of the many general considerations favoring five as the subgroup size, this size was adopted. It was considered essential that, wherever possible, all measurements be made at the point of production. As lots of five of these large parts did not accumulate at the machine, it was decided that one part would be measured out of approximately every 20 produced, and that a subgroup would consist of five such measurements

Table 2-1 $\bar{X}$ & R control chart data sheet											
Product: Terminal Block(TB)				Dept No. 78		Order# 54321					
Characteristic :width of slot				Specified limits		{0.8800 Max 0.8750 Min					
Unit of Measurement 0.0001 in. over 0.8000											
Subgroup No.	1	2	3	4	5	6	7		$\bar{X}$	R	
	a	772	756	756	744	802	783	747	1	770	85
	b	804	787	773	780	726	807	766	2	750	54
	c	779	733	722	754	748	791	753	3	751	51
	d	719	742	760	774	758	762	758	4	765	36
	e	777	734	745	774	744	757	767	5	780	76
Total		3851	3752	3756	3826	3778	3900	3791	6	780	50
Average, $\bar{X}$		770	750	751	765	756	780	758	7	758	20
									8	771	38
Range, R		85	54	51	36	76	50	20	9	748	16
Date		3/7	3/7	3/7	3/8	3/8	3/8	3/9	10	717	36
Subgroup No.		8	9	10	11	12	13	14	11	737	38
									12	740	20
	a	788	757	713	716	747	749	771	13	769	38
	b	750	747	730	730	727	762	767	14	772	20
	c	784	741	710	752	763	778	785	15	768	13
	d	769	746	705	735	734	787	772	16	777	27
	e	762	747	727	751	730	771	765			
Totals		3853	3738	3585	3684	3700	3847	3860			
Average $\bar{X}$		771	748	717	737	740	769	772		12129	621
Range, R		38	16	25	36	36	38	20	Calculations of limits		
Date		3/9	3/9	3/10	3/10	3/10	4/2	4/2			
Subgroup No.		15	16								
	a	771	767	$\bar{X} = \frac{12129}{16} = 758$ $\bar{R} = \frac{621}{16} = 39$							
	b	758	769	$A_2 \bar{R} = 0.58(39) = 23$							
	c	769	770	$UCL_{\bar{X}} = \bar{X} + A_2 \bar{R} = 758 + 23 = 781$							
	d	770	794	$LCL_{\bar{X}} = \bar{X} - A_2 \bar{R} = 758 - 23 = 735$							
	e	771	786	$UCL_R = D_4 \bar{R} = 2.11(39) = 82$							
Total		383	388	$LCL_R = D_3 \bar{R} = 0$							
Average, $\bar{X}$		768	777								
Range, R		13	27								
Date or time		4/3	4/3								
<p>To find the actual values of <math>\bar{X}</math>, <math>X</math> insert 0.8 before the numbers in this table; e.g. for <math>UCL_{\bar{X}} = 781</math> the actual value is <math>UCL_{\bar{X}} = 0.8781</math>.</p> <p>To find the actual values of <math>R</math>, <math>\bar{R}</math> insert 0.00 before the numbers in this table; e.g. for <math>\bar{R} = 39</math> the actual value is <math>\bar{R} = 0.0039</math></p>											

The type of form used for recording the data is illustrated in Table 2-1. It was chosen as a result of the decision to measure many of the dimensions to the nearest ten thousandth of an inch; it was believed that with so many significant figures, delays and errors would be introduced by any type of form calling for much mental arithmetic. If measurements had been made

only to thousandths of an inch, the other type of form would have been appropriate. This is illustrated in Table 2-2, in which the same measurements as in Table 2-1 have been recorded to the nearest thousandth of an inch.

Table2-2 another form of recoding data for $\bar{X}$ & R charts										
Material or part name		RECORD SHEET FOR $\bar{X}$ & R CHART								
Terminal block		Plant No 1-23-45								
Characteristic measured		Width of slot					Plant :6 Dept: 78			
Unit of measurement		0.001 in. Over 0.800					<i>Red by H. B.</i>			
Samp. No.	date	measurements					$\bar{X}$	R		
		A	B	C	D	E				
1	3/7	77	80	78	72	78	77.0	8		
2		76	79	73	74	73	75.0	6		
3		76	77	72	76	74	75.0	5		
4	3/8	74	78	75	77	77	76.2	4		
5		80	73	75	76	74	75.6	7		
6		78	81	79	76	76	78.0	5		
7	3/9	75	77	75	76	77	76.0	2		
8		79	75	78	77	76	77.0	4		
9		76	75	74	75	75	75.0	2		
10	3/10	71	73	71	70	73	71.6	3		
11		72	73	75	74	75	73.8	3		
12		75	73	76	73	73	74.0	3		
13	4/2	77	77	78	79	77	77.0	4	Operator's check measurements	
14		77	77	78	77	76	77.0	2	have been	
15	4/3	77	76	77	77	77	76.8	1	made on hot part.	
16		77	77	77	79	79	77.8	2	Instructed to wait until part	
17									has cooled before making	
18									check measurement and to	
19									center process at 0.8775 inches	
20										
	Total	S					1212.8	61		
$\bar{\bar{X}} = \frac{1212.8}{16} = 75.8 \quad \bar{R} = \frac{61}{16} = 3.8$ $A_2 \bar{R} = 0.58(3.8) = 2.2$ $D_4 \bar{R} = 2.11 \times 3.8 = 8.0$							$UCL_{\bar{X}} = 75.8 + 2.2 = 78.0$ $LCL_{\bar{X}} = 75.8 - 2.2 = 73.6$ $UCL_R = 8.0, LCL_R = 0$			

The method of inspection to secure data for each  $\bar{X}$  and R chart was stated in written instructions. In the case of the slot width of the terminal block, this was to measure the width with a micrometer at two specified positions in the slot. The recorded slot width was the average of these two measurements.

### Starting the Control Charts.

The actual measurements for the first 16 subgroups are shown in Table 2-1. This number of subgroups corresponds to a production order for 1,600 of these terminal blocks. Averages and ranges were calculated as shown in Table 2-1 and were plotted as shown in Figures 2-14&2-15



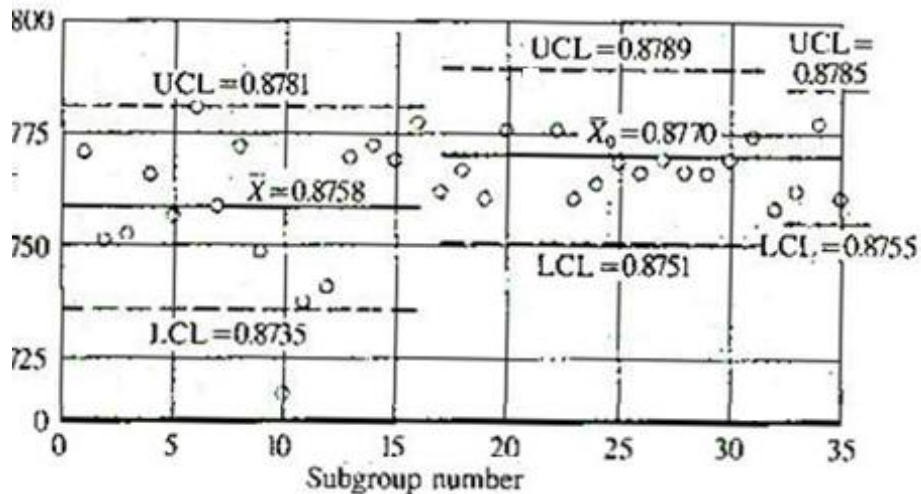


Fig.2-14  $\bar{X}$  chart for Example 2.13

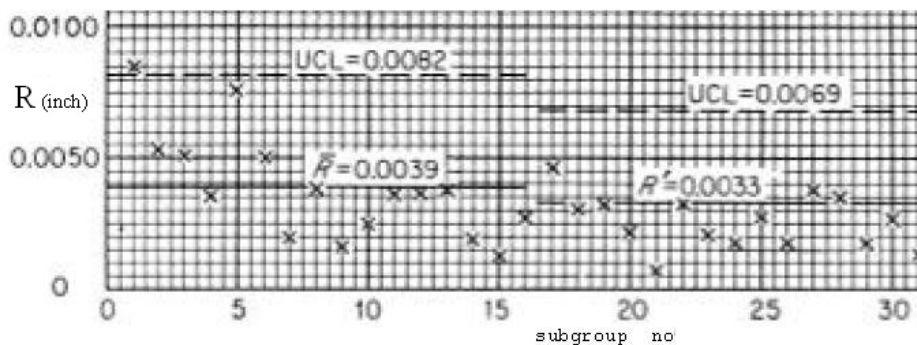


Fig. 2-15 R chart for Example 2.13

At the time of the twelfth subgroup, before the completion of this production order and before the calculation of central line or control limits, the quality control inspector noticed that the machine operator was occasionally checking his performance by a micrometer measurement on width of slot on a terminal block which had just come off the machine. As the block was still hot from the milling operation, this dimension as measured by the operator was too high because of the expansion of the metal due to temperature. Moreover, the operator was influenced by the unilateral tolerance to aim at a dimension at or very slightly above the nominal dimension of 0.8750 in.

Even without a central line or control limits, it was evident from the chart and the data sheet that this was producing many slots that were too narrow. After the twelfth subgroup the operator was instructed to make his check measurements on parts that had cooled to room temperature and to aim at a dimension of 0.8775, halfway between the upper and lower tolerance limits. This was reflected in the results in subgroups 13 to 16.

### Determining the Trial Control Limits

Calculation of trial limits as made after the first 16 subgroups which completed the production order. As shown in Table 2-1, this was done using the  $A_2$  and  $D_4$  factors and formulas from Table U. These trial control limits are shown for the first 16 subgroups in the control charts in Table 2-2.

### Drawing Preliminary Conclusions from the Charts.

Subgroup 1 is above the upper control limit on the R chart. Subgroup 10 is below the lower control limit on the  $\bar{x}$  chart. Moreover, the last 10 of the 16 points on the R chart all fall below the central line. It is evident that the measurements obtained are not the result of a constant system of chance causes.

If subgroup 1 is eliminated from consideration, R for the remaining 15 subgroups is  $536/15 = 36$ . This gives as the revised upper control limit  $D_4\bar{R} = 2.11 \times 36 = 76$

Subgroup 5 falls exactly on the control limit.

A common experience on hand-operated machines, where the dispersion of a controlled process is dependent in part on the care taken by the operator, is that the introduction of the control chart increases the care taken by the operator and thus reduces process dispersion. For this reason the ranges of the first few subgroups may not be representative of what may be expected as time goes on. The general appearance of this R chart with its run of the last 10 points below R suggests this as probably true of the slot width. Hence a second revision of R, with subgroup 5 eliminated, seems reasonable. This gives  $\bar{R} = 460/14 = 33$ ; that is  $\bar{R} = 0.0033$  in.

From this second revision of R an estimate may be made of  $\sigma$ , the process standard deviation that might be anticipated if the process were controlled in the future. This estimate of  $\hat{\sigma} = \bar{R}/d_2 = 0.0033/2.326 = 0.0014$  in. If this should be the value of  $\sigma$ , the natural tolerance or spread of the controlled process,  $6\sigma$ , will be  $6(0.0014) = 0.0084$ . This spread may be compared with the tolerance spread:  $U - L = 0.8800 - 0.8750 = 0.0050$ .

It is evident that the natural tolerance of this process is considerably greater than the specified tolerance. Unless the process dispersion can be reduced, it is evident that even though the process can be brought into control a high percentage of nonconforming product will be produced.

It is evident that the dispersion is too wide and the process average ( $\bar{\bar{X}} = 0.8758$ ) is too low. It is also evident that the process average is capable of adjustment; the instructions given to the operator after subgroup 12 seemed definitely to raise the average. At first glance, it would seem that the most desirable thing to do is to aim at minimum rejections by trying to hold the process in control at an average midway between the upper and lower specification limits, namely, at 0.8775.

However, this conclusion fails to give weight to the fact that a slot that is too narrow can be widened but a slot that is too wide cannot be narrowed. In other words, rework is less costly than spoilage. It is therefore desirable to center the process at a level that results in few slots over the upper specification limit of 0.8800 in., even though a number may be under the lower specification limit of 0.8750 in. The upper 3-sigma limit on individual values might be placed at 0.8800 to make a trial calculation of the aimed-at  $X'_0$ . If this is done  $X'_0 + 3\hat{\sigma} = 0.8800 \rightarrow X'_0 + 3(0.0014) = 0.8800 \rightarrow X'_0 = 0.8758$ .

By chance, this is exactly the  $\bar{\bar{X}}$  of the first 16 subgroups. It is evident that such a centering will continue to result in considerable rework.

Experience on similar jobs indicates that it is reasonable to expect that process dispersion may be further reduced. Hence it seems wise to center the process somewhat above 0.8758. Just how much above depends on how much improvement is expected and on the relative costs of spoilage and rework. A figure of 0.8770 was selected.

### Continuing to Use the Charts.

For the continuation of the control chart for the next production order, which started several weeks later, the central line was set as  $X'_0 = 0.8770$ . The three-sigma control limits were based on assuming  $\hat{\sigma} = 0.0014$ . Using Table U

$$UCL_{\bar{X}} = X'_0 + A\hat{\sigma} = 0.8770 + 1.34(0.0014) = 0.8789,$$

$$\text{Central line} = X'_0 = 0.8770,$$

$$LCL_{\bar{X}} = X'_0 - A\hat{\sigma} = 0.8770 - 1.34(0.0014) = 0.8751,$$

$$UCL_R = D_2\hat{\sigma} = 4.92(0.0014) = 0.0069,$$

$$(\text{Central Line})_R = d_2\hat{\sigma} = 2.326(0.0014) = 0.0033,$$

$$LCL_R = D_1\hat{\sigma} = 0.$$

(As  $\sigma$  was estimated from an  $\bar{R}$  of 0.0033, the same limits would have been obtained using the formulas  $\bar{\bar{X}} \pm A_2\bar{R}$  and  $D_4\hat{\sigma}$  and  $D_3\hat{\sigma}$ , with this  $\bar{R}$ .)

These limits are shown for subgroups 17 to 32 on the control charts of Fig. 2-14&2-15. The averages and ranges of these subgroups were as follows: (As in Table 2-1,  $\bar{X}$  is in units of 0.0001 in. above 0.8000 and R is in units of 0.0001 in.):

Subgroup No.	$\bar{X}$	R
17	761	47
18	766	31
19	760	32
20	775	22
21	788	7
22	775	32
23	760	21
24	763	18
25	768	27
26	766	17
27	769	38
28	766	35
29	766	17
30	769	26
31	774	14
32	774	14
totals	12284	408

None of the pieces inspected for control-chart inspection in subgroups 17 to 32 fell outside the specification limits. The average values for these 16 subgroups are

$$\bar{\bar{X}} = \frac{12284}{16} = 768 \quad (i.e. \text{ 0.8768 in})$$

$$\bar{R} = \frac{408}{16} = 26 \quad (i.e. \text{ 0.0026 in})$$

It is evident that there has been a further narrowing of the process dispersion. This should be recognized by a revision of control limits starting with subgroup 33.

As there seems to be no reason for a change in the aimed-at average, these revised control limits should be computed from an  $X'_0$  of 0.8770 and an R of 0.0026, using the factors from Table U.

$$UCL_{\bar{X}} = X'_0 + A_2 \bar{R} \ll 0.8770 + 0.58(0.0026) = 0.8785$$

$$LCL_{\bar{X}} = X'_0 - A_2 \bar{R} = 0.8770 - 0.58(0.0026) = 0.8755$$

$$UCL_R = D_4 \bar{R} = 2.11(0.0026) = 0.0055$$

$$LCL_R = D_3 \bar{R} = 0$$

These limits are shown on the control chart of Figs.2-14&2-15 as applying to subgroups 33, 34, and 35. With  $\bar{R}$  reduced to 0.0026, the estimate of  $\sigma$  is now  $\hat{\sigma} = \frac{\bar{R}}{d_2} = \frac{0.0026}{2.326} = 0.0011$  in.

If control can be maintained at this level,  $X'_0 + 3 \hat{\sigma} = 0.8803$  and  $X'_0 - 3 \hat{\sigma} = 0.8737$ . This indicates that a small amount of spoilage and a moderate amount of rework will still be produced; however, the situation is greatly improved as compared to that which existed before the start of the control chart. As time went on it proved possible to maintain control and to decrease  $\bar{R}$  (and  $\sigma$ ) further to the point where nearly all the product fell within specification limits.

In situations where the specification limits are as tight as this in relation to the process dispersion, it is not appropriate to use the control chart for acceptance as a substitute for 100% inspection. Neither was this a situation in which the tolerance limits could be widened; the required fit of the part properly controlled the specifications despite the fact that the natural tolerance of the process seemed to be wider than the specified tolerance. Grant & Leavenworth(1988) has added the following a comment on the example; interested reads could refer the this reference.

### **E n d      O f      E x a m p l e      2 . 1 3 .**

After reading the following example solve problem 2.9 of this chapter.

#### **Example 2.14**(Based on Naghandarian, 1993 p192)

Certain dimension of a part is specified as  $2.050 \pm 0.020$ . If the part falls below LSL it must be scrapped, if above USL, rework required. To monitor the production process, S chart and  $\bar{X}$  chart with limits  $\bar{X} \pm A_3 \bar{S}$  are used. For 20 subgroup of sample size 5,  $\sum \bar{X} = 41.340$ ,  $\sum \bar{S} = 0.144$ . The charts indicate the process is in control.. Assume the process is normally distributed. What could be concluded about the capability of this process if the specification is  $2.050 \pm 0.020$ . What could be done to improve the process?

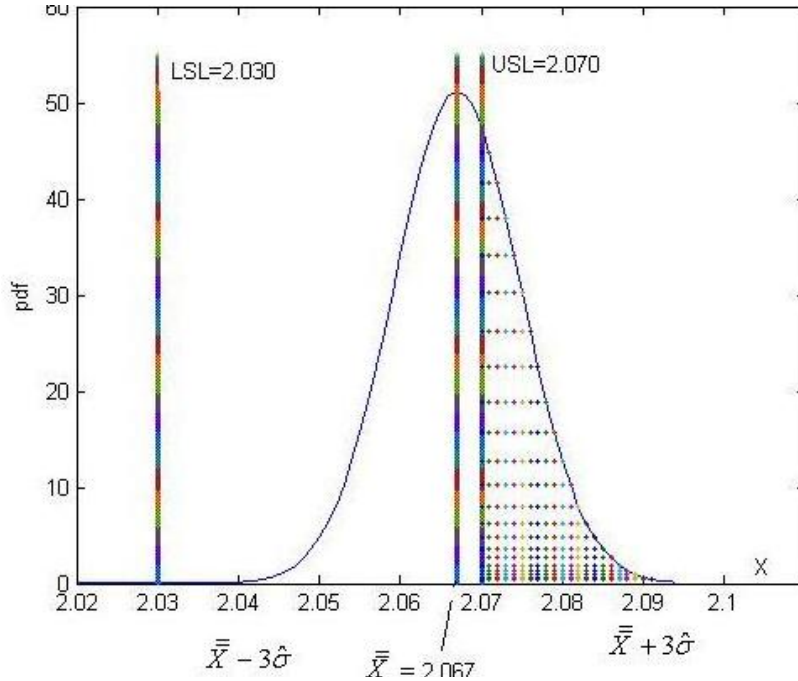
#### **Solution**

$$USL - LSL = 0.040, \hat{\mu} = \bar{\bar{X}} = \frac{41.340}{20} = 2.067, \bar{S} = \frac{.144}{20} = 0.0072$$

$$\text{For } n=4 \text{ from Table U } c_4 = 0.9213, \text{ then : } \hat{\sigma} = \frac{\bar{S}}{c_4} = \frac{.0072}{0.9213} = 0.0078,$$

$$6\sigma = 6 \times 0.0078 = 0.0468 ,$$

The mean of the process ( $\bar{\bar{X}}$ ) is different from the nominal mean ( $\frac{U+L}{2}$ ); the natural tolerance or the spread of the process,  $6\sigma$ , is greater than the specification tolerance i.e. USL-LSL (Fig. 2-16).



**Fig. 2-16 Spread of the process, the specification tolerance and rework percent (hatched surface)**

$$USL = 2.050 + 0.020 = 2.070 \quad ,$$

$$LSL = 2.050 - 0.020 = 2.030 \quad ,$$

$$\hat{\mu} + 3\hat{\sigma} = \bar{\bar{X}} + 3\hat{\sigma} = 2.067 + 3(0.0078) = 2.0904 \quad ,$$

$$\hat{\mu} - 3\hat{\sigma} = \bar{\bar{X}} - 3\hat{\sigma} = 2.067 - 3(0.0078) = 2.043 \quad .$$

Let  $X$  denote the quality characteristic being monitored, then at the time being the percent of the product needing rework is:

$$\Pr(X > 2.070) = \Pr(Z > \frac{2.070 - 2.067}{0.0078}) =$$

$$1 - \text{normcdf}(2.070, 2.067, 0.0078) = 0.3503$$

i.e. nearly 35% of the products need to be reworked.

$LSL = 2.030 < \mu - 3\sigma = 2.043$  and the spoilage percent is negligible:

$$\Pr(X < 2.030) = \text{normcdf}(2.030, 2.067, 0.0078) = 1.0498 \times 10^{-6}$$

To improve the process 2 alternatives are considered:

Changing the process mean to a value calculated from the following alternatives:

a)Coinciding the lower process limit with lower specification limit i.e.

$$\mu_0 - 3\sigma = LSL$$

or

b)Coinciding the upper process limit with upper specification limit i.e.

$$\mu_0 + 3\sigma = USL .$$

#### Alternative a)

$$\mu_0 - 3\sigma = LSL$$

$$\mu_0 - 3(\cdot/\cdot\cdot78)=2/030 \Rightarrow \mu_0 =2/0534$$

If the process mean is  $\mu_0 = 2.0534$ , the percent of rework would be:

$$\Pr(X > 2.070) = \Pr(Z > \frac{2.070 - 2.0534}{0.0078}) =$$

$$1 - \text{normcdf}(2.070, 2.0534, 0.0078) = 0.0167$$

$$\Pr(X < 2.030) = \text{normcdf}(2.030, 2.0534, 0.0078) = 0.0013$$

With this value as the process mean the percent of rework and the percent of spoilage would be 1.67% and 0.13% respectively.

#### Alternative b)

$$\mu_0 + 3\sigma = USL \quad \mu_0 + 3(\cdot/\cdot\cdot78)=2/070 \Rightarrow \mu_0 =2/0466,$$

With this process mean, the rework percent would be:

$$\Pr(X > 2.070) =$$

$$1 - \text{normcdf}(2.070, 2.0466, 0.0078) = 0.0013$$

and the spoilage percent:

$$\Pr(X < 2.030) = \text{normcdf}(2.030, 2.0466, 0.0078) = 0.0167$$

The following table compares the current situation with the 2 alternatives:

	Alternative		
	current	a	b
Mean	2.067	2.0534	2.0466
Rework percent	35.03	1.67	0.13
Spoilage percent	0.0000	0.13	1.67

All in all, alternative *a* is better(why?).

**Example 2.15**

$\bar{X}$  and S control charts are being used to monitor the process of filling a can whose contents is normally distributed. For 18 subgroups of size 8 :  $\sum s = 8.24$  ,  $\sum \bar{X} = 59.58$  . Find the 3-sigma limits of the charts.

**Solution**

for  $\bar{X}$  chart:

$$\sum \bar{X} = 59.58 \quad n = 8, k = 18 \quad \bar{\bar{X}} = \frac{\sum \bar{X}}{k} = \frac{59.58}{18} = 3.31$$

$$\sum s = 8.24 \quad \bar{s} = \frac{\sum s}{k} = \frac{8.24}{18} = 0.458$$

$$\hat{\sigma} = \frac{\bar{s}}{c_4} = \frac{0.458}{0.9650} = 0.475$$

$$UCL_{\bar{X}} = \bar{\bar{X}} + 3\sigma_{\bar{X}} = \bar{\bar{X}} + \frac{3\hat{\sigma}}{\sqrt{n}} = 3.31 + \frac{3 \times 0.475}{\sqrt{8}} = 3.81$$

$$\text{Central line } \bar{\bar{X}} = 3.31$$

$$LCL_{\bar{X}} = \bar{\bar{X}} - 3\sigma_{\bar{X}} = \bar{\bar{X}} - \frac{3\hat{\sigma}}{\sqrt{n}} = 3.31 - \frac{3 \times 0.475}{\sqrt{8}} = 2.81$$

**Example 2.16**

A production process is normally distributed with  $\mu = 60$  ,  $\sigma = 8$ . Calculate the limits of  $s, R, \bar{X}$  for using random samples of size  $n=5$ .

$$\mu = 60 \quad \sigma = 8 \quad n = 5$$

$$UCL_{\bar{X}} = \mu + \frac{3\sigma}{\sqrt{n}} = 60 + \frac{3 \times 8}{\sqrt{5}} = 60 + 10.73 = 70.73$$

$$C.L. = \mu = 60$$

$$LCL_{\bar{X}} = \mu - \frac{3\sigma}{\sqrt{n}} = 60 - \frac{3 \times 8}{\sqrt{5}} = 60 - 10.73 = 49.27.$$

**Example 2.17**

$R, \bar{X}$  charts are used to monitor a dimension of a product. The result of 30 subgroups of size 4 are  $\sum R = 0.320$  ,  $\sum \bar{X} = 41.34$  . Find the limits for each chart and estimate the standard deviation of the product, assuming the process is normally distributed in control.



**Solution**

$$\sum \bar{X} = 41.34 \quad n = 4 \quad k = 30 \quad \bar{\bar{X}} = \frac{\sum \bar{X}}{k} = \frac{41.34}{30} = 1.378$$

$$\sum R = 0.320 \quad \bar{R} = \frac{\sum R}{k} = \frac{0.320}{30} = 0.011$$

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_2 \bar{R} = 1.378 + (0.73)(0.011) = 1.386$$

$$CL = \bar{\bar{X}} = 1.378$$

$$LCL_{\bar{X}} = \bar{\bar{X}} - A_2 \bar{R} = 1.378 - (0.73)(0.011) = 1.367$$

$$UCL_R = D_4 \bar{R} = 2.28 \times 0.011 = 0.0251$$

$$CL = \bar{R} = 0.011$$

$$LCL_R = D_3 \bar{R} = 0 \times \bar{R} = 0$$

$$\hat{\sigma} = \frac{\bar{R}}{d_2} = \frac{0.011}{2.059} = 0.0053$$

**Example 2.18**

In a quality inspection, 30 subgroup of size 25 has been taken such that

$$\sum \bar{X} = 2419.2, \quad \sum s = 107.6.$$

The process is roughly normally distributed

a) Estimate  $\sigma$  if the process is in control and.

b) Calculate the limits of  $\bar{X}$  and S charts.

c) If the specification limits set by the design department is  $85 \pm 10$ , what percent of the products satisfy the design limits

**Solution**

a)

$$k = 30 \quad \bar{s} = \frac{\sum s}{k} = \frac{107.6}{30} = 3.586$$

$$n = 25 \quad \hat{\sigma} = \frac{\bar{s}}{c_4} = \frac{3.586}{0.9896} = 3.623$$

b)

$$\bar{\bar{X}} = \frac{\sum \bar{X}}{k} = \frac{2419.2}{30} = 80.64$$

$$UCL_{\bar{X}} = \bar{\bar{X}} + 3\sigma_{\bar{X}} = \bar{\bar{X}} + \frac{3}{\sqrt{n}}\hat{\sigma} = \bar{\bar{X}} + \frac{3}{\sqrt{n}} \times \frac{\bar{s}}{c_4} = 80.64 + \frac{3}{\sqrt{25}} \times \frac{3.586}{0.9896} = 82.80$$

$$C.L. = \bar{\bar{X}} = 80.64$$

$$LCL_{\bar{X}} = \bar{\bar{X}} - 3\sigma_{\bar{X}} = \bar{\bar{X}} - \frac{3}{\sqrt{n}}\hat{\sigma} = \bar{\bar{X}} - \frac{3}{\sqrt{n}} \times \frac{\bar{s}}{c_4} = 80.64 - \frac{3}{\sqrt{25}} \times \frac{3.586}{0.9896} = 78.46$$

$$UCL_s = B_4\bar{s} = 1.44 \times 3.586 = 5.164$$

$$C.L. = \bar{s} = 3.586$$

$$LCL_s = B_3\bar{s} = 0.56 \times 3.586 = 2.008$$

c)

$$\sum_{i=1}^{30} \bar{X}_i = 2419.2 \quad \hat{\mu} = \frac{2419.2}{30} = 80.64 \quad \Pr\left(\frac{75 - \hat{\mu}}{\hat{\sigma}} < Z < \frac{95 - \hat{\mu}}{\hat{\sigma}}\right) =$$

$$\Pr\left(\frac{-5.64}{3.623} < Z < \frac{14.36}{3.623}\right) = \Pr(-1.55 < Z < 3.96) = 0.0605 = 6.05\%$$

## 2.12 Performance Measures of a control chart

There are some measures considered in evaluating the performance of control charts; three popular of which are the type I and type II error probabilities ( $\alpha$  &  $\beta$ ), the expected value of the run-length distribution, called the *average run length* (ARL). Among these, the ARL is an easily interpreted, well-defined measure, whose definition follows.

### 2.12.1 Average Run Length (ARL) Performance of Control Charts

Average Run Length (ARL) is the average number of samples required to detect an out-of-control status by a control chart. Here it is assumed when a point falls outside the control chart, the process is out of control. This measure of performance is sometimes called the waiting time to signal "out of control" status.

Let

$p$  is the probability of that any subgroup falls outside a Shewhart control chart i.e. the probability of "detecting the change as success" and

$X$  = the no. of consecutive independent samples as the number of independent trials required to reach the first success.

Then  $X$ , the number of samples required to detect the change, is a geometric random variable with parameter  $p$  and mean  $E(X) = ARL = \frac{1}{p}$ .

### 2.12.1.1 Calculation of the ARL for $\bar{X}$ control chart

When using an  $\bar{X}$  chart with 3-sigma limits  $\mu_0 \pm 3\frac{\sigma}{\sqrt{n}}$ , if the process

mean shifts from  $\mu_0$  to  $\mu_1 = \mu_0 + k\sigma$

where  $\sigma$  is known & constant and  $k \in \mathbb{R}$ , the mean of the samples taken from the process would have a normal distribution:

$\bar{X} \sim N(\mu_0 + k\sigma, \frac{\sigma}{\sqrt{n}})$  and the type II probability for the chart is calculated

as follows:

$$\beta = \Pr(LCL < \bar{X} < UCL \mid \mu \neq \mu_0 \text{ or } \mu = \mu_1 = \mu_0 + k\sigma) =$$

$$\Pr\left(\frac{\mu_0 - 3\frac{\sigma}{\sqrt{n}} - (\mu_0 + k\sigma)}{\frac{\sigma}{\sqrt{n}}} < Z < \frac{\mu_0 + 3\frac{\sigma}{\sqrt{n}} - (\mu_0 + k\sigma)}{\frac{\sigma}{\sqrt{n}}}\right) = \Pr(-3 - k\sqrt{n} < Z < +3 - k\sqrt{n})$$

$$\Rightarrow \beta = \phi_Z(3 - k\sqrt{n}) - \phi_Z(-3 - k\sqrt{n}).$$

Then the probability of detecting the change (i.e. detecting  $\mu \neq \mu_0$ ) is

$$1 - \beta.$$

and

$$ARL = \frac{1}{1-\beta} \text{ in } \bar{X} \text{ control chart.}$$

A general formula about this case is in Derman&Ross(1997) page 85.

It should be added that), when monitoring a process with  $\bar{X}$ -control chart even if the process is in control (Montgomery& Rungers,1994), on the average for every  $\frac{1}{0.0027} \cong 370$  produced samples, one falls outside the chart. This could be a false alarm of out-of-control. Montgomery(2005) states that in each control chart on the average the number of subgroups produced until one falls outside the control chart is  $\frac{1}{\alpha}$ .

### Example 2.19

(Based on Derman & Ross,1997 page 86)

The mean of a process is monitored by 3-sigma-limit  $\bar{X}$  control chart. If the standard deviation of the process remains constant, but the mean shifts from  $\mu_0$  to  $\mu_0 - \frac{1}{2}\sigma$ , find the average no of samples of volume  $n=9$  to detect this change.

**Solution**

As you have already observed, for an  $\bar{X}$  control chart when the mean shifts from  $\mu_0$  to  $\mu_0 + k\sigma$ ,

$$\beta = \varphi_Z(3 - k\sqrt{n}) - \varphi_Z(-3 - k\sqrt{n}).$$

For this case :

$$ARL = \frac{1}{1 - \beta} \quad \beta = \Pr(-3 - k\sqrt{n} \leq Z \leq 3 - k\sqrt{n}) \quad k = -\frac{1}{2} \quad n = 9 \Rightarrow$$

$$\beta = 0.9332 \Rightarrow ARL = 14.97 \cong 15$$

Therefore to pick up the shift, 15 subgroups are needed on average.

**2.13 Change Point**

In this section the aim is to get familiar with out-of-control signal point called change point, when a change or deviation occurs in the process parameter related to a control. After the *signal* has been given by the control chart, investigations has to done to find the cause, to prevent recurring if possible and to bring the process under control again. In many cases the change in the process parameter has begun before the time of the signal. Knowing the right time when the change has started to occur in the process, which is called change point, is of importance for performing the corrective action to eliminate the cause of out-of-control signals. Figure 2-17 shows a control chart which has produced an out-of-control signal of shift in the parameter by the 36th subgroup, while the shift has begun at the time of 23rd subgroup; in other words the change point occurs at sample # 23. control processes.

The changes (occurring in the process) affecting the process parameters may be classified into single-step change, multiple-step change, drift and monotonic change; defined below (from Amiri & Allahyari, 2012):

**Single -step change**

A step change for a process parameter occurs when the value of the parameter suddenly changes and then remains unchanged again until corrective actions are taken.

**Multiple- step change**

A multiple step change occurs when several changes occur at different times before the signal is given by the control chart. This type of change can occur because of one or more influential process input variable(s) at different times.

**Drift**

Drift is a change type in which the process drifts off target, either linearly or nonlinearly at an unknown point in time. The trend continues until some corrective action is taken to bring the process back in control.

**Monotonic change**

In monotonic changes, the type of change is unknown a priori, but the direction of shifts is the same, increasing or decreasing. This type of change is more general than the others because it encompasses the single step, multiple step, and drift pattern changes.

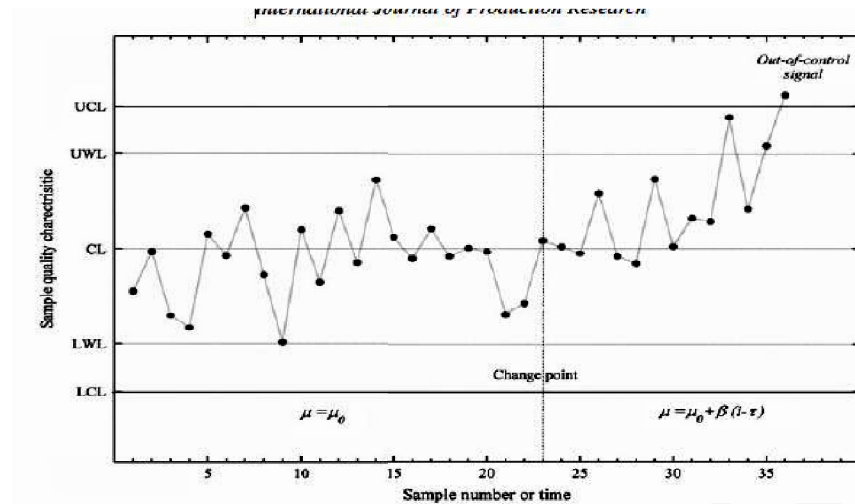


Figure 1. A typical control chart for monitoring a process; the out-of-control signal is illustrated at  $t = 36$  while the process mean begins drifting linearly at  $t = 23$ .

**Fig. 2-17 The out-of control signal is at 36<sup>th</sup> sample ; the change point occurs at 23<sup>rd</sup> sample** (after Kazemi et al, 2013)

Each of the types of the change in the process could be modeled mathematically; for example the general model for a kind of step change occurring in a process mean is as follows:

$$\mu_t = \begin{cases} \mu_1 & 1 < t < \tau_1 \\ \mu_r & \tau_1 + 1 < t < \tau_2 \\ \mu_r & \tau_2 + 1 < t < \tau_3 \\ \vdots & \\ \mu_{k-1} & \tau_{k-2} + 1 < t < \tau_k \\ \mu_k & \tau_{k-1} + 1 < t < T \end{cases}$$

where

$\mu_t$  = the mean of the process in the  $t^{\text{th}}$  sample(period),

$T$  = Total number periods,

$\tau_j$  = the  $j^{\text{th}}$  change point in the process mean.

### Example 2.20

To monitor the mean of a process,  $T=445$  subgroups of size  $n=5$  were produced from the process; the last sample indicates the process is out of control. At the beginning, the process was normally distributed with mean

$\mu_0 = 100$  and  $\sigma = 5$ . The central line, the upper and lower control limits and also the upper and lower warning limits for  $\bar{X}$  control chart are as follows:

$$UCL = \mu + 3 \frac{\sigma}{\sqrt{n}}$$

$$UWL = \mu + 2 \frac{\sigma}{\sqrt{n}}$$

$$CL = \mu$$

$$LWL = \mu - 2 \frac{\sigma}{\sqrt{n}}$$

$$LCL = \mu - 3 \frac{\sigma}{\sqrt{n}}$$

Replacing  $\mu = \mu_0 = 100$ ,  $\sigma = 5$  and  $n = 5$ :

$$UCL = 106.7082$$

$$UWL = 104.4721$$

$$CL = 100$$

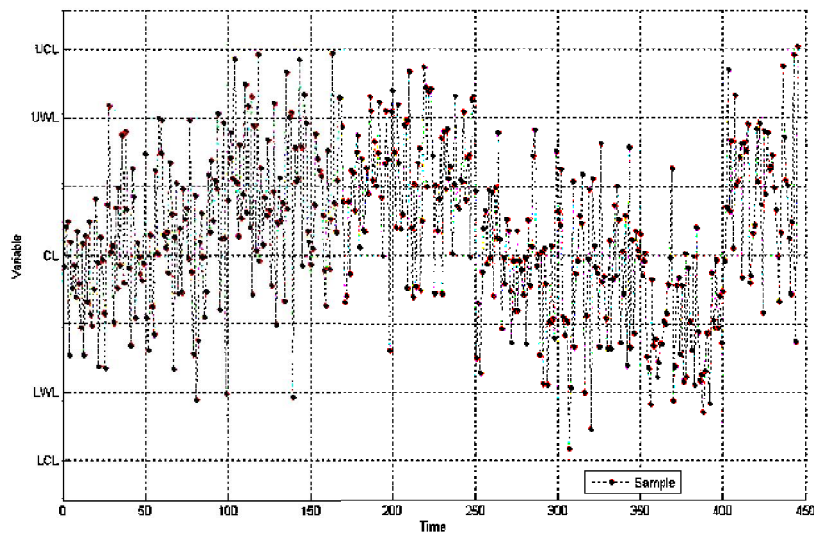
$$LWL = 95.5278$$

$$LCL = 93.2918$$

Suppose that due to changing the operator s, the mean has encountered the following change:

$$\mu_t = \begin{cases} \mu_0 & \text{if } 1 < t \leq 100 \\ \mu_0 + 0.5\sigma & \text{if } 100 < t \leq 250 \\ \mu_0 & \text{if } 250 < t \leq 350 \\ \mu_0 - 0.5\sigma & \text{if } 350 < t \leq 400 \\ \mu_0 + 0.5\sigma & \text{if } 400 < t \leq 445 \end{cases}$$

Figure 2-18 shows the  $\bar{X}$  control chart with 445 subgroups taken from the process of the example. The process has more than one change point in its mean.



**Fig. 2-18  $\bar{X}$  control chart for the example (showing more than one change point).**

## Exercises

**2.1.** Using Table U read the values of  $A$ ,  $A_1$ ,  $A_2$  and  $A_3$  for 20 subgroups of size 8.

**2.2.** Plot  $\bar{X}$  and S chart for the 25 subgroup of the first problem of the exercises of chapter 1 and answer the following questions:

- Is the process in control?
- The dimension specified by the design department is  $395 \pm 3$ . Do all the 125 parts satisfy these limits? What could be concluded about the capability of the normally distributed process used for the production of part between  $395 \pm 3$ .
- What control limits do you suggest for continuing the charts in the future?

**2.3.**  $\bar{X}$  and S control charts are being used for monitoring a dimension of a product. All the samples are randomly distributed between the limits of the charts. A sudden change in the process has no effect on the  $\sigma$  but causes the process mean to change from  $\mu$  to  $\mu + 1.5\sigma$ . Answer the following question, assuming the distribution of the process is normal both before and after the change:

- If the sample size is  $n=3$ , what percent of the points do you expect to fall outside the  $\bar{X}$  chart limits  $\mu \pm 3 \frac{\sigma}{\sqrt{n}}$ ? Answer 34.3%
- If the sample size is  $n=5$ , repeat part a. Answer 63.8%

c) If the sample size is  $n=5$ , repeat part a. Answer 89.3%

**2.4.** A Textile Manufacturing Group needs a kind of fibers such that 95% of the products have at least the tension strength of 1.80 units. manufacturer takes the responsibility of producing the fiber. A contract is held between the group and the manufacturer. The standard deviation of the process used by the manufacturer is 0.015. What is the minimum aimed-at mean for the process to satisfy the contract. Assume the process continues to be in control and the distribution of tension strength is normally distributed.

**2.5.**  $\bar{X}$  and  $S$  control charts have been run on a process for sufficient time using subgroups of 5 parts. During the past nearly one month, no point has fallen outside the charts, and the one-month results indicate  $\bar{\bar{X}} = 0.7503$  and  $\bar{S} = 0.0018$ .

a) Estimate the value of  $\sigma$  for the process.

b) If ten points out of the last 11 points fall above the central line of  $S$  chart, is still the process under the statistical control?

**2.6.** (Grant & Leavenworth, 1998 page 146 problem 4-14)

A certain product is manufactured with a specification of  $120.0 \pm 5.0$  for a quality characteristic. At present, the estimated process average is 120.0 and the standard deviation is 1.50.

a) Compute the 3-sigma control limits for an  $\bar{X}$  and  $R$  chart based on subgroups of size 4. Answer: 122.25, 117.5 & 7.05, 0

b) What percent of the product will not meet specifications if the process average suddenly shifts to 121.0, assuming a normally distributed product.

Answer: 0.39%

**2.7.** (Grant & Leavenworth, 1998 page 146 problem 4-16)

A control chart has been used to monitor a certain process during a considerable period of time. The process is sampled in groups of 4 at intervals of about 2 hours and the  $\bar{X}$  control chart has limits of 121.0 & 129.0 with the target  $\mu_0 = \bar{X}_0$  at 125.0.

a) If the product is sold to a user who has specification of  $127.0 \pm 8.0$ , what percent of the product will not meet this specification assuming a normally distributed output? Ans: 1.22%

b) If the target value of this process can be shifted without effect on the process standard deviation, what target value would minimize the amount of product being produced outside the specification values? Ans: 127

c) At this new target value, what percent of the product will not meet the required specifications? Ans: 0.27%



**2.8. (Grant & Leavenworth, 1998 page 146 problem 4-18)**

In the extensive example if the slot width were below the lower specification limit (0.8735), the part could be reworked to bring it within specification. However, if above 0.8800, it would have to be scrapped. After the first 16 subgroups, an estimate was made that the standard deviation ( $\sigma$ ) of a controlled process would be 0.0014. This called for a  $\mu$  of 0.8785 as the process average.

**a)** If the distribution of slot width were normal with  $\sigma$  of 0.0014, approximately what percentage of rework could be expected if  $\mu$  were 0.8758? Ans: 28.4%

**b)** With  $\mu$  at 0.8770, what percentage of spoilage would be expected? What percentage of rework? Ans: 1.62%. 7.78%.

**c)** In what way do the answers in (a) and (b) suggest an economic basis for establishing the process average in cases of this type?

**d)** How would your decision as to process average to be aimed at in the extensive example be influenced if you knew that terminal blocks (TBs) with slot widths up to 0.8830 had been accepted by the plant salvage committee for use in the airplane and had been used satisfactorily?

**2.9. (Grant & Leavenworth, 1998 page 146 problem 4-19)**

The following data were obtained over a ten-day period to initiate  $\bar{X}$  and R control charts for a quality characteristic of a certain manufactured product that had required a substantial amount of rework. All the figures apply to product made on a single machine by a single operator. The subgroup size was 5. Two subgroups were taken per day.

Sample No.	$\bar{X}$	R	Sample No.	$\bar{X}$	R
1	177.6	23	11	179.8	9
2	176.6	8	12	176.4	8
3	178.4	22	13	178.4	7
4	176.6	12	14	178.2	4
5	177.0	7	15	180.6	6
6	179.4	8	16	179.6	6
7	178.6	15	17	177.8	10
8	179.6	6	18	178.4	9
9	178.8	7	19	181.6	7
10	178.2	12	20	177.6	10
			sum	3569.2	196

- a) Determine central lines and trial limits for the  $\bar{X}$  and R control charts.      Ans: 172.8, 178.5, 184.2; 0, 9.8, 20.8
- b) What preliminary conclusions about statistical control can you draw from your observation and analysis of the data and of the control charts? Justify these conclusions. Does it seem reasonable that the introduction of the control charts may have had some influence on the process. Why or why not?
- c) The specified requirements for the quality characteristic are given as  $171 \pm 11$ . If a product falls below the LSL of 160, it must be scrapped, whereas it falls above USL of 182 it may be reworked. Because scrapping an article is much more costly than rework, it is desired to hold scrap to a low figure without causing excessive rework. The process average be shifted by a relatively simple machine adjustment. What would you suggest as the aimed-at value for process centering in the immediate future? Why? Ans: 171.2

**2.10.** (Based on Grant & Leavenworth, 1988 page 151 problem 4-40)  
After 25 subgroups of 4 items each have been checked on a certain machining operation  $\sum \bar{X} = 301.2500 \text{ mm}$  &  $\sum S = 0.5757 \text{ mm}$ .

- a) Compute central lines and control limits for  $\bar{X}$  &  $S$  charts based on these data.
- b) Assuming that this process is in control, estimate the process standard deviation
- c) Specifications on this dimension are  $12.000 \pm 0.100 \text{ mm}$ . What proportion, if any, of the product falls outside the specifications, assuming a normal distribution applies.
- d) Assume that the process shifts to a new mean of 11.990 mm without affecting  $\sigma$ . What is the probability that this shift will not be affected on the first subgroup drawn after the shift occurs?

**2.11.** (Grant & Leavenworth, 1988 page 148 problem 4-21)

Control charts for  $\bar{X}$  &  $S$ , based on a subgroup size of 4, are to be used to control a process. The standard deviation of this process is 10. An aimed-at value of the mean ( $\mu_0$ ) is to be 250.

- a) Determine control limits for the  $\bar{X}$  &  $S$  charts
- b) Determine the probability of a point falling within the  $\bar{X}$  chart control limits if the actual  $\mu$  is  $0.5\sigma$  below the aimed-at value of 250.

If it is  $1.0\sigma$  below 250.

If it is  $2.0\sigma$  below 250.

If it is  $2.5\sigma$  below 250.

- c) Sketch these probabilities as a function of the location of the true mean (horizontal scale) and comment on the usefulness of this diagram in relation to the detection of shifts in the mean.

**2.12.** A certain manufacturing process with  $\sigma = 0.0010$  has been operating in control at a mean of  $\mu = 0.0360$  mm with  $\sigma = 0.0010$  mm. The products are sold to 2 different customers A & B. The specification required by A and B are  $0.0360 \pm 0.0040$ , B,  $0.0380 \pm 0.0040$ , A.

a) Assuming a normal distribution, what proportion of the product does not meet the specification of A.

b) If the process is centered on  $\mu_0 = 0.0370$ , find the percent of the spoilage for both A & B. Is this change suitable? Why?

**2.13.**  $\bar{X}$ ,  $R$ ,  $S$  control charts are plotted for the samples of size 4 taken from a bag of balls which are normally distributed with  $\mu = 40$  &  $\sigma = 4$ . Find the 3-sigma control limits for the charts and the central lines. After sampling, a frequency distribution is prepared for the  $\bar{X}$  values. Estimate the mean and standard deviation of this distribution. A frequency distribution is also prepared for the standard deviation ( $S$ ) values, estimate the mean and the standard deviation of this distribution.

Hint:  $\sigma_S = \sigma \sqrt{1 - c_4^2}$ ,  $E(S) = c_4 \sigma$ ,  $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$ ,  $E(\bar{X}) = \mu$

**2.14.** The  $\bar{X}$  control chart for a certain quality characteristic is given to you. This chart contains 50 subgroups all near the central line, none of them near the 3-sigma limits. In fact when you draw the 1-sigma limits (i.e. one third of the distance between the central line and the 3-sigma limits) all samples fall within these narrow limits. Does the chart

make you suspicious of happening a difficulty in the process? Why? What kind of explanations could be presented for these kinds of charts?

**2.15.** Why an increase in the variability of the process without increase in its mean, make the R, S charts indicate lack of control?

**2.16.** After 20 subgroups of size 10 from a production process

$\sum_{i=1}^{20} \bar{X}_i = 731.4, \sum_{i=1}^{20} s_i = 59.16$ . Assuming a normal distribution applies, what percent of the products satisfy the specifications  $35.00 \pm 10.00$ ?

**2.17.** According to government regulations, at least 98% of the products must have a weight equal to the weight printed on the label  $s, \bar{X}$  charts are used. After 20 subgroups of size 10,  $\sum \bar{X} = 731.4, \sum s = 9.16$ . find the 3-sigma limits for the charts and estimate  $\sigma$ , assuming the process is in control. If the weight printed on the label is 36 oz and the process is normally distributed; determine whether the product weight satisfy the government requirement.

**2.18.**  $\bar{X}$  and R control charts are used to control the concentration of the dissolved iron, measured in part per million (ppm), in a certain solution. 125 hourly specimens, were taken and classified into 25 subgroups of size 5. From these data  $\sum R = 84$ ,  $\sum \bar{X} = 390.8$ . Find the 3-sigma limits of the charts and estimate  $\sigma$ , assuming that  $\sigma$  is in control and the concentration is normally distributed.

**2.19.** According to the given specification, the concentration of the iron in problem 2.18 must not exceed 10 ppm. If the data are normally distributed and no change has occurred in the mean and the standard deviation of the products, what portion of the product is expected to exceed this limit.

**2.20.**  $\bar{X}$  and S control charts are used to control the impedance of a kind of electrical coil. After 30 subgroups of size 5,  $\sum \bar{X} = 58395$  ohms,  $\sum S = 548$  ohms.

- Calculate the central line and the control limits for the 2 charts, and estimate  $\sigma$  if the X's are normally distributed.
- What percent of the produced coils have the impedance of  $2000 \pm 150$ , if the normal distribution applies?

**Reconciliation between people is the best alms.**

# Chapter 3

## Some Points on Subgrouping & Process Capability

## Chapter 3 Some Points on Subgrouping & Process Capability

### Aims

In this chapter some advices for subgrouping are given and the indices such as  $C_p$  and  $C_{pk}$ , commonly used to evaluate a process ability to produce output within given upper and lower tolerances, are defined.

### 3.1 Subgrouping

(Extracted from :Grant&Leavenworth,1988 page 155):

The basis of subgrouping calls for careful study, with a view to obtaining the maximum amount of useful information from any control chart. The following points should be considered while subgrouping:

The most obvious rational basis for subgrouping is the order of production.

Subgroups should be selected in a way that makes each subgroup as homogeneous as possible and that gives the maximum opportunity for variation from one subgroup to another. Particularly if the primary purpose of keeping the charts is to detect shifts in the process average, one subgroup should consist of items produced as nearly as possible at one time; the next subgroup should consist of items all produced at a single later time; and so forth. For this purpose either  $n$  items constituting one subgroup are produced in succession at say about 8 o'clock;  $n$  items constituting the next subgroup are produced in succession at about 9 o'clock; none of the items produced between the 8 and 9 o'clock subgroups are measured for purposes of the control charts. Or subgroups are taken after every say 1000 parts. It is good to act in a such way not to let the operator to know the exact time of sampling. For example if we take hourly samples at 9 AM., 10 AM, etc, we should take samples at 9:10 or 9:05 or 9:50 or 10.05...

#### 3.1.1 Decision on the Size of Subgroups.

Dr Shewhart suggested four as the ideal subgroup size( $n$ ). In the industrial use of the control chart, five seems to be the most common size. Because the essential idea of the control chart is to select subgroups in a way that gives minimum opportunity for variation within a subgroup, it is desirable that subgroups be as small as possible. On the other hand, a size of four is better than three or two on statistical grounds; the distribution of  $\bar{X}$  is nearly normal for subgroups of four or more even though the samples are taken from a non-normal universe; this fact is helpful in the interpretation of control-chart limits. Of course the priority of 4 and 5 over 2 and 3 is rejected when more expenditure is needed.

### 3.1.2 The time interval between subgroups

No general rule could be given for the time interval of the subgroups; it depends on such factors as the cost we can afford, in control or out of control status of the process. We might not reduce the interval due to high sampling expenditure.

### 3.1.3 The number of subgroups

The number of subgroups depend on some factors such as the population size ;however , if possible, 25 is recommended for the number of subgroups to be collected before trial control limits are calculated.

## 3.2 Process Capability Index

To evaluate the capability of an in-control production process, the so-called process capability indices, denoted by PCIs, are used in quality control. The indices compare the output of an *in-control process* to the specification limits.

### Capability Index for Normally distributed process

The indices used to evaluate an in-control and normally distributed process include  $C_p$  &  $C_{pk}$ ,  $C_{pm}$ ,  $C_{Npm}$ ,  $C_{Npk}$ ,  $C_{Np}$ ; among which  $C_p$  &  $C_{pk}$  are well known.

#### 3.2.1 Definition of $C_p$

If the production process is normally distributed and is in control about the nominal value  $\theta = \frac{U + L}{2}$  where U and L are upper and lower specification limits (this requires the process mean coincides  $\theta$ ),  $C_p$  is defined as follows:

$$C_p = \frac{U - L}{6\sigma},$$

where

U-L is the specification range or tolerance,

$\sigma$  is the process standard deviation and  $6\sigma$  is the process range or natural tolerance,

It is evident that to the extent that the quantity of  $\sigma$  is less than the quantity of  $(U - L)/6$ , the quantity of  $C_p$  will be reduced.

It should be added that:

- usually  $\sigma$  is not available; if the process is in control and normally distributed it is estimated by  $\hat{\sigma} = \frac{\bar{S}}{c_4}$  or  $\hat{\sigma} = \frac{\bar{R}}{d_2}$

- for using  $C_p$  as process capability index,  $\hat{\mu} = \bar{\bar{X}}$  has to coincide with  $\theta = \frac{U+L}{2}$ .

-if

$C_p < 1$ , the process is not good,

$1 \leq C_p < 1.33$  the process acts well,

$C_p \geq 1.33$  the process acts very well,

$1.66 \leq C_p < 2$  the process is excellent,

$C_p \geq 2$  the process is very excellent,

-many American companies suggest a minimum target value of 1.33 for  $C_p$ ; a minimum value of 1.66 for  $C_p$  has also been suggested for critical characteristics such as safety, strength,... Table 3-1 shows the portion of products falling out of U,L for several  $C_p$ 's assuming the process is normally distributed. It could be easily verified that for a given  $C_p$ , if the process is normally distributed with mean equal to  $\theta = \frac{U+L}{2}$  that is  $(\hat{\mu} = \bar{\bar{X}} = \theta)$ , the portion of products that fall outside U,L could be calculated from

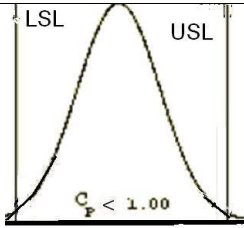
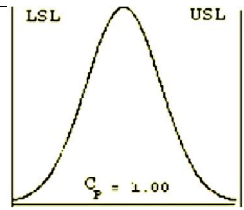
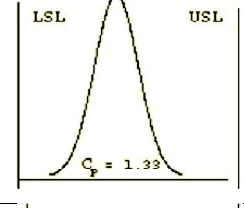
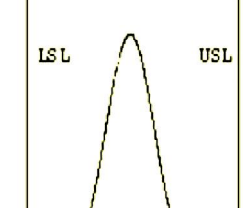
$$p = 1 - \Pr(-3C_p < Z < +3C_p),$$

or using MATABL from  $p = 1 - \text{normcdf}(3 * C_p) + \text{normcdf}(-3 * C_p)$ .

<b>Table 3-1</b> The portion of products falling outside U&L, given a normal and in control process about $\theta = \frac{U+L}{2}$							
U-L	$3\sigma$	$4\sigma$	$6\sigma$	$8\sigma$	$9,78\sigma$	$10\sigma$	$12\sigma$
$C_p$	0.5	0.67	1	1.33	1.63	1.66	2
Proportion falling outside U&L (% or ppm)	13.36 %	4.44 %	0.27 %	66 ppm	1 ppm	0.64 ppm	0.002 ppm
Observations			3-sigma Production	Min Cp for many American companies		Minimum Cp for critical characteristics	6-sigma production

Figure 3-1 shows some capability indices.



Process	portion outside U&L	Recommended Actions
	$\geq 5\%$	Heavy process control, Rework
	0.27%	Relatively heavy process control,
	66ppm	reduced Inspection, selected use of control charts
	1 ppm	spot checking, selected use of control charts

**Fig. 3.1 4 values of capability index**

Figure 3-2 shows four different normally distributed processes with mean  $\theta$  having different  $C_p$ 's producing a product. The more the  $C_p$ , the more the proportion fall inside the specifications limit.

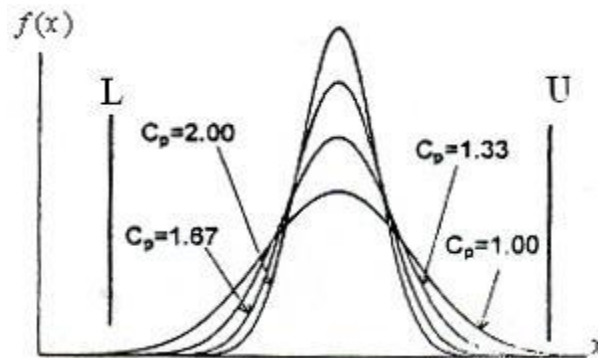


Fig. 3.2 4 processes with 4 values of capability indices<sup>1</sup>

### Cp for single specification

If only the lower specification or the upper specification (L or U) is given, the following indices are used:

$$C_{pL} = \frac{\mu - L}{3\sigma} \quad \text{or} \quad C_{pU} = \frac{U - \mu}{3\sigma}$$

Note that

- $\mu$ , the process mean, is estimated as  $\hat{\mu} = \bar{\bar{X}}$ .
- The values of  $C_{pU}$  or  $C_{pL}$  less than 1 are undesirable.

#### 3.2.1.1 Interpretation of $\frac{1}{C_p}$

Since  $6\sigma = \frac{1}{C_p}(U - L)$ , then the process range is equal to  $\frac{1}{C_p}$  of the specification range. In other words  $\frac{1}{C_p} \times 100$  indicates the portion of the specification range which covers the process range. For example if  $C_p$  is 1.55, the process range ( $6\sigma$ ) takes  $\frac{1}{C_p} \times 100 = 64.5\%$  of the specification range i.e. U-L.

if  $C_p$  is 1, the process range takes  $\frac{1}{C_p} \times 100 = 100\%$  of U-L.

if  $C_p$  is 0.5,  $\frac{1}{C_p} \times 100 = 200\%$  i.e. the process range is as twice as of U-L.

### 3.2.2 Definition of $C_{pk}$

In defining  $C_p$  it was assumed the process was in control and normally distributed and the process mean was equal to nominal value i.e.

$\hat{\mu} = \bar{\bar{X}} = \frac{U + L}{2}$ . If  $\bar{\bar{X}} \neq \frac{U + L}{2}$  and we would like to evaluate the

<sup>1</sup> For a detailed discussion refer to references such as [www.qualityamerica.com](http://www.qualityamerica.com)

process capability, another index denoted by  $C_{pk}$  and defined as follows is used.

$$C_{pk} = \min \left\{ C_{pU} = \frac{U - \mu}{3\sigma}, C_{pL} = \frac{\mu - L}{3\sigma} \right\}$$

It could be shown that the above relationship is equivalent to the following relationship:

$$C_{pk} = C_p(1 - k),$$

where

$$\theta = \frac{U + L}{2}, k = \frac{\left| \frac{\theta - \mu}{\frac{U - L}{2}} \right|}{1}$$

$C_{pk} = C_p(1 - k)$  shows that always  $C_{pk} \leq C_p$  and if  $C_{pk} = C_p$  then  $k = 0 \Rightarrow \mu = \theta$ .

The larger  $C_{pk}$  is, the less likely it is that any item will be outside the specification limits.

When  $C_{pk}$  is negative it means that a process will produce output that is outside the customer specification limits.

Figure 3.3 compares 4 normally distributed processes whose means are not equal to the target value  $\theta$ . They have different  $C_{pk}$ 's. The more the  $C_{pk}$ , the less the proportion fall outside the specification limits.

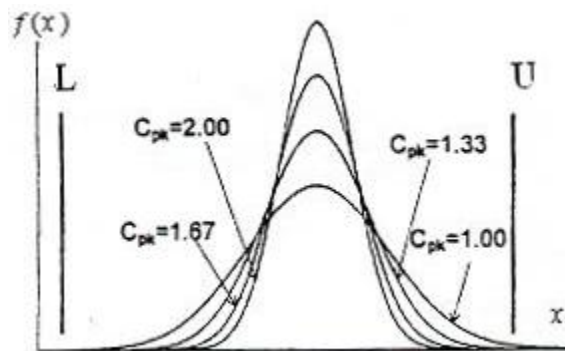


Fig. 3.3 4 processes with 4 values of  $C_{pk}$  producing the same product.

### 3.2.2.1 Explanation of $C_{pk}$ values

$C_{pk}$  = Negative number: Your products will be outside  $U \& L$ .

$C_{pk} = 0.5$ : a bad process.

$C_{pk} = 1$ : there are some nonconforming products

$C_{pk} = 2$ : Great! You have a great clearance

$C_{pk} = 3$ : Excellent! You have excellent clearance.

### 3.2.3 Definition of $\sigma$ level

Sometimes  $\min\left\{\frac{U-\mu}{\sigma}, \frac{\mu-L}{\sigma}\right\}$  is called  $\sigma$  level. If the process is normally distributed, to the extent this quantity is more than 3, the more capable the process will be.

#### Example 3.1

Assume a production process is normally distributed with a standard deviation equal to one twelfth of the specification range ( $U - L = 12\sigma$ ) and a mean equal to the nominal value  $\hat{\mu} = \bar{X} = \theta = \frac{U+L}{2}$  (Fig. 3-4). In this case  $C_p = C_{pk} = 2$  and the portion of products falling outside  $U \& L$  is 0.002 ppm calculated from normally distributed.

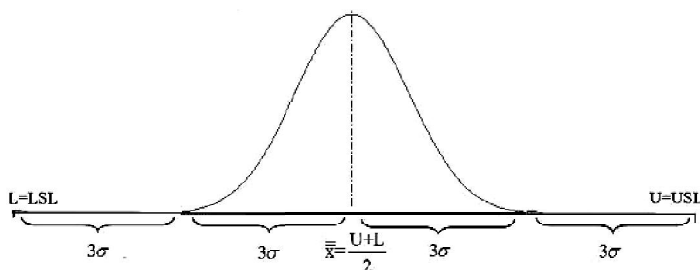


Fig 3-4. The process mean coincides nominal value.

Now suppose, the process mean ( $\mu$ ) has shifted to the right side of nominal value ( $\theta = \frac{U+L}{2}$ ) as much as  $1.5\sigma$ , while the type of distribution and its standard deviation remains constant (Fig. 3.5). In this case the capability index of the process is equal to

$$C_{pk} = \min\left\{\frac{U-\mu}{3\sigma}, \frac{\mu-L}{3\sigma}\right\} = \min\left\{\frac{4.5\sigma}{3\sigma}, \frac{7.5\sigma}{3\sigma}\right\} = 1.5.$$

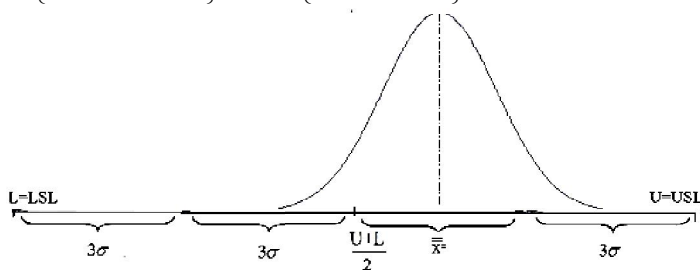


Fig. 3-5 Shift of the process mean to the right of  $\theta = \frac{U+L}{2}$ .

Then based on normal distribution, the portion of the products falling outside U,L is

$$1 - \Pr(L < X < U) = 1 - \Pr\left(\frac{L - \mu}{\sigma} < Z < \frac{U - \mu}{\sigma}\right) = 1 - \Pr(-7.5 < Z < 4.5) = 0.0000035$$

that is 3.5 ppm. Therefore in a normally distributed production process with  $C_p = 2$  if the process mean shifts  $1.5\sigma$  from  $\theta = \frac{U+L}{2}$ , the portion of products falling outside the specification limits is still insignificant.

### Example 3.2

The specification limits a product is  $30 \pm 10$ , and from 10 subgroup of size 5 we have  $\sum_{j=1}^{10} S_j = 20.21$ ;

If the production process is under control and normally distributed with mean equal to  $(U+L/2)$ , what can be said about the capability of the process?

### Solution

$$C_p = \frac{U - L}{6\sigma} \quad \hat{\sigma} = \frac{\bar{S}}{C_4} = \frac{20.21}{0.94} = 2.15 \rightarrow C_p = \frac{20}{6 * 2.15} = 1.55$$

The process is a good one and  $\frac{1}{C_p} \times 100 = 64.5\%$  i.e. the process uses 64.5% of the process specification limits.

Find the process capability If the process is normally distributed and

$$\sum_{i=1}^{10} \bar{X}_i = 331.9.$$

### Solution

In part b we have to calculate  $C_{pk}$  (why?)

$$C_{pk} = \min\left\{\frac{U - \bar{X}}{3\hat{\sigma}}, \frac{\bar{X} - L}{3\hat{\sigma}}\right\} = \left\{\frac{40 - 33.19}{3 \times 2.15}, \frac{33.19 - 20}{3 \times 2.15}\right\} = 1.05$$

Supposing  $\sigma$  is unchanged, the portion of the products falling outside U,L is

$$1 - \Pr\left(\frac{20 - 33.19}{2.15} < Z < \frac{40 - 33.19}{2.15}\right) = 0.0008$$

The mean of the process has shifted from nominal mean  $\left(\frac{U+L}{2}\right)$  and 0.08% of the products has fallen outside U,L.

### 3.3 Some notes on $C_{pk}$ & $C_p$

- $C_p$  resembles the potential capability of the process and always  $C_{pk} \leq C_p$ .

-if  $C_{pk} = C_p$ , the process mean coincides the nominal mean  $(\frac{U+L}{2})$ .

-  $C_{pk} < C_p$  indicates the process mean has shifted from the nominal mean  $(\frac{U+L}{2})$ .

- to the extent that the quantity of  $C_{pk}$  is less than quantity of  $C_p$ , the process mean has shifted further from the nominal mean  $(\frac{U+L}{2})$ .

Some companies recommend  $C_p$  of 2.

-As well as  $C_{pk}$ ,  $C_p$  some other capability indices such as  $C_{pmk}$ ,  $C_{pm}$  and their extensions  $C_{Npk}$ ,  $C_{Npm}$ ,  $C_{Np}$  have been introduced.

For example  $C_{pm}$  is defined for a normally distributed production process as follows:

$$C_{pm} = \frac{U - L}{6\sqrt{E(X_i - T)^2}} = \frac{U - L}{6\sqrt{E(X_i - \mu)^2 + (\mu - T)^2}} = \frac{U - L}{6\sqrt{\sigma_x^2 + (\mu - T)^2}}$$

or

$$C_{pm} = \frac{C_p}{\sqrt{1 + \frac{(\bar{\bar{X}} - T)^2}{\sigma_x^2}}}$$

where

$T = X'_0$  The target value; T is not necessarily equal to  $\theta = \frac{U + L}{2}$ , though often it is

$\mu$  The process mean estimated with  $\hat{\mu} = \bar{\bar{X}}$

$\sigma_x$  The process standard deviation

$\mu - T$  The deviation of the process mean from the target,

It is worth mentioning that Vanmann(1995) proposes he following two-variable relationship for calculating  $C_p, C_{pk}, C_{pm}, C_{pmk}$  :

$$C_p(u, v) = \frac{d - u \times |\mu - \theta|}{v \sqrt{\sigma_x^2 + (\mu - T)^2}},$$

where

$d$  half of the tolerance( $=\frac{U-L}{2}$ )

$L$	The lower specification limit
$T = X_0'$	The target value
$u, v$	Binary variables (0,1)
$U$	The upper specification limit
$\bar{\bar{X}}$	Estimate $\mu$
$\theta$	Normal value, $\theta = \frac{U + L}{2}$
$\mu$	Process mean
$\sigma$	Process standard deviation

Then:

$$C_p(0,0) = C_p, \quad C_p(1,0) = C_{pk}, \quad C_p(0,1) = C_{pm}, \quad C_p(1,1) = C_{pmk}.$$

## Exercises

### 3.1

Consider two production processes used to produce a product with  $L = 96.5$  and  $U = 103.5$ . The first process produces products which are normally distributed with mean 99 and variance 1; the products of the other process are uniformly distributed on the interval (97,103). The target value is  $T = X'_0 = 100$ . Can you calculate the process capability for the 2 processes?

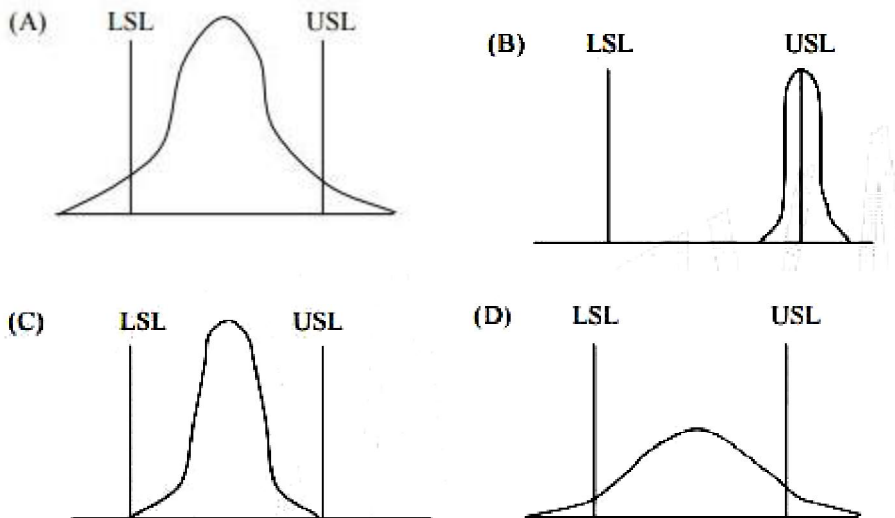
### 3.2

A customer requires a  $C_p \geq 1.33$  for a specification of  $15.5 \pm 2.0$ . If the supplier's process has a mean of 15.0 and a standard deviation of 0.5, describe the process capability. (<http://asq.org/cert/resource/pdf/sample-exam/cqe-sample-exam.pdf>)

Answer:  $C_{pk}$  not acceptable and product requirements does not meet specifications.

### 3.3

Given identical specification limits, which of the following process distributions has the highest  $C_p$  value? Why? (source: <http://asq.org/cert/resource/pdf/sample-exam/cqe-sample-exam.pdf>)



### 3.4

a) Will  $C_{pm}$ ,  $C_{pk}$  and  $C_p$  be the same, if  $T = X'_0$  coincides process mean?

b) If the process mean coincides  $\theta = \frac{U+L}{2}$ , will  $C_{pm}$  and  $C_p$  be equal?



**3.5**

The following sample has been taken from the production process of certain product with specification of  $100 \pm 0.30$ . Find the process capability index.

$$x_1 = 99.88 \quad x_2 = 100.11 \quad x_3 = 99.99 \quad x_4 = 100.02 .$$

**Many people live either in the future or in the past; the present time is God's present to us**

# Chapter 4

## A review of Probability and Statistics Key Concepts

## Chapter 4 A review of Probability and Statistics Key Concepts

### Aims

This chapter aims to introduce some of the fundamental concepts and results of probability and statistics theory. Some distributions as well as the central limit theorem regarding the limiting distribution of the mean of a sample and a founding stone of the extreme value theory i.e. the Fisher–Tippett theorem regarding the limiting distribution of the maximum & minimum of a sample are described. A graphical technique called *Q-Q plot* is also introduced for determining if a statistical distribution fits a series of data.

### 4.1 Review of definitions

#### Random experiment, sample space, event

**4.1.1 Random experiment:** An experiment whose outcome cannot be predicted with certainty, although the possible outcomes are known.

**4.1.2 Sample space(SS):** The collection of all possible outcomes of a random experiment.

**4.1.3 Event :** A subset of sample space.

An event is said to have occurred if one of its elements has occurred. Example for an experiment, sample space and event are given below:

Experiment	Sample space	Event
Casting a die	$\{1,2,3,4,5,6\}$	$A = \{2,3\}$
Ambient air Temperature	$\{t \mid -10 \leq t \leq 45\}$	$B = \{15 \leq t \leq 20\}$
Asking the GPA of students	$\{x \mid 0 \leq x \leq 20\}$	$C = \{14 < x < 17\}$

#### 4.1.4 Frequency, Relative Frequency and Probability

Suppose a random experiment with  $k$  possible outcomes has been repeated  $n$  times and the frequency of the  $i^{\text{th}}$  outcome is equal to  $f_i$ ,  $i = 1, 2, \dots, k$ ; its relative frequency of the outcome is defined as  $f_i/n$ . The following table shows the frequency and the relative frequency of each outcome.

outcome	Frequency	relative frequency
$x_1$	$f_1$	$f_1/n$
$x_2$	$f_2$	$f_2/n$
$\vdots$	$\vdots$	$\vdots$
$x_k$	$f_k$	$f_k/n$
Sum	$n$	1

#### 4.1.5 A priori or classic probability

Consider a random experiment has  $S.S = \{x_1, x_2, \dots, x_k\}$  as its sample space. If the outcomes  $x_i, i = 1, 2, \dots, k$  are equally likely, then the classic probability of occurring  $x_k$  denoted by  $p_k(x_k)$  is defined as :

$$p_X(x_k) = \frac{1}{k}.$$

#### 4.1.6 A posteriori or statistical probability

If a random experiment has the sample space  $S.S = \{x_1, x_2, \dots, x_k\}$ , a posteriori or statistical probability is the relative frequency when the experiment is run for a largish number of times i.e.

$$\Pr(X = x_i) = \lim_{n \rightarrow \infty} \frac{f_i}{n}.$$

#### 4.1.7 Conditional probability

The probability of Event A conditioned on B is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(AB)}{P(B)}.$$

Therefore:

$$P(A.B) = P(A \cap B) = P(A|B).P(B) = P(B|A).P(A).$$

### Example 4.1

Suppose we have 4 defective and 6 non-defective light bulbs. What is the probability of both non-defective in a sample of size 2 when sampling is done without replacement?

### Solution

Let Events E and F be defined as:

Event E: the 1<sup>st</sup> outcome is non defective

Event F: the 2<sup>nd</sup> outcome is non defective,  
then

$$\Pr(F \cap E) = \Pr(F|E)\Pr(E) = \frac{6}{10} \times \frac{5}{9} = \frac{1}{3}.$$

## 4.1.8 Chain rule for factorization in probability theory

In probability theory, chain rule for factorization allows us to factor a joint probability or density function into a product of much smaller conditional probabilities or conditional density functions.

### 4.1.8.1 Chain rule factorization of joint probability

Given  $n$  events  $E_1, \dots, E_n$  the joint probability of the events is factorized as

$$\Pr(E_1, \dots, E_n) = \Pr(E_n | E_{n-1}, \dots, E_1) \dots \Pr(E_2 | E_1) \Pr(E_1);$$

And also

given a set of random variables  $X_1, \dots, X_n$ , define the parents of  $X_i$  to be the variables  $X_j$  with  $j < i$ . Then any given probability  $\Pr(X_1, \dots, X_n)$  is equal to the product of all  $X_i$ 's conditioned on "its parents"

$$\Pr(X_1, \dots, X_n) =$$

$$\Pr(X_n | X_{n-1}, \dots, X_1) \cdot \dots \Pr(X_3 | X_2, X_1) \cdot \Pr(X_2 | X_1) \cdot \Pr(X_1)$$

or

$$L = \Pr(X_n = x_n, \dots, X_1 = x_1) =$$

$$= \Pr(X_n = x_n | X_{n-1} = x_{n-1}, \dots, X_1 = x_1) \Pr(X_{n-1} = x_{n-1} | X_{n-2} = x_{n-2}, \dots, X_1 = x_1) \dots \Pr(X_2 = x_2 | X_1 = x_1) \Pr(X_1 = x_1)$$

or

$$L = \Pr(X_n = x_n, \dots, X_1 = x_1) = \prod_{i=1}^n \Pr(x_i | \text{its parents})$$

In practice it is not necessary to condition on  $X_{n-1}, \dots, X_1$ ;  $m$  terms are enough i.e. to condition on  $X_{n-1}, \dots, X_{n-m}$ :

$$L =$$

$$\Pr(X_n = x_n, \dots, X_1 = x_1) = \Pr(X_n = x_n | X_{n-1} = x_{n-1}, \dots, X_{n-m} = x_{n-m}) \Pr(X_{n-1} = x_{n-1} | X_{n-2} = x_{n-2}, \dots, X_{n-m-1} = x_{n-m-1}) \dots \times \Pr(X_{m+1} = x_{m+1} | X_m = x_m, \dots, X_1 = x_1) \times \Pr(X_m = x_m, \dots, X_1 = x_1)$$

$m$  depends on the particular and its value is something to be tackled.

#### 4.1.8.2 Chain rule factorization of joint probability density

Let

$$L = f_{X_n, X_{n-1}, \dots, X_1}(x_n, x_{n-1}, \dots, x_1) = \prod_{i=1}^n f_{X_i | \text{parents}}(x_i)$$

but in practice  $m$  terms are enough to condition on:

$$L \cong \prod_{i=1}^n f_{X_i | X_{i-1}=x_{i-1}, \dots, X_{i-m}=x_{i-m}}(x_i) \times f_{X_m, X_{m-1}, \dots, X_1}(x_m, x_{m-1}, \dots, x_1)$$

In case of independence

$$\Pr(X_n = x_n, \dots, X_1 = x_1) = \Pr(X_n = x_n) \dots \Pr(X_1 = x_1)$$

$$f_{X_n, \dots, X_1}(x_n, \dots, x_1) = f_{X_n}(x_n) \dots f_{X_1}(x_1)$$

For example :

$$f_{x_4, x_3, x_2, x_1}(x_4, x_3, x_2, x_1) = f_{X_4 | x_3, x_2, x_1}(x_4) f_{X_3 | x_2, x_1}(x_3) f_{X_2 | x_1}(x_2) f_{X_1}(x_1)$$

#### 4.1.9 Permutation and combination

Suppose  $r$  objects are chosen from  $n$  objects without replacement and are put into  $r$  cells, the number of objects that could be placed into each cell is:

$n$	$n-1$	...	$n-r+1$
-----	-------	-----	---------

##### Permutation

A permutation, also called an "arrangement number" or "order" is one of the many different ways or forms in which these cells exist or can be arranged.

In general, the number of permutations of  $n$  things taken  $r$  at a time is given by the formula

$$(n)(n-1) \dots (n-r+1) = \frac{(n)(n-1) \dots (n-r+1)(n-r)(n-r-1) \dots (1)}{(n-r) \dots (1)} = \frac{n!}{(n-r)!}$$

In particular if  $r=n$

$$\frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

i.e. the number of different ways  $n$  objects can be arranged is  $n!$ .

### Combination

Out of  $\frac{n!}{(n-r)!}$  permutations, the sets without regard to the order of drawing are called combinations. The number of different combinations of  $r$  objects that may be drawn from a lot of  $n$  objects is given by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

### Example 4.2

The combination of  $r=80$  drawn from  $n=100$  objects is equal to

$$k = \binom{100}{20} = \frac{100!}{20! \times 80!}.$$

### Handling factorial of large numbers

If for a particular  $n$ ,  $n!$  is not possible to be calculated by hand or a software such as MATLAB, there are tables in some books which gives the logarithm of  $n!$ . For example  $k$  in the above relationship could be calculated from:

$$\log k = \log 100! - \log 80! - \log 20! = 157.97 - 118.8547 - 18.3861 = 20.7292$$

$$.k = 10^{20.7292} = 5.36043457 \times 10^{20}$$

The logarithm of  $100!$ ,  $80!$  and  $20!$  have been read from Table H of Grant & Leavenworth (1988).

For  $n > 100$  sterling's formula i.e.

$$n! \cong \left(\frac{n}{e}\right)^n \sqrt{2\pi n},$$

is also useful.

For example

$$1500! \cong \left(\frac{1500}{2.71828}\right)^{1500} \times \sqrt{2 \times 3.14 \times 1500} = (551.82)^{1500} \times 97.05.$$

## 4.2 Introducing some statistical distributions

The aim of this section is to remind certain properties of some distributions used in statistical quality control.

### 4.2.1 Hyper geometric distribution

A sample of size  $n$  is randomly selected without replacement from a population of  $N$  items. In the population,  $k$  items can be classified as successes, and  $N - k$  items can be classified as failures. The probability of the number of success in the sample is given by the probability function of hyper geometric statistical distribution :

$$\Pr\{X = d|N, D, n\} \text{ or } \Pr(d|N, D, n) = \frac{\binom{D}{d} \binom{N-D}{n-d}}{\binom{N}{n}}$$

where integers  $N, D, n, d$  denote

$N$  Population (lot, batch,...)size

$D$  The number of items in the population that are classified as successes

$n$  Sample size

$D$  The number of items in the sample, classified as successes

$$\binom{N}{n} = \frac{N!}{n! \times (N-n)!}$$

The mean and variance of hyper geometric distribution are:

$$E(X) = \frac{D \times n}{N},$$

$$\text{Var}(X) =$$

$$\frac{D(N-D)(n)(N-n)}{N^2(N-1)} = np(1-p) \left( \frac{N-n}{N-1} \right) \quad p = \frac{D}{N}$$

In MATLAB, the command `hygecdf(d,N,D,n)` calculates the cumulative probabilities of the distribution and command `hygepdf(d,N,D,n)` returns the value of probability density function at  $d$ .

#### Example 4.3

In a lot of size 50 there are 4 defective products; a random sample of size 5 is drawn from the lot.

a) what is the probability of 1 defective product in this sample?

b) what is the probability of less than 3 defectives in the sample?



**Solution****a)**

$$N = 50 \quad n = 5 \quad D = 4 \quad d = 1$$

$$\Pr\left\{X = 1 \left| \begin{array}{l} N = 50 \\ D = 4 \\ n = 5 \end{array} \right. \right\} = \frac{\binom{4}{1} \binom{46}{4}}{\binom{50}{5}} = \frac{\left(\frac{4!}{1! \times 3!}\right) \left(\frac{46!}{4! \times 42!}\right)}{\left(\frac{50!}{5! \times 45!}\right)} \cong 0.31$$

MATLAB : hygepdf(1, 50, 4, 5)=0.3081

**b)****Pr(X<3)**

$$= \Pr\left(d = 0 \left| \begin{array}{l} N = 50 \\ D = 4 \\ n = 5 \end{array} \right. \right) + \Pr\left(d = 1 \left| \begin{array}{l} N = 50 \\ D = 4 \\ n = 5 \end{array} \right. \right) + \Pr\left(d = 2 \left| \begin{array}{l} N = 50 \\ D = 4 \\ n = 5 \end{array} \right. \right) =$$

$$= \frac{\binom{4}{0} \binom{46}{5}}{\binom{50}{5}} + \frac{\binom{4}{1} \binom{46}{4}}{\binom{50}{5}} + \frac{\binom{4}{2} \binom{46}{3}}{\binom{50}{5}} = \frac{1 \times 1370754}{2118760} + \frac{4 \times 163185}{2118760} + \frac{6 \times 15180}{2118760} = \frac{2114574}{2118760} = 0.998$$

MATLAB: hygecdf(2, 50, 4, 5)=0.9980.

**4.2.2 Binomial distribution**

Suppose  $100p'$  is the proportion defective in the process of a special product of a factory ; in other words the probability of observing a defective product in the population is  $p'$ . The probability of observing  $r$  defectives in a sample of size  $n$  taken randomly from the population(process) is given by the probability density function of a binomial distribution:

$$\Pr(X = r) = \binom{n}{r} (p')^r (1 - p')^{n-r} \quad 0 < p' < 1.$$

In MATLAB

the command **binopdf(r,n,p')** computes this probability and **binocdf(r,n,p')** calculates  $\Pr(X \leq r)$ .

**Example 4.4**

Suppose 20 percent of a population is defective. What is the probability of observing 3 defectives in a sample of size 10.

$$\Pr(X = 3) = \binom{10}{3} (0.2)^3 (0.8)^7 = 120 \times 0.008 \times 0.2097 = 0.2013$$

Using MATLAB:  $\Pr(X = 3) = \text{binopdf}(3, 10, 0.2) = 0.2013$ .

If  $X$  is the number of defective items in a sample of size  $n$  drawn from a population having  $p'$  as its defective proportion then  $X$  is binomially distributed and the mean and standard deviation of  $X$  is given by :

$$E(X) = np' \quad \sigma_X = \sqrt{np'(1-p')}.$$

in the sample of size 10 in the above example, the mean and standard deviation of the number of defectives in the sample are:

$$E(X) = np' = 10 \times 0.2 = 2$$

$$\sigma_X = \sqrt{np'(1-p')} = \sqrt{10 \times 0.2 \times 0.8} \cong 1.27.$$

#### 4.2.2.1 Mean and standard deviation of sample proportion( $p$ )

Let the proportion of nonconforming products in a sample be denoted by  $p$ , then

$$p = \frac{X}{n} \quad E(p) = E\left(\frac{X}{n}\right) = \frac{1}{n} \times E(X) = \frac{np'}{n} \Rightarrow \quad E(p) = p'$$

$$Var(p) = \frac{1}{n^2} Var(X) = \frac{1}{n^2} (np'q') = \frac{1}{n} p'q' \Rightarrow \quad \sigma_p = \frac{\sqrt{p'q'}}{\sqrt{n}}$$

If the mean of the proportions of the defectives of the samples is  $\bar{p}$ , the proportion of defectives in the process (population) is estimated by:  $\hat{p}' = \bar{p}$ .

#### Example 4.5

Forty samples were drawn from a population with  $p' = 17.034\%$ ; the following table shows the frequencies. Calculate the mean and variance of the number of defectives both from

a) the following empirical information and from b) the theory.

Order(i)	1	2	3	4	5	
Number of defectives ( $d_i$ )	0	1	2	3	4	sum
Number of samples having $d_i$ defectives ( $f_i$ )	5	16	9	7	3	40

#### Solution

a) The mean of the defectives in each of the 5 samples of size 10 is

$$\frac{\sum_{i=1}^5 f_i d_i}{\sum_{i=1}^5 f_i} = \frac{0 \times 5 + 1 \times 16 + 2 \times 9 + 3 \times 7 + 4 \times 3}{40} = 1.675.$$

The proportion of defectives in the population estimated from the empirical information is  $\hat{p}' = \bar{p} = \frac{1.675}{10} = 0.1675$ . The standard deviation of the defectives in sample is:

$$s = \sqrt{\frac{\sum f_i d^2 - \bar{d}^2 \sum f_i}{\sum f_i - 1}} = \sqrt{\frac{\sum f_i (d_i - \bar{d})^2}{\sum f_i - 1}} = 1.57 .$$

b) From theory using a binomial distribution with  $n=10$  &  $p'=0.17034$ :  
the mean of defectives in a sample is:

$$E(X) = np' = 10 \times 0.17034 = 1.7034,$$

The standard deviation is

$$\sigma_X = \sqrt{np'q'} = \sqrt{10 \times 0.17034(1 - 0.17034)} = 1.189.$$

Where X is the defectives in a sample

#### 4.2.2.2

#### *Approximating hyper geometric distribution with binomial distribution*

If the size of the population(N) is largish( $N > 10n$ ), the probabilities of hyper geometric distribution could be estimated using binomial distribution

formula  $\Pr(X = r) = \binom{n}{r} \times p^r \times (1 - p')^{n-r}$  by replacing  $p'$  with  $\frac{D}{N}$  .

### 4.2.3 Poisson distribution

The Poisson distribution is a discrete distribution that expresses the probability of a given number of events(X ) happening in a specified time period. The events occur with a known average( $\lambda$ ) over that time period and independently of the time since the last event. The probability function of the distribution is:

$$\Pr(X = x) = \frac{(\lambda)^x e^{-\lambda}}{x!} \quad \lambda > 0, \quad x = 0, 1, 2, 3, \dots$$

$$E(X) = \text{Var}(X) = \lambda$$

#### 4.2.3.1 *Approximating Binomial Distribution with Poisson*

When probability tables or statistical soft-wares are not available or they do not give the desired probabilities of a random variable with binomial distribution , the probabilities can be approximated with Poisson or normal distribution. Although approximating binomial distribution with Poisson distribution is common, however this approximation when the sample size is large and the proportion is low gives good results and when  $n \geq 100$ ,  $np' \leq 10$  gives very good results. If a sample of size n is drawn from population having  $p'$  as a desired class proportion, the probability of

number of elements of the desired class(X) found in the sample is given approximately by :

$$\Pr(X = r) \cong \frac{(np')^r e^{(-np')}}{r!} \quad \lambda = np' .$$

#### 4.2.4 Normal distribution

The “normal” or “Gaussian” distribution is a continuous probability distribution that has a two-parameter probability density function as given below:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < +\infty$$

If  $X \sim N(\mu, \sigma)$  i.e. the random variable X is normally distributed with parameters  $\mu, \sigma$ , then

$$E(x) = \mu, \quad Var(X) = \sigma^2,$$

$$\begin{aligned} \Pr(a \leq X \leq b) &= \Pr\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right) = \Pr\left(Z \leq \frac{b-\mu}{\sigma}\right) - \Pr\left(Z \leq \frac{a-\mu}{\sigma}\right) \\ &= \phi_Z(b) - \phi_Z(a) = normcdf(b) - normcdf(a) \end{aligned}$$

##### 4.2.4.1 Approximating Binomial Distribution with Normal

we will now focus on using the normal distribution to approximate binomial probabilities. When  $p'$  is close to 0.5 the distribution tends to be symmetric as the normal distribution is symmetric. In other words when the proportion( $p'$ ) is near to 0.5 the approximation gives better results than when it is near to 0 or 1. However for any given proportion( $p'$ ), as the sample size increases, the distribution tends to the shape of normal distribution. The greater the sample size( $n$ ) the better the approximation. Hald(1952) shows that when

$$np'(1-p') > 9 \quad \text{or} \quad n > \frac{9}{p'(1-p')}$$

the binomial distribution is acceptably close to the normal distribution (Johnson, 2000 page339) . Another common practice for choosing the normal distribution to approximate binomial is when

$$\begin{aligned} n(1-p') &\geq 5, \quad np' \geq 5 \quad \text{or} \\ \min [np', n(1-p')] &\geq 5 \quad \text{or} \\ n &> \min \left\{ \frac{5}{p'}, \frac{5}{1-p'} \right\}. \end{aligned}$$

To approximate the probabilities of a binomial distribution with normal we could use either of the followings:

$$\Pr(r|n, p') \cong \Phi_z \left( \frac{r - np' + 0.5}{\sqrt{np'q'}} \right) - \Phi_z \left( \frac{r - np' - 0.5}{\sqrt{np'q'}} \right)$$

$$\text{Or } \Pr(r|n, p') \cong \frac{1}{\sqrt{2\pi np'q'}} e^{-\frac{(r - np')^2}{2np'q'}}.$$

where  $\Phi_z$  is cdf of standard normal distribution.

Nowadays, with the development of softwares such as MATLAB, the need for approximation has reduced significantly.

### 4.2.5 Lognormal Distribution

Lognormal distribution with parameters  $(\sigma, \mu)$  has the following density function and mean and variance:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} \quad x \geq 0$$

$$E(X) = e^{\mu + \frac{\sigma^2}{2}} \quad \text{Var}(X) = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$$

Its probabilities could be calculated from MATLAB command *logncdf* or from standard normal distribution:

$$\Pr(X < a) = \Pr(\ln X < \ln a) = \Pr(Z < \frac{\ln a - \mu}{\sigma})$$

Fig. 4-1 Shows some normal and lognormal distributions as samples.

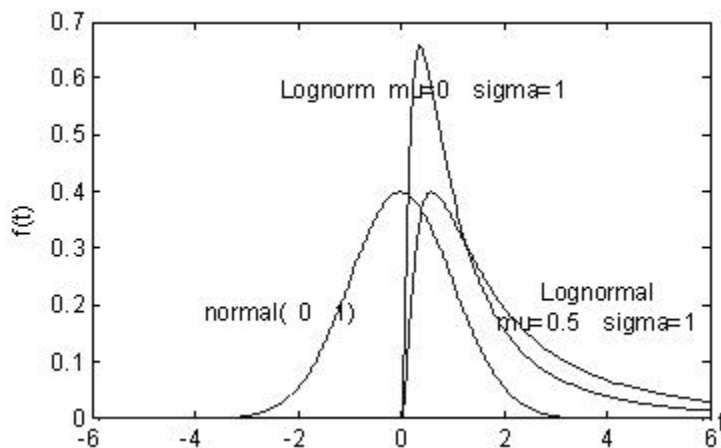


Fig. 4.1 Sample lognormal and normal distributions

#### 4.2.5.1

#### *Calculation of the parameters of a lognormal distribution from the mean and variance*

Given a lognormal distribution with mean and variance  $E(X)$  &  $Var(X)$ ,

$$\sigma^2 = \ln [Var(X)/E^2(X)]$$

$$\mu = \ln E(X) - \frac{\sigma^2}{2}.$$

In reliability theory, lognormal distribution is one of the distributions used for product life or repair time, in economics it is used for income.

### 4.2.6 Exponential Distribution

Exponential distribution is a one parameter continuous distribution whose density function is :

$$f(t) = \lambda e^{-\lambda t} = \frac{1}{\theta} e^{-\frac{t}{\theta}},$$

Where  $\lambda$  is the parameter and  $\theta$  is the distribution mean.

If random variable  $T$  is exponentially distributed with parameter  $\lambda$  then:

$$F(t) = \Pr(T \leq t) = 1 - e^{-\lambda t} = 1 - e^{-\frac{t}{\theta}} = \text{expcdf}(t, \theta)$$

$$E(T) = \theta = \frac{1}{\lambda}, \quad Var(T) = \theta^2 = \frac{1}{\lambda^2}$$

### 4.3 Limit theorems

There are several limit theorems dealing with the limiting distributions of mean, sum, maximum and minimum ...of a large number of random variables. It is usually assumed that the random variables are either independent, or almost independent. In this section we will mention two limiting theorems i.e. Central Limit Theorem and Fisher-Tippett theorem. In chapter 1 a common version of Central Limit Theorem was stated as follows:

#### 4.3.1 Central Limit theorem(Lindeberg & Levy Theorem)

If  $X_1, \dots, X_i, \dots, X_n$  are independent and identically distributed random variables with mean  $\mu$  and standard deviation  $\sigma$  ( $0 < \sigma < \infty$ ), regardless of the distribution of the variables, as  $n$  increases the distribution of the

variables mean( $\bar{X}$ ) tends to the normal distribution  $N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$  and also the

distribution of  $\sum_{i=1}^n X_i$  tends to the normal distribution  $N\left(n\mu, \sigma\sqrt{n}\right)$ .

Now another version is described.

### 4.3.2 Central Limit Theorem(Lyapounov Conditions)

Let  $X_1, \dots, X_i, \dots, X_n$  be a sequence of independent random variables with arbitrary distributions and means  $\mu_1, \dots, \mu_n$  and finite variances  $\sigma_1^2, \dots, \sigma_n^2$  ( $0 < \sigma_i^2 < \infty$ ). Also let random variable  $Y_n$  be defined as

$$Y_n = \frac{\sum_{i=1}^n X_i - \sum_{i=1}^n \mu_i}{\sqrt{\sum_{i=1}^n \sigma_i^2}}$$

Note that  $E(Y_n) = 0$  and  $\text{Var}(Y_n) = 1$ .

Then, under the following conditions

$$\lim_{n \rightarrow \infty} \left( \frac{\sum_{i=1}^n |X_i - \mu_i|^3}{\left(\sum_{i=1}^n \sigma_i^2\right)^3} \right) = 0 \quad |X_i - \mu_i|^3 < \infty \quad i = 1, \dots, n$$

as  $n$  increase, We have

$$\Pr_{n \rightarrow \infty} (Y_n \leq x) = \Phi_Z(x)$$

where

$\Phi_Z(x)$  is the cumulative density function of standard normal for a given  $x$ .

### 4.4Fisher -Tippet Theorem

Let the sequence of random variables  $X_1, \dots, X_n$  are independent and identically distributed, then as  $n$  increases, the distribution of their maximum ( $\max(X_i)$ ) tends to a GEV distribution; and their minimum tends to another GEV.

### 4.5Pickands' Theorem

What is known as Pickand's theorem (Pickands, 1975) could be stated as follows:

Suppose that we have a sample of  $n$  mutually independent and identically distributed (iid) random variables with common but unknown continuous distribution function  $F(x)$ , then the distribution of the data

exceeding a large threshold  $u$  converges towards a generalized Pareto distribution (GPD). In other words the conditional distribution of  $X$  given that  $X \geq u$  tends to a GPD. This theorem, which is applicable to a wide range of continuous distributions, is used in a method called POT method in the theory of extreme value analysis.

## 4.6 Estimation of Mean, Standard Deviation and Variance of a Production Process

Let  $k$  random samples with sample sizes  $n_1, \dots, n_k$  have been drawn from a production process or a population, then the mean, standard deviation and variance are estimated as described below:

### 4.6.1 Point Estimate of Process Mean

If from a process,  $k$  sample with means  $\bar{X}_i$ ,  $i = 1, 2, \dots, k$  is available then the estimate of the mean of the process is given by:

$$\hat{\mu} = \bar{\bar{X}} = \frac{\sum n_i \bar{X}_i}{\sum n_i}.$$

### 4.6.2 Confidence Interval (Interval Estimate) for the Process Mean

If the process is normally distributed or almost normally distributed, and the sample sizes  $n_1, \dots, n_k$  are different but close; then a confidence interval for the process mean is given by:

$$\bar{\bar{X}} \pm \frac{\bar{S}}{\sqrt{\bar{n}}} t_{n-1, \frac{\alpha}{2}}, \quad \bar{n} = \frac{\sum_{i=1}^k n_i}{k}, \quad \bar{S} = \frac{\sum_{i=1}^k S_i}{k}$$

Where  $S_1, \dots, S_k$  are standard deviations of the  $k$  samples drawn from the process.

### 4.6.3 Estimate of the Process Standard Deviation

If the process is normally distributed and in statistical control, then standard deviation is estimated by:

$$\hat{\sigma} = \frac{\bar{S}}{c_4}$$

where  $c_4$  is read from Table U for  $n = \bar{n} = \sum_{i=1}^k n_i / k$ .



#### 4.6.4 Estimate of the Process Variance

Let  $S_i^2$   $i, = 1, \dots, k$  be the variances of  $k$  samples taken from a process or population with variance  $\sigma^2$ , and the process is in control, then the estimate of  $\sigma^2$  is:

$$\widehat{\sigma^2} = \frac{\sum_{i=1}^k (n_i - 1) S_i^2}{(\sum n_i) - k};$$

Needless to say if the sample sizes are equal the estimate of  $\sigma^2$  would be

$$\widehat{\sigma^2} = \overline{S^2} = \frac{\sum_{i=1}^k S_i^2}{k}.$$

### 4.7 Goodness of Fit

What should be done to evaluate fitting a statistical distribution to a series of data or a frequency distribution? Two ways for dealing with this purpose are: 1) goodness of fit tests such as Pearson Chi-square general goodness of fit test, Bartlett goodness of fit test for exponential distribution and 2) a graphical device named Q-Q plot. The tests are easily found in text books; Appendix A at the end of this chapter deals with Bartlett test. The Q-Q plot is described here. It should be added that before using the test or the plot, drawing the histogram and the calculation of coefficient of skewness of the dataset are useful to evaluate the symmetry of the distribution of the dataset. This coefficient is a measure that studies the degree and the direction of departure from symmetry.

#### 4.7.1 Q-Q plot

Quantile-Quantile(Q-Q) plot is a graphical device to observe whether a particular distribution fits a dataset or not. In this graph the observed data and the corresponding data obtained from the distribution are plotted against each other in an X-Y coordinate plane. The better the population follows the distribution, the closer the points to the angle bisector of the first quarter of the X-Y plane. The procedure for preparing a Q-Q plot is as follows:

Sort The sample of data from minimum to maximum, giving rank 1 through  $n$ :  $X_{(1)}, \dots, X_{(n)}$

Allocate a number  $F(i)$ , called plotting position calculated from one of the following formulae to each  $x_{(i)}$ . In fact  $F(i)$  is a number near to relative

frequency and an estimate for the cumulative distribution function at  $x_{(i)}$ .

There are many formulae for plotting position including the followings:

#### A) Gumbel Plotting position

One of the first formulae for Plotting position was given by Gumbel:

$$F(i) = \frac{i}{n+1}, \quad i = 1, \dots, n$$

#### B) Plotting position for normal distribution

There are some formulae for the normal case including (Besterfield, 1990 page 52):

$$F(i) = \frac{i - 0.5}{n}$$

or (Goda, 2000 page 287):

$$F(i) = \frac{i - 0.375}{n + 0.25}.$$

#### C) Plotting position for Weibull distribution with parameters A, B, C

(Goda, 2000 page 287):

$$F(i) = \frac{i - a}{n + b} \quad a = 0.20 + \frac{0.27}{\sqrt{C}} \quad b = 0.20 + \frac{0.23}{\sqrt{C}}$$

#### D) Plotting position for Exponential Distribution

Since Exponential distribution could be considered a Weibull with  $C=1$  then:

$$F(i) = \frac{i - 0.47}{n + 0.43}$$

For each  $F(i)$ ,  $i=1, \dots, n$  calculate  $\hat{x}_{(i)}$ ,  $i=1, \dots, n$  from  $F_x[\hat{x}_{(i)}] = F(i)$  where  $F_x$  is the cumulative distribution function of the distribution under study.

Plot the pairs  $(x_{(i)} \& \hat{x}_{(i)})$  in an X-Y coordinate plane, and fit a line to the points. The closer this line to the angle bisector of the first quarter of the plane, the better fits the distribution to the dataset. It is worth knowing that the better the distribution fits the data set the closer the correlation coefficient of  $x_{(i)} \& \hat{x}_{(i)}$  to 1; but the vice versa is not necessarily true i.e.

if the correlation coefficient of  $x_{(i)} \& \hat{x}_{(i)}$  is close to 1, necessarily the distribution does not fit the dataset well.

The correlation coefficient is calculated by the following formula:

$$R = \frac{n \sum x_{(i)} \hat{x}_{(i)} - \sum x_{(i)} \sum \hat{x}_{(i)}}{\sqrt{n \sum x_{(i)}^2 - (\sum x_{(i)})^2} \sqrt{n \sum \hat{x}_{(i)}^2 - (\sum \hat{x}_{(i)})^2}}$$

#### Example 4.5

The following table shows a sorted random sample,  $x_{(i)}$  's, from a population. Is the sample a representative of normal distribution?

#### Solution

To answer, a Q-Q plot is drawn. The mean and variance of the distribution is estimated as follows:

$$\hat{\mu} = \bar{X} = 54.81 \quad , \quad \hat{\sigma} = \frac{s}{c_4} = 11.7287$$

$F(i)$ ,  $i=1,...,n$  was computed using  $F(i) = \frac{i - 0.375}{n + 0.25}$  as the plotting position, and inserted in the table. Then the corresponding  $\hat{x}_{(i)}$  is calculated by equating the  $F(i)$  to the normal standard cumulative distribution, and calculating  $\hat{x}_{(i)}$  from these equations.

$$\Pr(X < \hat{x}_{(i)}) = \Pr(Z < \frac{\hat{x}_{(i)} - \hat{\mu}}{\hat{\sigma}}) = F(i) \quad \hat{\mu} = \bar{X}, \quad \hat{\sigma} \cong \frac{s}{c_4}.$$

sample calculation follows:

For  $i = 1$  :

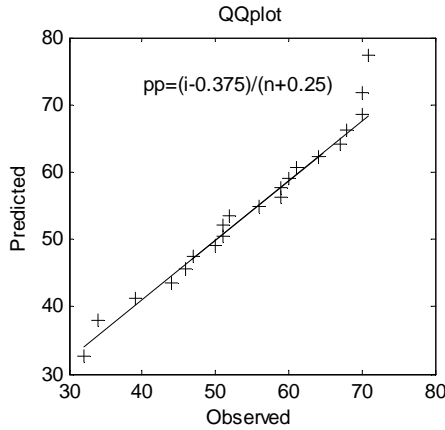
$$\Pr(Z < \frac{\hat{x}_{(i)} - \hat{\mu}}{\hat{\sigma}}) = F(i) \quad \Pr(Z < \frac{\hat{x}_{(1)} - 54.81}{11.7287/0.9876}) = 0.0294 \Rightarrow \hat{x}_{(1)} \cong 32.37$$

or  $\hat{x}_{(1)} = \text{norminv}(0.0294, 54.81, 11.8751) = 32.3698$ .

The following table contains all the results

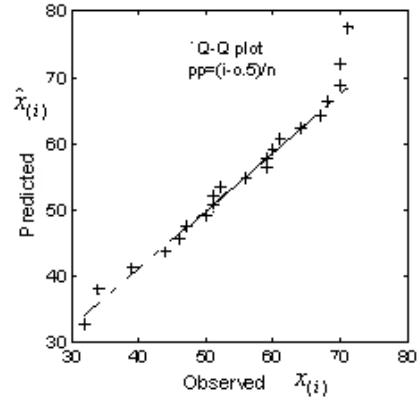
Rank(i)	$x_{(i)}$	F(i)	$\hat{x}_{(i)}$	Rank(i)	$x_{(i)}$	F(i)	$\hat{x}_{(i)}$
1	32	0.0294	32.37	12	59	0.5471	56.21
2	34	0.0765	37.84	13	59	0.5941	57.64
3	39	0.1235	41.06	14	60	0.6412	59.10
4	44	0.1706	43.51	15	61	0.6882	60.64
5	46	0.2176	45.55	16	64	0.7353	62.28
6	47	0.2647	47.34	17	67	0.7824	64.08
7	50	0.3118	48.98	18	68	0.8294	66.11
8	51	0.3588	50.52	19	70	0.8765	68.56
9	51	0.4059	51.98	20	70	0.9235	71.78
10	52	0.4529	53.41	21	71	0.9706	77.25
11	56	0.5000	54.81				

$x_{(i)}$  's and  $\hat{x}_{(i)}$  's have been plotted against each other in Fig. 4.2 and a line fitted to them.



**Fig. 4.2 Q-Q plot with**

$$F(i) = \frac{i - 0.375}{n + 0.25}.$$



**Fig. 4.3 Q-Q plot with**

$$F(i) = \frac{i - 0.5}{n}.$$

Since the points are near to the fitted line and the line is close to the angle bisector of the first quarter of the X-Y coordinate plane, it is concluded that the normal distribution fits the dataset.

In MATLAB, the command *qqplot* could be utilized to make a Q-Q plot from the dataset  $X$ .<sup>1</sup> The Q-Q plot of Fig. 4.3 was made by this command. The difference of the two plots is not significant.

The correlation coefficient( $r$ ) between  $\hat{x}_{(i)}$ ,  $x_{(i)}$  is calculated by

`R=corrcoef(X,Xhat);r=R(1,2)`

where

$X$  is the vector containing  $X(i)$ ,  $i=1,2,3..$

$Xhat$  is the vector containing  $\hat{x}(i)$ ,  $i = 1,2,..$

which gives 0.9826. This value, being near to 1, together with the Q-Q plot of Fig. 4-2 or Fig. 4-3 indicate that normal distribution is a good fit for the data best fit.

<sup>1</sup>>> X=[....data]; pd=makedist('normal', mu, sigma);qqplot(X,pd)

## Some solved problems:

### Example 4.6

25 percent of a kind of fan have a life of more than 100,000 hours. In a sample of size 45 from this product, what is the probability that 13 fans work more than 100,000 hours.

### Solution

Let  $X$  = Number of fans with life greater than 100000 hr in the sample.

a) With binomial distribution (Exact Solution)

$$p' = 0.25 \quad n = 45$$

$$P(X = 13) = \binom{45}{13} (.25)^{13} (.75)^{45-13} = \text{binopdf}(13, 45, .25) = 0.1093.$$

b) With Poisson Distribution

Verify that the conditions for approximation with Poisson distribution are not satisfied.

c) With Normal Distribution

Since  $np' > 5$ ,  $n(1 - p') > 5$ , then the conditions for approximation with normal distributions are satisfied.

$$p' = 0.25 \quad n = 45$$

$$np' = 11.25 \quad \sqrt{np'q'} = 2.905$$

$$\Pr(X = r | n, p') \cong \frac{1}{\sqrt{2\pi} \sqrt{np'q'}} e^{-\frac{(r - np')^2}{2np'q'}} = 0.1145$$

In another way:

$$P(X = r) \text{ or } \Pr(r | n, p') \cong \Phi_z \left( \frac{r - np' + 0.5}{\sqrt{np'q'}} \right) - \Phi_z \left( \frac{r - np' - 0.5}{\sqrt{np'q'}} \right)$$

$$P(X = r) \cong \phi_z(z_2) - \phi_z(z_1)$$

$$z_2 = \frac{r_2 - np' + 0.5}{\sqrt{np'q'}} = \frac{13 - 11.25 + 0.5}{\sqrt{45 \times 0.25 \times 0.75}} = \frac{2.25}{\sqrt{8.4375}} = \frac{2.25}{2.905} = 0.77$$

$$z_1 = \frac{r_1 - np' - 0.5}{\sqrt{np'(1-p')}} = \frac{13 - 11.2 - 0.55}{\sqrt{45 \times 0.25 \times 0.75}} = 0.43$$

$$\Pr(X = 13) = \phi_z(z_2) - \phi_z(z_1) = 0.7794 - 0.6664 = 0.113.$$

**Example 4.7**

From a production process a batch of size 10000 is prepared, this process produces 0.1% nonconforming items or defectives.

- a) What is the probability of existing exactly 10 defective products in the batch,  
 b) find r=umber of defectives in the lot in such a way that the probability of existing more that this value as defective products in the batch is 0.05.

**Solution**

a)

Exact solution

$$P(X = r) = \binom{n}{r} (p')^r q'^{n-r} \quad n=10000 \quad p'=0.001 \quad r=10$$

$$\Pr(X=10) = \text{binopdf}(10, 10000, 0.001) = 0.1252$$

approximation using the following normal distribution pdf :

$$\Pr(X = r | n, p') \cong \frac{1}{\sqrt{2\pi} \sqrt{np'q'}} e^{-\frac{(r - np')^2}{2np'q'}} =$$

$$\begin{aligned} \Pr(X = 10 | n = 10000, p' = 0.001) \\ \cong \text{normpdf}(10, 10000 * (0.001), \text{sqrt}(10000 * (0.001) * (.999))) = 0.1262 \end{aligned}$$

b) Using normal approximation

$$n=10000 \quad p'=0.001 \quad \Pr(X>r)=0.05 \quad r=?$$

$$E(X) = 10000 \times .001 = 10$$

$$\sigma_x = \sqrt{np'q'} = \sqrt{10000 \times .001 \times .999} = 3.161$$

$$\Pr(X > r) = 1 - \Pr\left(Z < \frac{r - np'}{\sigma_r}\right) = 0.05 \quad \Pr(Z < \frac{r - 10}{3.161}) = .95$$

$$\frac{r - 10}{3.161} = 1.645 \Rightarrow r = 15.199$$

$$\text{or } r = \text{norminv}(.95, 10, 3.161) = 15.199 \rightarrow 15$$

$$\text{binoinv}(0.95, 10000, 0.001) = 15 \text{ gives the exact answer.}$$

**Example 4.8**

A batch contains 56 non-defective and 4 defective products . A random sample of 5 is selected. What is the probability of observing no defective, 3 defectives and less than 3 defectives in the sample. Do not use approximation; find exact solution.

**Solution**

The exact distribution is hyper geometric

$$N = 56 + 4 = 60 \quad D = 4 \quad n = 5$$

$$\Pr \left( r = 0 \left| \begin{array}{l} N = 60 \\ D = 4 \\ n = 5 \end{array} \right. \right) = \frac{\binom{4}{0} \binom{56}{5}}{\binom{60}{5}} = \frac{4 \times 3819816}{5461512} = 0.699$$

Or  $\text{hygepdf}(0,60,4,5)=0.6994$

$$\Pr \left( r = 3 \left| \begin{array}{l} N = 60 \\ D = 4 \\ n = 5 \end{array} \right. \right) = \frac{\binom{4}{3} \binom{56}{2}}{\binom{60}{5}} = \frac{4 \times 1540}{5461512} = 0.001$$

Or  $\text{hygepdf}(3,60,4,5)= 0.0011$

$$\Pr \left( r \leq 2 \left| \begin{array}{l} N = 60 \\ D = 4 \\ n = 5 \end{array} \right. \right) = \Pr \left( r = 0 \left| \begin{array}{l} N = 60 \\ D = 4 \\ n = 5 \end{array} \right. \right) + \Pr \left( r = 1 \left| \begin{array}{l} N = 60 \\ D = 4 \\ n = 5 \end{array} \right. \right) + \Pr \left( r = 2 \left| \begin{array}{l} N = 60 \\ D = 4 \\ n = 5 \end{array} \right. \right) =$$

$\text{hygecdf}(2,60,4,5)= 0.9989$ .

**Example 4.9**

40 random samples were drawn from a production process and the proportion of the defectives in each sample was calculated; the following table shows the results;

Sample proportion defectives( $p_i$ )	0.04	0.03	0.02	0,01	0
Frequency( $f_i$ )	3	7	9	16	5

a) Estimate the proportion of defectives in the process and the standard deviation of this proportion.

b) If the samples are of size 10 and the proportion of the defectives in the process is  $p'=17.034\%$ . What is theoretically the mean and standard deviation of the proportion of defectives in each sample?

**Solution**

a)the mean of proportion( $\bar{p}$ ) is estimated:

$$\bar{p} = \frac{(3 \times 0.04) + (7 \times 0.03) + (9 \times 0.02) + (16 \times 0.01) + (0)}{40} = \frac{0.67}{40} = 0.01675$$

and the standard deviation:

$$s = \sqrt{\frac{\sum_{i=1}^5 f_i (p_i - \bar{p})^2}{\sum_{i=1}^5 f_i - 1}} = 0.157 \quad \sigma_{rms} = \sqrt{\frac{\sum_{i=1}^5 f_i (p_i - \bar{p})^2}{\sum_{i=1}^5 f_i}} = 0.155$$

b)

Theoretically the mean of the sample proportion defective( $p$ ) is  $E(p) = p' = 0.17034$

and its standard deviation is

$$\sigma_p = \frac{\sqrt{p'q'}}{\sqrt{n}} = \sqrt{\frac{(0.17034)(1-0.17034)}{10}} = 0.1189$$

At the end of this chapter some concepts of descriptive statistics<sup>1</sup> are reviewed without illustration, which is easily available on the internet.

## 4.8 Measures of Central tendency

The central tendency measures of a set of measurements are some numbers that best summarize the entire set, numbers in some way "central" to the set. An application of central tendency measure is in the comparison of frequency distribution. These measures, including mean, mode and median, are described here.

### 4.8.1 Arithmetic mean

The arithmetic mean of a set  $x_1, \dots, x_n$  with frequencies  $f_1, \dots, f_n$ , denoted by  $\bar{X}$  is calculated as follows:

$$\bar{X} = \frac{\sum f_i x_i}{\sum f_i} = \sum p_i x_i \quad p_i = \frac{f_i}{\sum f_i}$$

The following table illustrates the calculations

$x_i$	$f_i$	$x_i f_i$	$\bar{X} = \frac{380}{68} = 5.59$
2	12	24	
3	16	48	
8	18	144	
9	19	171	
-1	2	-2	
-5	1	-5	
sum	68	380	

<sup>1</sup> Descriptive statistics deal with the presentation, numerical processing and graphical representation of a data set.



It worth mentioning that the mean of a random sample, or the mean of several  $\bar{X}$ 's, denoted by  $\bar{\bar{X}}$  are unbiased estimators for the population mean ( $\mu$ ):

$$\hat{\mu} = \bar{X} \text{ or } \bar{\bar{X}}.$$

## 4.8.2 Median

The median related to a data set, denoted by Me or  $\tilde{X}$  is a value that 50 percent of the total set falls below and 50 percent falls above it. If the dataset is odd, the number falling in the middle of the sorted set is the median; if the data set is even the arithmetic mean of the 2 numbers that fall in the middle of the sorted set is the median. The median of a distribution is a value that bisects the distribution into 2 equal parts.

### Example 4.10

To calculate the median of {190,180, 200,195,185}, first we sort the set as follows:{200,195,190,185,180}; the median is:Me=190

To calculate the median of {200,195,190,185,175,170}, since the set is sorted and even the median is:  $Me = \frac{190+185}{2} = 187.5$ .

### 4.8.2.1The Median of Grouped Data

If the variable is continuous and a the sample is not available , but instead it has been grouped into k classes of width c with given frequencies, the median is given by:

$$\tilde{X} = L + \frac{\sum_{i=1}^K \frac{f_i}{2} - F_c}{f} \times c$$

where

$f_i$  The frequency of  $i^{\text{th}}$  group

n The total frequency

L The lower bound of " median class "

Median class is the first class whose cumulative frequency is greater

or equal  $\frac{n}{2}$

K no of classes

$F_c$  the cumulative frequency before the median class

$f$  the frequency of the class median

c the class width

### 4.8.3 Mode

Mode is the value that has the highest frequency in a data set.

For the data set  $X_1, \dots, X_n$  with  $f_1, \dots, f_n$  the  $X_i$  with the highest frequency is the median.

#### 4.8.3.1 The Mode of Grouped Data

For grouped data, use the following formula to the mode:

$$Mode = L + \frac{d_1}{d_1 + d_2} \times c$$

where

- c The width of the class with the highest frequency( modal class )
- $d_1$  the difference between the frequency of the modal class and the frequency of the class preceding the modal class,
- $d_2$  the difference between the frequency of the modal class and the frequency of the class succeeding the modal class
- L Lower boundary of the modal class

#### 4.8.3.2 The Mode of a Distribution

If the probability density function(pdf) of the distribution of a random variable is given, to calculate the mode of the distribution, get the derivative of the function and equate it to zero. The mode results from the solution of this equation. Figure 4.4 shows the modes of two pdfs.

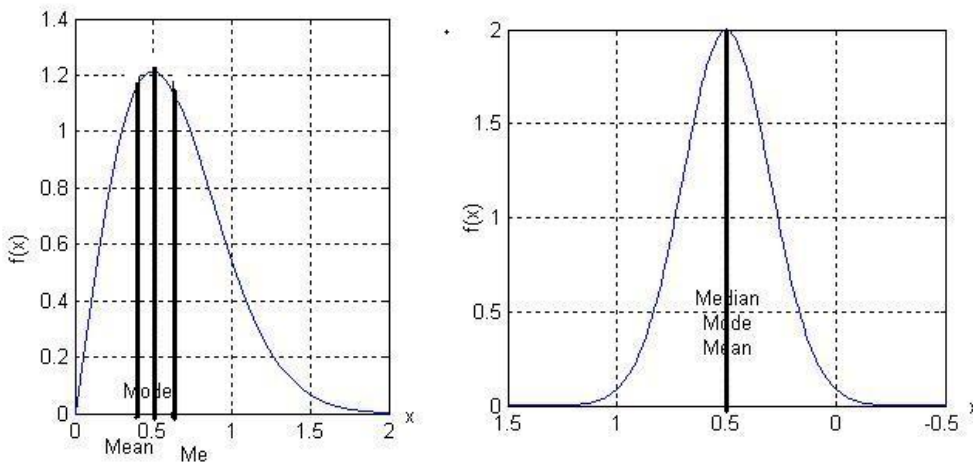


Fig. 4.4 the mode of 2 probability density functions

#### 4.8.4 Geometric Mean

The geometric mean is a measure of central tendency. The geometric mean (G) of the non-negative dataset  $x_1, \dots, x_n$  is defined as:

$$G = \sqrt[n]{x_1 x_2 \dots x_n} \quad \text{or} \quad G = \sqrt[n]{\prod_{i=1}^n x_i} \quad \text{or}$$

$$\log G = \frac{1}{n} \underbrace{\log(x_1 x_2 x_3 \dots x_n)}_{\log x_1 + \log x_2 + \dots + \log x_n} = \frac{1}{n} \sum_{i=1}^n \log x_i$$

If the weights  $w_i, i = 1, \dots, k$  are allocated to  $x_1, \dots, x_k$ , the formula for the geometric mean would be:

$$G = \left( \prod_{i=1}^k x_i^{w_i} \right)^{\frac{1}{\sum w_i}}.$$

A useful application of G as a measure of summary is when changes in the data occur in percentages. If there is any negative or zero values in the data, do not use this mean.

##### Example 4.11

The production of a factory in the last five years has been reported in ratios as follows indicating the production of each year to the production of its preceding year:

1.1, 0.9, 2.3, 1.3, 1.35.

What is the mean of production ratio in the past 5 years.

##### Solution

$$G = \sqrt[5]{x_1 x_2 \dots x_n} = (1.1 * 0.9 * 2.3 * 1.3 * 1.35)^{\frac{1}{5}} \cong 2$$

That is on the average the production in each year of the past 5 years has been doubled.

#### 4.9 Measures of dispersion

Measures of dispersion give us an indication of how broadly our data are spread out from their central tendency. Mostly used measures of statistical dispersion i.e. the variance, standard deviation and range are dealt here.

##### 4.9.1 Range(R)

The range of a sample ( $x_1, \dots, x_n$ ) is the difference between the highest and lowest scores in the data set:  $R = X_{\max} - X_{\min}$ .

### 4.9.2 Variance ( $S^2$ )

The variance of a set of data  $x_1, \dots, x_n$

Is :

$$s^2 = \frac{\sum (x_i - \bar{X})^2}{n-1} = \frac{\sum x_i^2 - 2(n\bar{X})(\bar{X}) + n(\bar{X}^2)}{n-1} = \frac{\sum x_i^2 - n\bar{X}^2}{n-1}$$

$S^2$  is an unbiased estimator of the population variance, however the following value is a biased estimator for the population variance.

$$\sigma_{rms}^2 = \frac{\sum (x_i - \bar{X})^2}{n} = \frac{\sum x_i^2}{n} - \bar{X}^2$$

If the data contains frequencies, then

$$s^2 = \frac{\sum f_i (x_i - \bar{X})^2}{\sum f_i - 1} = \frac{\sum f_i x_i^2 - n\bar{X}^2}{n-1} \quad n = \sum f_i$$

### 4.9.3 Standard deviation

$$s = \sqrt{\frac{\sum f_i (x_i - \bar{X})^2}{\sum f_i - 1}} = \sqrt{\frac{\sum f_i x_i^2 - \bar{X}^2 \sum f_i}{\sum f_i - 1}}$$

Neither  $S$  nor  $\sigma_{rms}$  given by:

$$\sigma_{rms} = \sqrt{\frac{\sum f_i x_i^2}{\sum f_i} - \bar{X}^2} = \sqrt{\frac{\sum f_i (x_i - \bar{X})^2}{\sum f_i}} = \sqrt{\sum p_i (x_i - \bar{X})^2} \quad p_i = \frac{f_i}{\sum f_i}$$

is an unbiased estimator for population standard deviation; but for the standard deviation of a normal distribution the following formulae are useful:

$$\hat{\sigma} = \frac{S}{C_4} \text{ or } \frac{\bar{S}}{C_4} \quad \hat{\sigma} = \frac{R}{d_2} \text{ or } \frac{\bar{R}}{d_2}$$

$C_4$  is read from Table U or calculated from MATLAB:

`n=...; c4=sqrt(2/(n-1)).*gamma(n/2)/gamma((n-1)/2)`

$d_2$  is also read from Table U or calculated in MATLAB from:

`n = ...; pd = makedist('normal',0,1); fun = @(x) (1-(1- cdf(pd,x)).^n- (cdf(pd,x)).^n); d2 = integral(fun,-inf,inf)`

However, as stated earlier, the estimator  $\hat{\sigma} = \frac{\bar{R}}{d_2}$  has a larger mean square error (MSE) than does  $\hat{\sigma} = \frac{\bar{S}}{C_4}$ ; some references strongly

suggest using  $\hat{\sigma} = \frac{\bar{s}}{C_4}$  to estimate the standard deviation of normal populations (Derman & Ross, 1997 page 92).

## APPENDIX A

### Bartlett's Goodness of fit test for exponential distribution

There are several tests concerning probability distributions (goodness of fit). Some of them could be used for all distributions, such as Pearson's chi-squared goodness of fit test. But there are tests developed for a specific distribution. One of the tests of the latter kind is described below.

#### Bartlett's test for exponential distribution

To deal with the following hypotheses

$H_0$ : The distribution is exponential

$H_1$ : The distribution is not exponential

using Bartlett's test do the followings:

Take a random sample of size at least 20 :  $t_1, \dots, t_r$   $r \geq 20$ , where

$t_i$  is the time of the  $i^{\text{th}}$  event; calculate the statistic B given by:

$$B = \frac{2r \left[ \ln \left( \left( \frac{1}{r} \right) \sum_{i=1}^r t_i \right) - \left( \frac{1}{r} \right) \sum_{i=1}^r \ln t_i \right]}{1 + \frac{r+1}{6r}}$$

which has a chi-squared distribution with  $r-1$  degrees of freedom under the null hypothesis. If B is outside  $\left[ \chi_{1-\frac{\alpha}{2}, r-1}^2, \chi_{\frac{\alpha}{2}, r-1}^2 \right]$ , reject  $H_0$ ;  $\alpha$  is the level of significance of the test.

#### Example A-1

The following random sample of size 20 was taken from the life of a kind of lamp.

32.0	6.2	84.9	42.6	99.1	36.3	96.5	31.1	20.9	50.1
10.7	88.6	84.6	11.5	1.8	2.5	4.6	14.2	87.7	30.4

Is the life of the lamps distributed with 0.95 level of confidence?  
If yes, give its mean and its pdf.

**Solution**

$H_0$ : The life distribution is exponential

$H_1$ : The life distribution is not exponential

The statistic is:

$$B = \frac{2r \left[ \ln \left( \left( \frac{1}{r} \right) \sum_{i=1}^r t_i \right) - \left( \frac{1}{r} \right) \sum_{i=1}^r \ln t_i \right]}{1 + \frac{r+1}{6r}}$$

$$\sum_{i=1}^{20} t_i = 50.1 + 20.9 + \dots + 88.6 + 10.7 = 836.3$$

$$\sum_{i=1}^{20} \ln(t_i) = \ln 50.1 + \ln 20.9 + \dots + \ln 88.6 + \ln 10.7 = 63.9385$$

$$B = \frac{2 \times 20 \left[ \ln \left( \frac{836.3}{20} \right) - \frac{63.94}{20} \right]}{1 + \frac{20+1}{6 \times 20}} = 19.34$$

$$\chi^2_{1-\frac{\alpha}{2}, r-1} = \text{chi2inv}(0.05, 19) = 10.1170$$

$$\chi^2_{\frac{\alpha}{2}, r-1} = \text{chi2inv}(0.95, 19) = 30.1415$$

The value of B is inside  $[\chi^2_{1-\frac{\alpha}{2}, r-1} = 10.1, \chi^2_{\frac{\alpha}{2}, r-1} = 30.1]$

Therefore  $H_0$  is not rejected i.e. one of the distributions fitting the lifetime is the exponential with mean  $\hat{\theta} = \frac{\sum_{i=1}^{20} t_i}{20} = \frac{836.3}{20} \cong 41.82$  and the density

$$\text{function } f(t) = \frac{1}{\theta} \times e^{\frac{-t}{\theta}} = \frac{1}{41.82} \times e^{\frac{-t}{41.82}}.$$

## APPENDIX B: Testing Hypothesis concerning exponential distribution mean( $\theta$ )<sup>1</sup>

The development of this hypothesis test is based on the fact that  $n$  items are placed on test and the test is truncated at the  $r^{\text{th}}$  failure ( $r < n$ ); and  $\frac{2r\hat{\theta}}{\theta}$  is distributed as a chi-square with  $2r$  degree of freedom;

Consider the hypotheses

$$H_0: \theta = \theta_0,$$

$$H_1: \theta > \theta_0.$$

Then for a significance level of  $\alpha$ , the probability of accepting  $H_0$  is

$$Pa = \Pr\left(\frac{2r\hat{\theta}}{\theta_0} \leq \chi_{\alpha, 2r}^2 \mid \theta = \theta_0\right) = 1 - \alpha$$

where

$\hat{\theta}$  is MTTF for the product given by (Kapur & Lamberson, 1977 page 252)

$$\hat{\theta} = \frac{\sum_{i=1}^r x_i + (n-r)x_{(r)}}{r}$$

where

$n$  : is the sample size

$x_i$  : time(cycles,...) to  $i^{\text{th}}$  failure  $i=1, 2, \dots, r$

$x_{(r)}$  : the maximum of time (or cycle,...) to failure among the  $r$  failed items ( $r \leq n$ )

The procedure for performing the test is :

Take a random sample of size  $n$ ,

Place the devices in the sample on test without replacement,

When the  $r^{\text{th}}$  failed item ( $r \leq n$ ) happened, stop the test,

Calculate the statistic  $\chi_0^2 = \frac{2r\hat{\theta}}{\theta_0}$

Reject  $H_0$  if (Kapur & Lamberson, 1977 page 264)

$$\chi_0^2 > \chi_{\alpha, 2r}^2.$$

The rejection criteria for other alternative Hypotheses are given in the following table :

---

<sup>1</sup> Extracted from Kapur & Lamberson, 1977 page 263

Testing Hypothesis concerning Exponential Distribution mean( $\theta$ )		
$H_0: \theta = \theta_0$		$\chi_0^2 = \frac{2r\hat{\theta}}{\theta_0}$
$H_1$	Reject $H_0$	Do not reject $H_0$
$\theta \neq \theta_0$	$\chi_0^2 \notin \left[ \chi_{1-\alpha/2, 2r}^2, \chi_{\alpha/2, 2r}^2 \right]$	$\chi_0^2 \in \left[ \chi_{1-\alpha/2, 2r}^2, \chi_{\alpha/2, 2r}^2 \right]$
$\theta < \theta_0$	$\chi_0^2 < \chi_{1-\alpha, 2r}^2$	$\chi_0^2 \in [\chi_{1-\alpha, 2r}^2, \infty)$
$\theta > \theta_0$	$\chi_0^2 > \chi_{\alpha, 2r}^2$	$\chi_0^2 \in [0, \chi_{\alpha, 2r}^2]$

Note that this test could be used for testing the failure rate of an exponential distribution which is the reciprocal of its mean.

#### Example B-1

Consider the data in the following table, which represents an ordered sample of cycles to failure for throttle return springs. 20 springs were tested under conditions similar to those encountered in actual use. The test was truncated at the time of the tenth failure among  $n=20$  springs.

$i$	1	2	3	4	5	6	7	8	9	10
Cycles to failure ( $\mathcal{X}_{(i)}$ ): Ordered	190,437	245,593	277,761	432,298	530,100	626,300	1,043,307	1,055,528	1,221,393	2,099,199

Suppose the cycle numbers are distributed exponentially and we are interested in testing whether the mean of this distribution is greater than 1million cycles. Estimate the mean cycle to failure and perform the appropriate test with  $\alpha = 0.05$ .

#### Answer

$$H_0: \theta \leq 10^6 \text{ cycles}$$

$$H_1: \theta > 10^6 \text{ cycles}$$



$$n=20, r=10$$

$$x_{(r)} = \max(X(1), \dots, X(r)) = 2,099,199$$

$$\chi_0^2 = \frac{2r\hat{\theta}}{\theta_0}$$

$$\hat{\theta} = \frac{\sum_{i=1}^r x_{(i)} + (n-r)x_{(r)}}{r}$$

$$\hat{\theta} = 2871391$$

If  $\chi_0^2 > \chi_{\alpha, 2r}^2$   $H_0$  is rejected

$$\chi_0^2 = \frac{(2)(10)(2871391)}{10^6} = 57.4 > \chi_{0.05, 20}^2 = 31.41$$

Then  $H_0$  is strongly rejected and we conclude  $\theta > 1000,000$  cycles.

## Exercises

### 4.1

If the probability of observing a defective in a sample is 0.025, what is the probability of observing 15 defectives in a sample of size fifty?

- Using binomial distribution
- Using Poisson Approximation
- Using normal approximation

### 4.2

A random sample of size 380 was taken from a production process with 3% defective proportion. What is the probability of observing 15 defective products in the sample using

- a) normal approximation
- b) Poisson approximation

Which approximation is better? Why?

### 4.3

A batch contains 4 defective and 56 non-defective products. A sample of size 5 is selected from this batch, what is the probability of observing 3, 2 and 1 defective(s) in the sample? What is the probability of observing less than 2 defectives in the sample?

### 4.4

The following samples were randomly taken from a population; using Q-Q plot determine if an exponential distribution fits the data?

0.9106	2.7064	0.2603	0.3373	3.1250	13.8574	4.7871	1.5136
4.8462	0.6998	0.2442	5.2073	0.2001	2.6728	8.5476	0.6205
5.9222	4.8475	3.9038	12.5448				

**If youth but knew, If old age but could,  
Si jeunesse savait, Si vieillesse pouvait,  
(French Proverb)**



# Chapter 5

## Two Attribute Control Charts: p Chart & np Chart

## Chapter 5 p & np Control Charts

### Aims

In this chapter, 2 control charts used for attributes i.e. p chart and np chart are described. p-chart is the chart for fraction rejected as non-conforming to specifications. np-chart is the chart for the number of defective items in the process. While both of these 2 charts could be used for monitoring fraction defectives, np chart is used when the sample sizes are the same.

### 5.1 Introduction

One the limitation of the control charts introduced earlier is that they could monitor variables (measurable quality characteristics) while many quality characteristics can be observed only as attributes. *Attribute characteristics* are either “good” or “bad.” In other words, they either conform to the prescribed specification or not. This chapter and the next deal with control charts for attributes, most common of them are:

1-p-chart 2-np-chart 3 C-chart 4-u-chart D-chart

This chapter is concerned with the first 2 control charts which deal with monitoring defectives in a production process and the next chapter with the other 3 charts which deal with number of defects in the product. Before dealing with these charts let us define the proportion defective in a sample. The proportion of defectives in a sample (p) is:

$p =$

The number of nonconforming products (defectives) observed in the sample divided by the sample size,

The percent defectives in a sample =  $100 \times p$ .

For a sample of size n drawn from a population or process with fraction defectives p', the mean and standard deviation of p would be:

$$\sigma_p = \sqrt{\frac{p'(1-p')}{n}}, \quad E(p) = p'$$

### 5.2 p control chart

The most common control chart for attributes is the p-chart. p is the sample the fraction defectives of sample of size, say, n drawn from the process. A simple rule used to determine the sample size for plotting control charts used for proportion defective (p and np charts) is :

Start testing the products individually as they are produced.

Do as much as observations until the first defective product is produced;

The no. of observations until reaching the first defective product, is selected as sample size  $n$ .

### 5.2.1 Steps for constructing a p-chart

–several random samples of size  $n_i, i = 1, \dots, k$  are taken from the process,

–the number of nonconforming products in each subgroup(sample) is determined( $r_i$ )

–  $p_i$ , proportion of nonconforming items in the  $i^{\text{th}}$  subgroup is calculated:

$$p_i = \frac{r_i}{n_i} \text{ and}$$

–a table like the one below is prepared:

Subgroup No. (i)	$n_i$	$r_i$	$p_i$
1	100	2	0.02
2	800	4	0.005
$\vdots$	$\vdots$	$\vdots$	$\vdots$
K	.	.	.

### p chart limits –standard( $p'_0$ ) known

If  $p'_0$ , the standard or aimed-at proportion defective for the process is given, calculate 3-sigma control limits as follows, as it is conventional in Dr. Shewhart's control chart to calculate 3-sigma limits unless otherwise stated:

$$UCL_p = E(p) + 3\sigma_p = p'_0 + \frac{3\sqrt{p'_0(1-p'_0)}}{\sqrt{n}},$$

$$CL = E(p) = p'_0,$$

$$LCL_p = E(p) - 3\sigma_p = p'_0 - \frac{3\sqrt{p'_0(1-p'_0)}}{\sqrt{n}}.$$

If the value obtained for  $LCL_p$  is negative, set  $LCL_p = 0$ .

### p chart limits –standard( $p_0'$ )unknown

If the standard or aimed-at proportion defective for the process is not given, the mean of proportion defectives of the samples ( $\bar{p}$ ) is substituted for  $p_0'$  in the above limits:

$$UCL = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = \bar{p} + \frac{3}{\sqrt{n}}\sqrt{\bar{p}(1-\bar{p})}$$

$$CL = \bar{p}$$

$$LCL = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = \bar{p} - \frac{3}{\sqrt{n}}\sqrt{\bar{p}(1-\bar{p})}$$

where

Total number of defectives observed in all  
subgroups

$$\bar{p} = \frac{\sum r}{n} = \frac{\text{Total number of defectives observed in all subgroups}}{\text{Total number inspected}}$$

$$\bar{p} = \frac{\sum n_i p_i}{\sum n_i} = \frac{\sum r_i}{\sum n_i};$$

If the sample sizes of the k given subgroups are equal, then:

$$n_i = n \quad \bar{p} = \frac{\sum np_i}{\sum n} = \frac{n \sum p_i}{kn} = \frac{\sum p_i}{k}$$

when the sample size is not constant, the control limits would not be straight lines, but all comments mentioned regarding in-control and out-of-control conditions in the discussion of the previous Shewhart's control charts are valid here.

#### Example 5.1

Using the following table(IshiKawa, 1983),construct a suitable control chart .

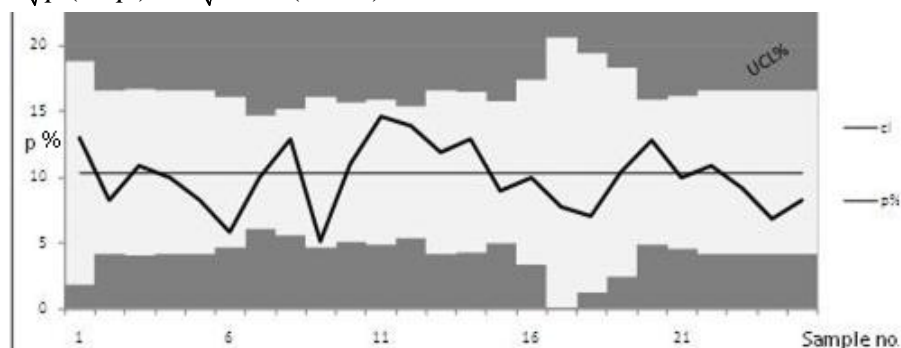
Subgroup No.	Sample size(n)	Number of defectives (np)	Proportion defective p(%)	UCL= $\bar{p} + 3\sigma_p$ (%)	LCL= $\bar{p} - 3\sigma_p$ (%)
1	115	15	13.0	18.8	1.8
2	220	18	8.2	16.5	4.1
3	210	23	10.9	16.6	4.0
4	220	2	10.0	16.5	4.1
5	220	18	8.2	16.5	4.1
6	255	15	5.8	16.0	4.6
7	440	44	10.0	14.6	6.0
8	365	47	12.9	15.1	5.5

Subgroup No.	Sample size(n)	Number of defectives (np)	Proportion defective p(%)	UCL= $\bar{p} + 3\sigma_p$ (%)	LCL= $\bar{p} - 3\sigma_p$ (%)
9	255	13	5.1	16.0	5.0
10	300	33	11.0	15.6	5.0
11	280	42	14.6	15.8	4.8
12	330	46	13.9	15.3	5.3
13	320	38	11.9	16.5	4.1
14	225	29	12.9	16.4	4.2
15	290	26	8.9	15.7	4.9
16	170	17	10.0	17.3	3.3
17	65	5	7.7	21.6	0
18	100	7	7.0	19.4	1.2
19	135	14	10.4	18.2	2.4
20	280	36	12.8	15.8	0.8
21	250	25	10.0	16.1	4.5
22	220	24	10.9	16.5	4.1
23	220	20	9.1	16.5	4.1
24	220	15	6.8	16.5	4.1
25	220	18	8.2	16.5	4.1
Sum	5925	610			

**Solution**

$$\text{Central line : } \bar{p} = \frac{\sum np}{\sum n} = \frac{610}{5925} = 0.103 (=10.3\%)$$

$$3\sqrt{\bar{p}(1-\bar{p})} = 3\sqrt{0.103(0.897)} = 0.912$$

**Fig. 5-1 p- chart for Example 5.1<sup>1</sup>**

$$\text{Upper Limit : } UCL = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = \bar{p} + \frac{3}{\sqrt{n}}\sqrt{\bar{p}(1-\bar{p})} = 0.103 + \frac{0.912}{\sqrt{n}}$$

$$\text{Lower limit : } LCL = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = 0.103 - \frac{0.912}{\sqrt{n}}$$

<sup>1</sup> With thanks to Mrs Hosna Khazad, former student of Dept. Of Industrial Eng. of the college of Engineering of Shahid Bahonar University of Kerman, Iran



The limits for each subgroup depends on the subgroup size. Therefore the upper and lower limits of the chart are not straight lines. The calculated limits and p for each subgroup are in the above table and chart is shown in Fig. 5-1.

As it is evident from Fig. 5-1, since the sample sizes are not equal, the control limits are not be straight lines and since the points have fallen within the control limits and are scattered randomly and no special pattern is observed it is concluded the proportion defectives in the process is in-control.

### Example 5.2

Calculate the 3-sigma limits of p-chart for the data given in the following table.

#### Solution

The control limits of each subgroup is calculated as follows:

$$UCL_p = \bar{p} + 3\sigma_p$$

$$LCL_p = \bar{p} - 3\sigma_p$$

$$\bar{p} = \frac{\sum np}{\sum n} = \frac{36}{720} = 0.05$$

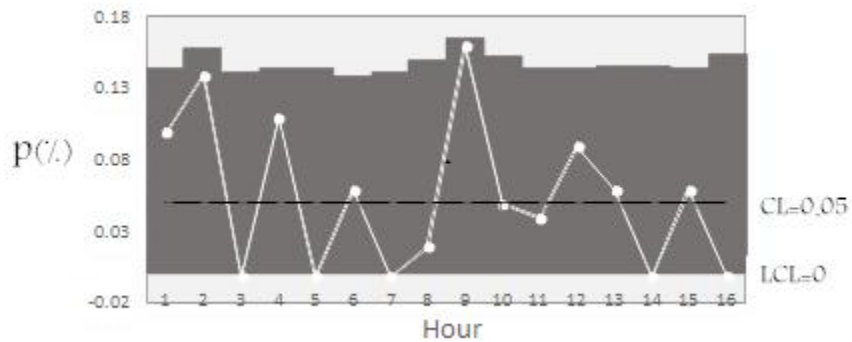
$$3\sigma_p = \frac{3\sqrt{\bar{p}(1-\bar{p})}}{\sqrt{n}}$$

$$3\sqrt{\bar{p}(1-\bar{p})} = 3\sqrt{0.05 \times 0.95} = 3 \times \sqrt{0.0475} = 0.65$$

The limits and p for each subgroup based on the above formulae are shown in the following table.

Hour	Sample size(n)	No of defectives (np)	$\frac{1}{\sqrt{n}}$	p	$3\sigma_p =$ $3\sqrt{\bar{p}(1-\bar{p})}$	$UCL_p =$ $\bar{p} + 3\sigma_p$	$LCL_p =$ $\bar{p} - 3\sigma_p$	
1	48	5	0.144	0.10	0.094	0.1438	0	
2	36	5	0.17	0.14	0.11	0.1583	0	
3	50	0	0.14	0	0.09	0.1419	0	
4	47	5	0.146	0.11	0.095	0.1448	0	
5	48	0	0.144	0	0.094	0.1438	0	
6	54	3	0.136	0.06	0.088	0.1385	0	
7	50	0	0.14	0	0.09	0.1419	0	
8	42	1	0.154	0.02	0.1	0.1503	0	
9	32	5	0.177	0.16	0.12	0.1649	0	
10	40	2	0.158	0.05	0.42	0.1528	0	
11	47	2	0.146	0.04	0.095	0.1448	0	
12	47	4	0.146	0.09	0.095	0.1448	0	
13	46	1	0.146	0.06	0.096	0.1458	0	
14	46	0	0.147	0	0.096	0.1458	0	
15	48	3	0.144	0.06	0.094	0.1438	0	
16	39	0	0.16	0	0.104	0.1541	0	
sum	720	$\sum np = 36$	$\bar{p} = \frac{\sum np}{\sum n} = \frac{36}{720} = 0.05$					$3\sqrt{\bar{p}(1-\bar{p})} = 0.65$

The following figure shows the corresponding  $p$ -chart.



**Fig. 5-2  $p$ - chart for Example 5.2<sup>1</sup>**

**End of Example 5.2.**

#### **The results and decisions based on the $p$ -chart**

After constructing a control chart for monitoring the proportion defectives of a process, some results could be obtained including the following:

1)

Detecting the subgroups falling outside the control limit; searching for and resolving the causes if possible. The causes may be attributable to such factors as a specific machine, worker or group of workers, a new batch of raw materials, a new supplier, a new method, to name a few possibilities.

2)

Deciding on the status of the proportion defectives of the process; whether it is in-control or out-of-control.

3)

Estimating the percent defectives of the process. If there are no sign of out control in the  $p$ -chart,  $\bar{p}$  is the best estimate of mean number conformities per product in the process(extracted from Besterfield ,1990, page 185)

4)

Decision on the control limits for future use.

The following g example illustrates the  $p$ -chart and the decisions.

<sup>1</sup> With thanks to Mr. Ali Maghsoudi, student of Dept. Of Industrial Eng. of the college of Engineering of Shahid Bahonar University of Kerman, Iran

**Example 5.3** (An extensive example ((Example 4-3 Grant and Leavenworth,1988, page 245)  
**ILLUSTRATION OF CALCULATIONS NECESSARY FOR CONTROL CHART  
 FOR FRACTION DEFECTIVE**

**Facts of the Case**

This example applies to a 4-month record of daily 100% inspection of a single critical quality characteristic of a part for an electrical device. It is intended chiefly as an illustration of how control limits are calculated with variable production, and of the setting and revision of standard values for fraction rejected.

When, after a change in design, the production of this part "was started early in June, the daily fraction defective was computed and plotted on a chart. At the end of the month, the average fraction defective  $\bar{p}$  was computed. Trial control limits were computed for each point. A standard value of fraction defective  $p'$  was then established to apply to future production. During July new control limits were computed and plotted daily based on the number of parts  $n$  inspected during the day. A single set of control limits was established for August, based on the estimated average daily reduction. At the end of August, a revised  $p'$  was computed to apply to September, and the control chart was continued during September with this revised value.

**Calculation of trial control limits**

Table 5-1 shows the number inspected and the number rejected as defective each day during June. The fraction defective each day is the number of parts rejected divided by the number inspected that day. For example, for June 6,  $p = \frac{31}{3350} = 0.0093 = 0.93\%$ .

At the end of the month, the average fraction defective  $\bar{p}$  is computed. It should be emphasized that the correct way to calculate  $\bar{p}$  is to divide the total number rejected in the period by the total number of parts inspected during that period. Whenever the subgroup size (in this case, the daily number inspected) is not constant, it is incorrect to calculate the simple average of  $p_i$  values. The standard deviation of the fraction rejected was calculated based on this value of  $\bar{p} = 0.0145$ . Note that the value of  $3\sqrt{\bar{p}(1-\bar{p})}$  was calculated only once to apply to all calculations of the control limits. Thus for June 6,  $\pm 3\sigma_p = \pm \frac{0.3586}{\sqrt{3350}} = \pm 0.0062$ .

The daily values of  $p_i$  and the control limits for each day are shown in Fig. 5-3.1. In this figure, per cent rejected (100p) rather than fraction rejected, has been plotted.

**Table 5.1**  
**Calculation of the trial control limits of p chart for June.**  
 Data on a single quality characteristic of a part of an electrical device

Date	$n_i$	Number rejected (r)	$p_i$	$3\sigma_p = \left(\frac{1}{\sqrt{n_i}}\right) \times 3\sqrt{\bar{p}(1-\bar{p})}$	$UCL_p = \bar{p} + 3\sigma_p$	$LCL_p = \bar{p} - 3\sigma_p$
June 6	335	31	0.0093	0.0062	0.0207	0.0083
7	335	113	0.0337	0.0062	0.0207	0.0083
8	1509	28	0.0186	0.0092	0.0237	0.0053
9	219	20	0.0091	0.0077	0.0222	0.0068
11	2678	35	0.0131	0.0069	0.0214	0.0076
12	3252	68	0.0209	0.0063	0.0208	0.0082
13	464	339	0.0730	0.0053	0.0198	0.0092
14	378	12	0.0032	0.0058	0.0203	0.0087
15	299	3	0.0010	0.0066	0.0211	0.0079
16	3382	17	0.0050	0.0062	0.0207	0.0083
18	3694	14	0.0038	0.0059	0.0204	0.0086
19	3052	8	0.0026	0.0065	0.0210	0.0080
20	3477	27	0.0078	0.0061	0.0206	0.0084
21	405	44	0.0109	0.0056	0.0201	0.0089
22	3042	70	0.0230	0.0065	0.0210	0.0080
23	1623	12	0.0074	0.0089	0.0234	0.0056
25	915	9	0.0098	0.0119	0.0264	0.0026
26	1644	1	0.0006	0.0088	0.0233	0.0057
27	1572	22	0.0140	0.0090	0.0235	0.0055
28	196	3	0.0015	0.0081	0.0226	0.0064
29	2440	3	0.0012	0.0073	0.0218	0.0072
30	2086	1	0.0005	0.0079	0.0224	0.0066
Totals	60680	880				

$$\bar{p} = \frac{\text{total number rejected}}{\text{total number inspected}} = \frac{\sum r_i}{\sum n_i} = \frac{880}{60688} = 0.0145$$

$$3\sqrt{\bar{p}(1-\bar{p})} = 3\sqrt{0.0145(1-0.0145)} = 0.3586$$

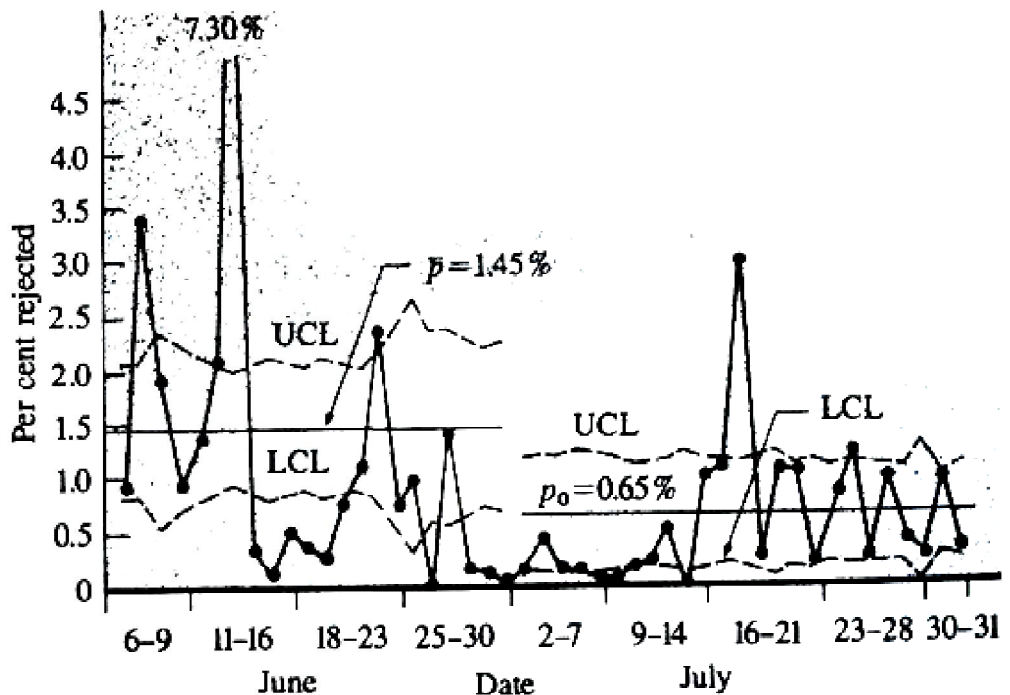
Because per cent defective is more readily understood by both shop and administrative personnel, it is usually desirable for fraction rejected to be converted to per cent rejected for all plotting.

#### Determination of aimed-at value of fraction rejected $p'_0$

If all the points fall within the trial control limits, the aimed-at value  $p'_0$  may be assumed to be equal to  $\bar{p}$ .

Here many points fell outside the trial control limits. In such cases, the decision as to the value of  $p'_0$  to be used calls for judgment as to what process average fraction rejected can be maintained in the future, provided the occasional assignable causes of bad quality can be eliminated. An aid to such judgment may be obtained by computing a revised value of

$\bar{p}$ , eliminating the days on which  $p$  fell above the upper control limit. With these days, June 7, 12, 13, and 22, eliminated, the remaining number of rejects is 290 and the remaining number inspected is 46,399. The revised  $\bar{p} = 290/46399 = 0.0063$ . After consideration of this and of the previous record on similar parts of slightly different design, it was decided to assume  $p'_0 = 0.0065$ .



**Fig 5.3.1 p-control chart for the months of June and July**  
( after Grant & Leavenworth, 1988 )

#### Calculation of control Limits based on standard Fraction rejected, $p'_0$

Table 5-2 gives the daily numbers inspected and rejected during July and shows the calculation of control limits based on the aimed-at fraction rejected.

This calculation appears to be almost identical with that shown in Table 5-1. The value of  $p'_0$  is used in the calculation of limits in Table 5-2 wherever  $\bar{p}$  was used in Table 5-1. The practical difference is that where  $\bar{p}$  is used, no control limits can be computed until  $\bar{p}$  is known, i.e. not until the end of the period. Where a standard or aimed at value  $p'_0$  is established in advance, the limits can be computed each day and drawn on the control chart as the day's point is plotted. In this way, the control chart provides a

basis for immediate action whenever a point goes outside the control limits.

<b>Table 5-2 Computation of daily control limits based on <math>p'_0</math> for July</b>						
Data on a single quality characteristic of a part of an electrical device						
Date	$n_i$	Number rejected (r)	$p_i$	$3\sigma_p = 3\sqrt{\frac{p'_0(1-p'_0)}{n_i}}$	$UCL_p$	$LCL_p$
July 2	2228	4	0.0018	0.0051	0.0116	0.0014
3	2087	9	0.0043	0.0053	0.0118	0.0012
5	2088	3	0.0014	0.0053	0.0118	0.0012
6	1746	2	0.0011	0.0058	0.0123	0.0007
7	2076	1	0.0005	0.0053	0.0118	0.0012
9	2164	1	0.0005	0.0052	0.0117	0.0013
10	2855	5	0.0018	0.0045	0.0110	0.0020
11	2560	5	0.0020	0.0048	0.0113	0.0017
12	2545	14	0.0055	0.0048	0.0113	0.0017
13	1874	1	0.0005	0.0056	0.0121	0.0009
14	2329	24	0.0103	0.0050	0.0115	0.0015
16	2744	30	0.0109	0.0046	0.0111	0.0019
17	2619	77	0.0294	0.0047	0.0112	0.0018
18	2211	5	0.0023	0.0051	0.0116	0.0014
19	1746	19	0.0109	0.0058	0.0123	0.0007
20	2628	28	0.0107	0.0047	0.0112	0.0018
21	2366	5	0.0021	0.0050	0.0115	0.0015
23	2954	23	0.0078	0.0044	0.0109	0.0021
24	2586	32	0.0124	0.0047	0.0112	0.0018
25	2790	8	0.0029	0.0046	0.0111	0.0019
26	2968	30	0.0101	0.0044	0.0109	0.0021
27	3100	13	0.0042	0.0043	0.0108	0.0022
28	1359	4	0.0029	0.0065	0.0130	0.0000
30	3940	39	0.0099	0.0038	0.0103	0.0027
31	3138	11	0.0035	0.0043	0.0108	0.0022
Total	61701	393				

#### Establishment of Control Limits Based on Expected Average Subgroup Size.

Although the correct position of 3-sigma control limits on a p chart depends on subgroup size (in this case, a subgroup is the number of parts inspected each day), the calculation of new limits for each new subgroup consumes some time and effort. Where the variation in subgroup size is not too great (for example, where the maximum and minimum subgroups are not more than 25% away from the average) it often may be good enough for practical purposes to establish a single set of control limits based on the expected average subgroup size. In this way, limits may be established at the start of a period (for instance, a month) and projected ahead for the entire period.

**Table 5-3 Record of daily fraction rejected with control limits computed on daily production and aimed-at value of  $p'_0$** 

Date	$n_i$	Number rejected (r)	$p_i$	Date	$n_i$	Number rejected (r)	$p_i$
+ 1 Aug	3068	6	0.0020	++1 Sept	2539	3	0.0012
2	776	3	0.0039	3	2425	16	0.0066
3	2086	16	0.0077	4	1537	9	0.0059
4	3652	10	0.0027	5	2852	17	0.0060
6	2606	3	0.0012	6	2953	16	0.0054
7	2159	21	0.0097	7	2649	5	0.0019
8	2745	27	0.0098	8	2835	4	0.0014
9	2606	3	0.0012	10	2752	6	0.0022
10	2159	21	0.0097	11	892	1	0.0011
11	2745	22	0.0080	12	3186	7	0.0022
13	3114	30	0.0080	13	2646	5	0.0019
14	1768	18	0.0102	14	2714	4	0.0015
15	3208	29	0.0090	15	2878	5	0.0017
16	2629	2	0.0008	17	2384	6	0.0025
17	3576	9	0.0025	18	2639	5	0.0019
18	2262	15	0.0066	19	3160	7	0.0022
20	3294	5	0.0015	20	1895	11	0.0058
21	3026	5	0.0017	21	4287	13	0.0030
22	2713	10	0.0037	22	2917	3	0.0010
23	2687	24	0.0089	24	2479	1	0.0004
24	3824	23	0.0060	25	1991	2	0.0010
25	3265	12	0.0037	26	3280	10	0.0030
27	1205	14	0.0116	27	2195	15	0.0068
28	3035	7	0.0023	28	2570	3	0.0012
29	2793	6	0.0021	29	3323	3	0.0009
30	3295	14	0.0042				
31	3227	18	0.0056				
Total	73523	373		Total	65978	177	
+For August: Estimated average daily production =2600 Aimed-at fraction rejected $p'_0 = .0065$ $3\sigma_p = \sqrt{\frac{p'_0(1-p'_0)}{n}} = \sqrt{\frac{0.0065 \times (1-0.0065)}{2600}} = 0.0047$ $UCL_p = p'_0 + 3\sigma_p = 0.0065 + 0.0047 = 0.0112$ $LCL_p = p'_0 - 3\sigma_p = 0.0065 - 0.0047 = 0.0018$				Average daily production for August $\frac{73523}{23} = 2723$ For August $\bar{p} = \frac{373}{73523} = .0051$ ++For September Estimated average daily production =2700 Aimed- at fraction rejected $p'_0 = .0051$ $3\sigma_p = 0.0041$ $UCL_p = 0.0051 + 0.0041 = 0.0092$ $LCL_p = 0.0051 - 0.0041 = 0.0010$			

At the end of July the situation was reviewed to consider the possibility of doing this. It was decided that daily output was well enough stabilized to justify the use of a single set of control limits during August. Average daily production during July had been  $61701/25 = 2468$ . The estimated average daily output during Month of August was 2600; this was assumed as the value of  $n$  for calculation of control limits. As  $\bar{p}$  during July had been

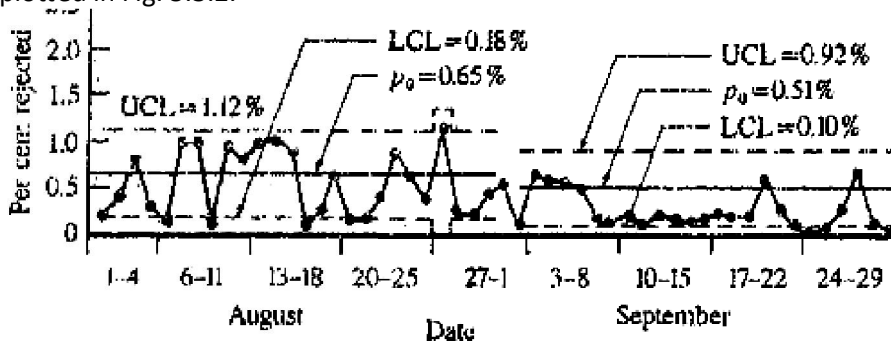
$393/61701 = 0.0064$ , no change was made in the aim-at  $p'_0$  of 0.0065. The calculations for the control limits for August are shown in Table 5-3.

Whenever control limits are set in this way on an expected average value of  $n$ , any points on the control chart that are either outside the limits or just inside the limits require more critical examination to see whether the limits as drawn really apply to these points. Whenever the subgroup size is larger than the assumed average value of  $n$ , the true limits are inside those drawn. Whenever the subgroup size is smaller, the true limits are outside.

Such a calculation was made for August 27, when  $p$  was 0.0116. This is above the upper control limit of 0.0112 which was computed for the assumed daily production of 2,600. A revised upper control limit for this day based on the actual production of 1205 is 0.0134. The revised limits for this day are indicated on the control chart (Fig. 5.3.2) ; they show that the point was actually not out of control.

#### Further Revision of $p'_0$

During August, the average fraction rejected  $\bar{p}$  was  $\frac{373}{73523} = 0.0051$ . No points fell above the upper control limit. This value, 0.0051, was therefore assumed as  $p'_0$  to apply to September. Control limits for September were based on an estimated average subgroup size of 2700. (Daily production during August had been  $\frac{73523}{27} = 2723$ . Daily values for September with calculated control limits are shown in Table 5-3 and plotted in Fig. 5.3.2.



**Fig 5.3.2 p-chart for Month of August and September**  
( after Grant & Leavenwoth, 1988)

The process quality during September improved even more. Although only two points fell below the lower control limit during the month, confirmation of the existence of a new better level of quality was given by an extreme run for eleven points -from September 7 to 19 — below the central line. For the month, the process average  $\bar{p}$  was  $177/65978 =$



0.0027. This justified a further downward revision of  $p'_0$  to 0.0027 for October. The data and the control chart for October are not shown here.

### End of Example 5.3.

#### Aimed –at process average and necessity for its revision

When plotting daily of weekly p-chart, it has to be noticed whether the calculated  $\bar{p}$  is appropriate as the central line. If all the points fell inside the chart, and no special pattern has been formed, the aimed-at process average( $p'_0$ ) is the calculated  $\bar{p}$ . If, for example, most of the points fall below the central line or above it, a revision has to be made on  $p'_0$ . In Fig 5-3.1 most of the points fell below the central line, and some outside the upper limit. Discarding the points that fall outside and have assignable cause, new  $\bar{p}$  is calculated and a new  $p'_0$  is set.

#### Tests based on the theory of runs

Notice that the tests based on the theory of run, described when discussing control charts for variables, are valid for all Dr Shewhart's control chart.

### 5.3 np Control Chart

np control chart is another control chart used to monitor non-conforming products. This chart is an alternative to p-chart, used whenever the sample sizes are equal. When sample sizes are unequal, p-control chart is used for monitoring rejects (non-conforming products); however when the sample sizes are equal, either of the 2 charts could be used.

If r is the number of non-conforming products in a sample of size n drawn from a population or process with fraction rejected  $p'$  and the fraction rejected in the sample is p, then

$$r=np \text{ and } E(r) = np', \text{ Var}(r) = np'(1-p').$$

The upper and lower control limits for np control chart is given by:

$$UCL_{np} = np' + 3\sqrt{np'(1-p')}$$

$$\text{Central line} = np'$$

$$LCL_{np} = np' - 3\sqrt{np'(1-p')}$$

Since a negative number of rejects is not possible, if the calculated value for the lower control limit is negative, the limit is set at 0.00.

If  $p'$ , the standard or aimed-at fraction rejected for the process, is not given  $\bar{p}$  is substituted for  $p'$ .

All comments mentioned earlier on in-control and out -of -control conditions of control charts are valid here. When no sign of out-of control

is observed on the chart, the process mean of nonconformities per unit product is under control and  $\bar{p}$  is a good estimate for the mean. When the upper and lower limits are calculated based on the standard value  $p'$  and there is state of in control, the process mean of nonconformities per unit product coincides  $p'$ .

#### Example 5.4<sup>1</sup>

Table 5-4 shows the number of defectives in 30 subgroups of size 100 drawn from a certain electroplating process. Construct a suitable control chart. Is the process operating in control or out of control?

Table 5-4 Number of defectives in 30 samples drawn from an electroplating process								
Sub-group No. (i)	Sample Size (n <sub>i</sub> )	Number of defectives	Sub-group No. (i)	Sample Size (n <sub>i</sub> )	Number of defectives	Sub-group No. (i)	Sample Size (n <sub>i</sub> )	Number of defectives
1	100	1	11	100	2	21	100	12
2	100	6	12	100	1	22	100	6
3	100	5	13	100	3	23	100	3
4	100	5	14	100	1	24	100	4
5	100	4	15	100	4	25	100	3
6	100	3	16	100	5	26	100	3
7	100	2	17	100	4	27	100	2
8	100	2	18	100	1	28	100	5
9	100	4	19	100	6	29	100	7
10	100	6	20	100	15	30	100	4
						sum	3000	129
						mean	100	4.3

#### Solution

The limits for the np control chart:

$$UCL_{np} = n\bar{p} + 3\sqrt{n\bar{p}(1-\bar{p})} = n\bar{p} + 3\sqrt{n\bar{p}} \sqrt{1-\bar{p}} = 4.30 + (6.22)(.98) = 4.3 + 6.09 = 10.39$$

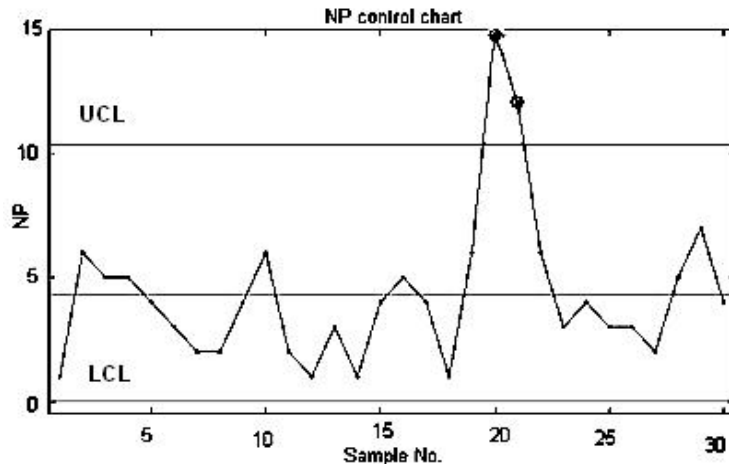
$$LCL_{np} = n\bar{p} - 3\sqrt{n\bar{p}(1-\bar{p})} = 4.3 - 6.09 < 0 \Rightarrow LCL_{np} = 0$$

$$CL = n\bar{p} = 100 \times \frac{129}{3000} = 4.30$$

In MATLAB the following instructions result in Fig. 5. 4 for this example;

<sup>1</sup>Extracted from Persian translation of Ishikawa( 1983 )with minor changes.

```
DATA=[1 6 5 5 4 3 2 2 4 6 2 1 3 1 4 5
4 1 6 15 12 6 3 4 3 3 2 5 7 4]; A=DATA;
st = controlchart(A,'charttype',{'np'},'unit', 100);fprintf('Parameter
estimates: mu = %g, sigma = %g\n',st.mu,st.sigma); xlabel('Sample No.');
```



**Fig. 5.5 np control chart for the data of Table 5-4 using MATLAB**

According to Fig. 5-4 which shows the control chart, the process percent of nonconforming items is out of control because some subgroups have fallen outside the chart.

## Exercises

**5.1** (P.7-1 Grant & Leavenworth, 1988 page 267)

An electronic company manufactures several types of cathode ray tubes on a mass production basis. During the past month, Type A has caused considerable difficulty. The following table contains data from 21 days of this troubling period. Compute the central line and 3-sigma control limits for a p-chart for this tube process. 100 units are inspected daily.

Day	fraction rejected (p)	Day	fraction rejected (p)
1	0.22	12	0.46
2	0.33	13	0.31
3	0.24	14	0.24
4	0.20	15	0.22
5	0.18	16	0.22
6	0.24	17	0.29
7	0.24	18	0.31
8	0.29	19	0.21
9	0.18	20	0.26
10	0.27	21	0.24
11	0.31		

Answer 0.392, 0.260, 0.128

## 5.2

To compute  $\bar{p}$ , it is not always correct to calculate the regular (unweighted) mean of p's. To show this, calculate the right value of  $\bar{p}$  for the table given below. Compare this with the regular  $\bar{p}$ . Why are they different? Why unweighted index is not a suitable index for proportion rejected of the population?

Batch No.	Number of inspected	Number out of specifications	P
1	1200	18	0.0150
2	750	40	0.053
3	1150	26	0.023
4	75	15	0.200
5	225	23	0.102
Sum	2400		

**5.4** ( 7-4 Grant & Leavenworth, 1988 page 268)

A manufacturer purchases small bolts in cartons that usually contain several thousand bolts. Each shipment consists of a number of cartons. As part of the acceptance procedure for these bolts, 400 bolts are selected at random from each carton and are subjected to visual inspection for certain non-conformities. In a shipment of 10 cartons, the respective percentages of rejected bolts in the samples from the 10 cartons are 0, 0, 0.5, 0.75, 0.2, 0.25, 0, 0.25, and 1.25. Does this shipment of bolts appear to exhibit statistical control with respect to the quality characteristics examined in this inspection?

Hint. Plot p-chart ( $LCL_p=0$ ,  $UCL_p=0.016$ )

**5.5** ( 7.5 Grant & Leavenworth, 1988 page 268)

An item is made in lots of 200 each. The lots are given 100% inspection. The record sheet for the first 25 lots inspected showed that a total of 75 items did not conform to specifications.

a)

Determine the trial limits for an np chart.

b)

Assume that all points fall within the control limits. What is your estimate of the process average fraction nonconforming  $\mu_p$ ?

c)

If this remains unchanged, what is the probability that the 26th lot will contain exactly 7 nonconforming units? That it will contain 7 or more nonconforming units?

Answers:

(a)  $UCL_{np}=8.157$   $LCL_{np}=0$  (b) 0.015 (c) 0.022, 0.034

**5.6** ( P.7-6 Grant & Leavenworth, 1988 page 268)

Daily inspection records are maintained on the production of a special-design electronic device. 100 items have been inspected each day for the past 21 days. A total of 546 items failed during a severe heat stress test.

The 4 highest and lowest values of p are:

Highest	Lowest
0.46	0.18
0.33	0.18
0.31	0.20
0.31	0.21

a) compute the central line and 3-sigma trial control limits for a p chart. Is the process operating in control?

Answer: 0.392 0.260 0.128

b) Recommend an aimed-at value for the mean and 3-sigma limits for continued use of the p chart.

Answer: 0.380 0.250 0.120

### 5.7

A production process has 1% as nonconformity proportion. Several samples of size 200 have been drawn from this process.

a) What is the expected number defective product in each sample?

b) Calculate the upper control limit for an np chart such that the probability of falling a sample above this limit is 5% .

### 5.8( P.7.15 Grant & Leavenworth, 1988 page 270)

A manufacturer wishes to maintain a process average of 1% nonconforming product or less. 1500 units are produced per day, and 2 days' runs are combined to form a shipping lot. It is decided to sample 250 units each day and use an np chart to control production.

(a) Find the 3-sigma control limits for this process.

(b) Assume that the process shifts from 1 to 4% nonconforming product. Find the probability that the shift will be detected as the result of the first day's sampling after the shift occurs.

### 5.9

Three samples of size 100, 150 and 1800 have been extracted from a process. What are the upper and lower control limits for a p-chart with aimed-at  $p'_0 = 2\%$ .

Answer:

Sample No.	$UCL_p = p'_0 + 3\sigma_p$	$LCL_p = p'_0 - 3\sigma_p$
1	0.062	0
2	0.054	0
3	0.03	0.01

**Learn truthfulness before speaking**



# Chapter 6

**Control Charts  
for Defects:**

**c , u & D  
charts**



## Chapter 6 C , u & D Control Charts

### Aims

This chapter deals with a set of attribute control charts for monitoring non-conformities or defects in a manufacturing process; i.e. C chart and u chart and D-chart are introduced. While both C and u control charts are used for monitoring a nonconformity, C chart is used when the sample sizes are the same. D-chart is a modification of the first 2 charts used to monitor simultaneously several different categories of defects of varying importance in a product.

### 6.1 C Control Chart

C control chart is an attribute control chart usually used to monitor the number of one kind of defect or nonconformity in a unit of product or in a group of items, based on the samples taken from the process at given times (hours, shifts, days, weeks, months, etc.); however, this chart sometimes allows the practitioner to assign each sample more than one kind of defect which are similar. The suitable distribution for the total count of a kind of defect in an individual unit products (or in individual samples of equal size) is Poisson with mean and variance  $\lambda$  and probability function:

$$\Pr(X = x) = \frac{\lambda^x \cdot e^{(-\lambda)}}{x!}$$

Where

$\lambda$  is the mean of the count of non-conformities per unit.

The upper and lower control limits for the C chart are calculated using the following formulas; if a negative value for the LCL was obtained, replace it with zero.

#### 6.1.1 C-chart limits –standard(C') known

If  $C'$ , a standard or aimed-at value for average number of nonconformities per sample size(n) is given then, where n is usually 1 or any constant value in all samples, ∴

$$UCL_C = C' + 3\sqrt{C'}$$

$$C.L. = C'$$

$$LCL_C = C' - 3\sqrt{C'}$$

### 6.1.2 C-chart limits –C' unknown

If no standard of aimed-at value for C' is available then

$$UCL_c = \bar{c} + 3\sqrt{\bar{c}}$$

$$CL = \bar{c}$$

$$LCL_c = \bar{c} - 3\sqrt{\bar{c}}$$

where

$\bar{c}$  is the average of observed non-conformities per sample .

when  $\bar{c}$  has been used, if all samples fall within the upper and lower limits and no special pattern is observed, conclude no special problem exists the process; otherwise there is a problem or some problems within the process which have to be identified. when C' has been used in the calculation of the limits, and the above conditions are hold, the mean of the count of the defect or nonconformity under study is in-control.

#### Example 6.1

Plot the c chart for the following data:

Date	Number inspected	Count of defects(C)
1/1	1	0
1/2	1	1
1/3	1	3

#### Solution

Since C' is not given,  $\bar{c}$  is used:

$$\bar{c} = \frac{0+1+3}{3} = 1.33$$

$$UCL_c = \bar{c} + 3\sqrt{\bar{c}} = \frac{4}{3} + 3\sqrt{\frac{4}{3}} = 4.8$$

$$\text{Central line: } \bar{c} = 1.33$$

$$LCL_c = \bar{c} - 3\sqrt{\bar{c}} = \frac{4}{3} - \frac{3\sqrt{4}}{\sqrt{3}} = \frac{4}{3} - \frac{6}{\sqrt{3}} = -0.9761 < 0 \rightarrow LCL_c = 0$$

Figure 6.1 shows the chart.

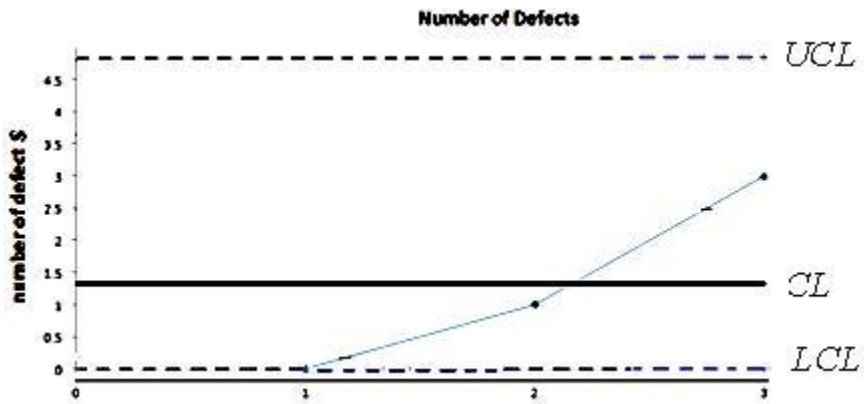


Fig. 6.1 the c chart for Example 6.1.

**Example 6.2**

The following table gives the number a kind of defect found in 20 samples of the same size drawn from a production process. Find  $\bar{c}$ , compute trial control limits and plot a c control chart. What value of aimed-at number of defects  $c_0$ , would you suggest for the following period?

Sample No.	Number of Defects(c)	Sample No.	Number of Defects (c)
1	7	11	6
2	5	12	3
3	3	13	2
4	4	14	7
5	3	15	2
6	8	16	4
7	2	17	7
8	3	18	4
9	4	19	2
10	3	20	3
Sum			82

**Solution**

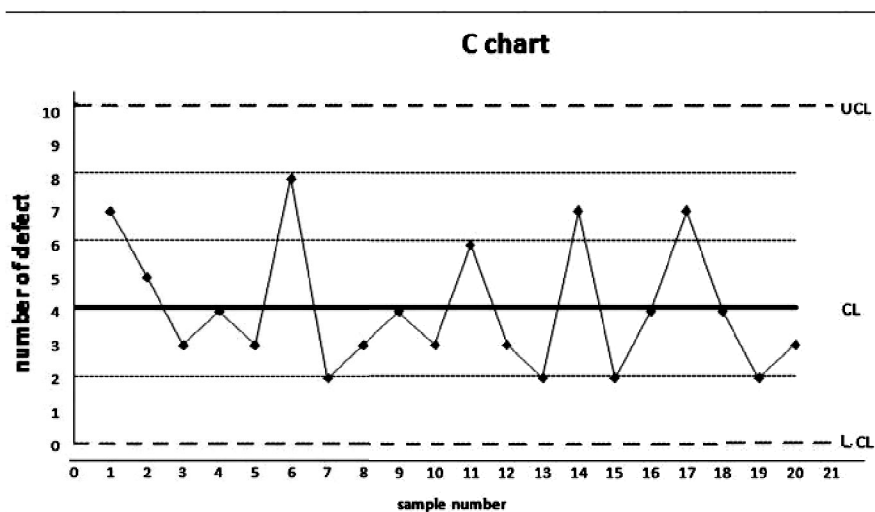
Since the sample sizes are the same C chart could be used.  $C'$  is not given then  $\bar{c}$  is used instead:

$$\bar{c} = \frac{7+5+\dots+3}{20} = \frac{82}{20} = 4.1$$

$$LCL_c = \bar{c} - 3\sqrt{\bar{c}} = 4.1 - 3 \times \sqrt{4.1} = -1.97 < 0 \rightarrow 0$$

$$UCL_c = \bar{c} + 3\sqrt{\bar{c}} = 4.1 + 3 \times \sqrt{4.1} = 10.17$$

The control chart for nonconformities is given in Fig. 6.2.



**Fig 6.2 Control chart for non conformities; data of Example 6.2**

Since none of the points is out of the limits and no other special out-of-control sign is observed,  $\bar{C}$  is a good estimate for average number of non-conformities in the process per  $n$  (sample size) products. Furthermore for future use the current limits are recommended.

## 6.2 u Control Chart

For monitoring the number of defects in the process, if the sample sizes are unequal,  $u$  control chart is used instead of  $c$  chart.  $u_i$  is the number of conformities per unit of the product in the  $i^{\text{th}}$  sample:  $u_i = \frac{c_i}{n_i}$  (or in general  $u = \frac{c}{n}$ ). Unit of the product could be a meter, a square meter, one component etc... In this chart we are allowed to define every say 5 m or 5m<sup>2</sup> as one unit.  $u$  chart is usually used when the sample sizes are different; however there is no restriction for using the chart when the sample sizes are the same. In the latter case the limits

would be straight lines, in the former case each subgroup would have its own limits. The upper and lower limits of u chart are calculated as follows:

### 6.2.1 u chart limits –standard(u') known

If  $u'$ , the standard or aimed-at number of nonconformities per unit product (1 meter, one component,...), is given, then:  
if the standard or aimed-at number of conformities per sample is denoted

by  $c'$ , we have  $u' = \frac{c'}{n}$  and  $\sigma_u = \sqrt{\frac{u'}{n}}$  because:

$$u = \frac{c}{n} \Rightarrow \text{Var}(u) = \frac{1}{n^2} \text{Var}(c) = \frac{c'}{n^2}$$

$$\frac{c'}{n} = u' \Rightarrow \text{Var}(u) = \frac{nu'}{n^2} = \frac{u'}{n} \Rightarrow \sigma_u = \sqrt{\frac{u'}{n}}$$

Then, the limits in this case would be:

$$UCL_U = u' + \frac{3\sqrt{u'}}{\sqrt{n_i}}$$

$$CL = u'$$

$$LCL_U = u' - \frac{3\sqrt{u'}}{\sqrt{n_i}}$$

where  $n_i$  is the size of  $i^{\text{th}}$  sample.

when the sample size is not constant, the control limits would not be straight lines, but all comments mentioned regarding in-control and out-of-control conditions in the discussion of previous Shewhart control charts are valid here.

### 6.2.2 u-chart limits –u' unknown

If  $u'$  is not available,  $\bar{u}$  calculated as follows, substitutes  $u'$ ,

$$\bar{u} = \frac{\sum c_i}{\sum n_i} = \frac{\text{Total observed defects in k samples}}{\text{Sum of k sample sizes}}$$

and the control limits are:

$$UCL_p = \bar{u} + \frac{3\sqrt{\bar{u}}}{\sqrt{n_i}}$$

$$CL = \bar{u}$$

$$LCL_p = \bar{u} - \frac{3\sqrt{\bar{u}}}{\sqrt{n_i}}$$

when  $LCL_p < 0 \rightarrow LCL_p = 0$ .

If all points fall within the limits and no special pattern is observed, conclude the variations are random and there is no problem in the process from the point of view of the number of conformities in the product. Otherwise there are problems whose causes have to be detected. And also if  $\bar{u}$  has been used as a standard value, conclude that the mean of the number of the nonconformity under study is according to the given standard.

In all Shewhart's control chart including u chart if the sample sizes are unequal, the control limits would not be straight lines and each sample would have its own limits. If the variation of the sample sizes is not that much, a mean sample size ( $\bar{n}$ ) could be computed and a common upper and lower control limits be calculated for all the samples, which would be straight lines; e.g. in the case of u chart:

$$\text{control limits} = \bar{u} \pm 3\sigma_u \quad \sigma_u = \sqrt{\frac{\bar{u}}{\bar{n}}}, \quad \bar{n} = \frac{\sum_{i=1}^k n_i}{k}.$$

### Example 6.3

Plot the suitable control chart for the following 20 samples. Is the process in control?

Sample No. (i)	Sample length in meter ( $n_i$ )	count of a special defect in the sample ( $c_i$ )	count of the defect per meter in the sample ( $u_i$ )	$UCL_U = \bar{u} + 3\sqrt{\frac{\bar{u}}{n_i}}$	$LCL_U = \bar{u} - 3\sqrt{\frac{\bar{u}}{n_i}}$
1	1	4	4.0	8.20	0
2	1	5	5.0	8.20	0
3	1	3	3.0	8.20	0
4	1	3	3.0	8.20	0
5	1	5	5.0	8.20	0
6	1.3	2	1.5	7.56	0
7	1.3	5	3.8	7.56	0
8	1.3	3	2.3	7.56	0
9	1.3	2	1.5	7.56	0
10	1.3	1	0.8	7.56	0
11	1.3	5	3.8	7.56	0
12	1.3	2	1.5	7.56	0
13	1.3	4	3.1	7.56	0
14	1.3	2	1.5	7.56	0
15	1.2	6	5.0	7.75	0
16	1.2	4	3.3	7.75	0
17	1.2	0	0	7.75	0
18	1.7	8	4.7	6.99	0
19	1.7	3	1.8	6.99	0
20	1.6	8	5	7.11	0
Sum	25.3	75	$\bar{u} = \frac{75}{25.3} = 2.965$	$\bar{u} \cong 3$	

### Solution

Since sample sizes are different, then for each sample the upper and lower control limits have been calculated using  $\bar{u} = \frac{75}{25.3} \cong 3$  and inserted into the above table. Figure 6-3 shows the chart for  $u$ . No point has fallen outside the limits and no special pattern is observed; therefore the variations are random and the process is in control from the viewpoint of the count of the special defect (nonconformity) per meter of the product; furthermore  $\bar{u}$  can be considered a good estimate of nonconformities per unit in the process.

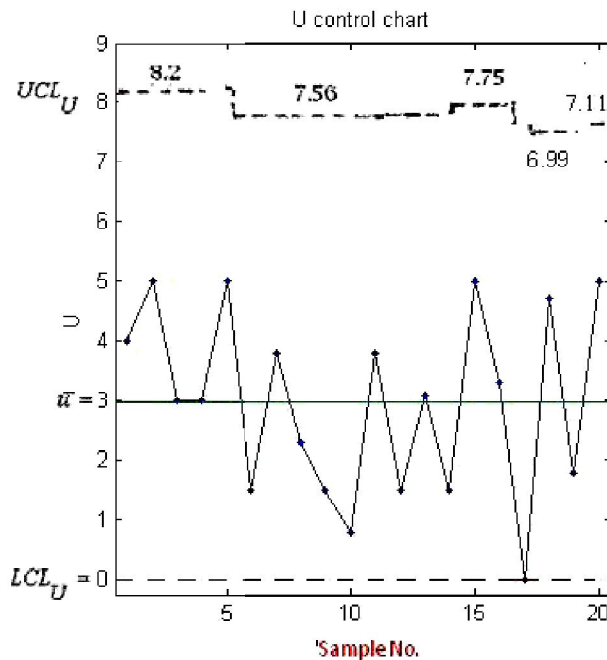


Fig. 6-3 u- chart for the data of Example 6.3

Notice that

- all comments mentioned regarding in-control and out -of -control conditions in the discussion of the previous Shewhart 's control charts are valid here.

- the tests based on the theory of runs, described when discussing control charts for variables, are valid for all Dr Shewhart's control chart including c chart and u chart.

### 6.3 Classification of Nonconformities and their Weights

Possible defects of a unit product are classified according to their seriousness. A common procedure divide the defects into 3 or 4 classes. Depending on factors such as the complexity of product a type of

classification is used. Two types of classification of defects are described here:

### 6.3.1 Three-class Defect Classification

The catalogue of standard Mil-STD-105D classifies defects or nonconformities into 3 classes<sup>1</sup>:

#### 6.3.1.1 Critical defect

A critical defect is a defect that judgment and experience indicate it would result in hazardous or unsafe conditions for individuals using, maintaining, or depending upon the product or a defect that judgment and experience indicate is likely to prevent performance of the tactical function of a major end item such as a ship, aircraft, tank, missile, or space vehicle.

#### 6.3.1.2 Major defect

A major defect is a defect, other than critical, that is likely to result in failure, or to reduce materially the usability of the unit of product for its intended purpose.

#### 6.3.1.3 Minor defect

A minor defect is a defect that is not likely to reduce materially the usability of the unit of product for its intended purpose, or is a departure from established standards having little bearing on the effective use or operation of the unit.

### 6.3.2 Four-class Defect Classification

The late H.F. Dodge classifies the defects into 4 classes as follows (Extracted from: Grant & Leavenworth ,1988, p. 293 and Stamatis,2003 page175 <sup>2</sup>)

#### 6.3.2.1 Class “A” Defects — Very serious.

This class

- Will render unit totally unfit for service;
- Will surely cause operating failure in service which cannot be readily fixed on the job
- Is likely to cause personal injury or property damage.

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<sup>1</sup> From Catalogue: MIL-STD-105—Sampling Procedures & Tables for Inspection by Attributes

<sup>2</sup>Stamati, D.H.2003, 6 Sigma and Beyond: Statistical Process Control, Vol 4 St Lucie Press



#### **6.3.2.2 Class "B" Defects — Serious.**

This class

- Will probably, but not surely, cause Class A operating failure
- Will surely cause trouble of a nature less serious than Class A operating failure, e.g. adjustment failure, operation below standard, etc.
- will surely cause increased maintenance or decreased life.

#### **6.3.2.3 Class "C" Defects — Moderately serious**

This class

- May cause operating failure in service.
- May cause trouble of a nature less serious than operating failure.
- Is likely to cause increased maintenance or decreased life.
- Forms major defects of appearance, finish, or workmanship.

#### **6.3.2.4 Class "D" Defects — Not serious.**

This class

- Will not cause operating failure of the unit in service.
- Forms minor defects of appearance, finish or workmanship.

### **6.3.3 Weighting**

Once a classification of all defects is established, demerits( weights) may then be assigned to each class of defect. Common weights are in the ranges (0,1) or (1,10) or (1,100). For example some researchers have used weights 1 for minor defect , 3 for major and 9 for critical in 3-class type (Besterfield,1990, page189) and weight 1,20,40,100 in 4-class type for class D, C, B and A respectively (Grant and leaven worth,1988,p1ge294).

## **6.4 D control chart**

A demerit control chart (D-chart )is a modification of the first 2 charts and is used to monitor simultaneously different classes of defects of varying importance in a product. An application of this chart is in the field of Total Quality Control.

After classifying and weighting the defects, D chart is plotted. The variable defined for this chart is denoted by letter D. In a 3-class type of classification, D for each sample is given by (Besterfield,1990 page 189):

$$D = W_c u_c + W_{ma} u_{ma} + W_{mi} u_{mi}$$

where

$u_c$	Count of critical defects per unit in the sample
$u_{ma}$	Count of major defects per unit in the sample
$u_{mi}$	Count of minor defects per unit in the sample
$W_c$	Weight(penalty) assigned to critical defects
$W_{ma}$	Weight assigned to major defects
$W_{mi}$	Weight assigned to minor defects

D which is an index of the defect per unit could be called demerit per unit in the sample. D control chart plots all demerits per unit product.

### 6.4.1 D chart Upper and Lower control limit

Depending on whether a standard value for the index of defects in the product, denoted by  $D_0$ , is specified or not 2 cases are distinguished below:

#### 6.4.1.1 D chart limits: $D_0$ known

For the 3-class Type defects, if the standard nonconformities per unit for the defects in the process (i.e.  $u_{oc}$ ,  $u_{oma}$ ,  $u_{omi}$ ) are known, the central line of the control D chart is computed from:

$$D_o = W_c u_{oc} + W_{mi} u_{oma} + W_{ma} u_{omi}$$

and the limits are calculated from:

$$UCL_D = D_o + 3\sigma_{0D}$$

$$CL = D_o$$

$$LCL_D = D_o - 3\sigma_{0D}$$

If  $LCL < 0$  set  $LCL = 0$ .

where

$u_{oc}$	The standard nonconformities per unit for the critical defect in the process
$u_{oma}$	The standard nonconformities per unit for the major defect in the process
$u_{omi}$	The standard nonconformities per unit for the minor defect in the process
$\sigma_{0D}$	$= \sqrt{\frac{W_c^2 u_{oc}^2 + W_{ma}^2 u_{oma}^2 + W_{mi}^2 u_{omi}^2}{n}}$
$D_o$	=The standard or aimed-at index per unit product = $D_o = W_{mi} u_{omi} + W_{ma} u_{oma} + W_c u_{oc}$

To enter the  $i^{th}$  subgroup into the chart, calculate  $D_i$  as follows and show point  $D_i$  on the chart ;

$$D_i = W_c \left( \frac{\text{Count of critical defects in } i^{\text{th}} \text{ subgroup}}{n_i} \right) + W_{ma} \left( \frac{\text{Count of major defects in } i^{\text{th}} \text{ subgroup}}{n_i} \right) + W_{mi} \left( \frac{\text{Count of minor defects in } i^{\text{th}} \text{ subgroup}}{n_i} \right)$$

or  $D_i = W_c (u_c)_i + W_{ma} (u_{ma})_i + W_{mi} (u_{mi})_i$ ,

#### 6.4.1.2 D chart limits: $D_0$ unknown

When the values of  $u_{oma}$ ,  $u_{omi}$ ,  $u_{oc}$  are not known for the process, replace  $u_{oc}$ ,  $u_{oma}$ ,  $u_{omi}$  with the following variables respectively, when calculating the central line and the limits :

$$\bar{u}_{ma} = \frac{\text{Count of major defects in } k \text{ subgroups}}{\text{sum of } k \text{ subgroup sizes}}$$

$$\bar{u}_{mi} = \frac{\text{Count of minor defects in } k \text{ subgroups}}{\text{sum of } k \text{ subgroup sizes}}$$

$$\bar{u}_c = \frac{\text{Count of critical defects in } k \text{ subgroups}}{\text{sum of } k \text{ subgroup sizes}}$$

and replace  $D_0$  by  $\bar{D} = \frac{\sum_{i=1}^k D_i}{k}$ .

**Example 6.4** (Based on Besterfield, 1990 page 190)

Assume that  $W_c = 9$ ,  $W_{ma} = 3$ ,  $W_{mi} = 1$  are used for a three-class weighting system. Determine the central line, and control limits for a D control chart, when  $u_{oc} = 0.08$ ,  $u_{oma} = 0.5$ ,  $u_{omi} = 3$ . The data of six samples are given in the following table.

Sample No( i )	Date	n	Count of critical defect	Count of major defect	Count of minor defect
1	1/1	40	1	15	130
2	2/1	40	1	22	153
3	3/1	40	1	10	173
4	4/1	40	0	15	119
5	5/1	40	1	15	130
6	6/1	40	2	26	160

**Solution**Calculation of aimed-at  $D_o$ 

$$D_o = W_{mi}u_{0mi} + W_{ma}u_{0ma} + W_c u_{0c} = 3 + 3(0.5) + 9(0.08) = 5.22$$

$$n=40 \Rightarrow \sigma_{0D} = \sqrt{\frac{W_{mi}^2 u_{0mi} + W_{ma}^2 u_{0ma} + W_c^2 u_{0c}}{n}} = 0.59$$

$$UCL_D = D_o + 3\sigma_{0D}$$

$$LCL_D = D_o - 3\sigma_{0D}$$

Since the sample sizes are equal ,the upper and lower control limits are straight lines:

$$UCL_D = 5.22 + 3(0.59) = 6.99$$

$$D_o = 5.22$$

$$LCL_D = 5.22 - 3(0.59) = 3.45$$

Using the data of the problem, the variable for the  $i^{\text{th}}$  sample was calculated from

$$D_i = W_c (u_c)_i + W_{ma} (u_{ma})_i + W_{mi} (u_{mi})_i = 9(u_c)_i + 3(u_{ma})_i + 1(u_{mi})_i,$$

where

$$(u_c)_i = \frac{(X_c)_i}{n_i} = \frac{\text{count of critical defects in } i^{\text{th}} \text{ sample}}{i^{\text{th}} \text{ sample size}},$$

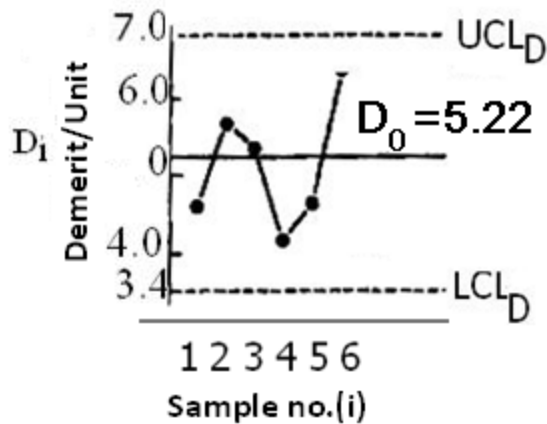
$$(u_{ma})_i = \frac{(X_{ma})_i}{n_i} = \frac{\text{count of major defects in } i^{\text{th}} \text{ sample}}{i^{\text{th}} \text{ sample size}},$$

$$(u_{mi})_i = \frac{(X_{mi})_i}{n_i} = \frac{\text{count of minor defects in } i^{\text{th}} \text{ sample}}{i^{\text{th}} \text{ sample size}}.$$

The results are in the last column,

Date	Sample No.(i)	Sample size	Count of critical defects $(X_c)_i$	Count of major defects $(X_{ma})_i$	Count of minor defects $(X_{mi})_i$	Demerit per unit product $\{ D_i = 9(u_c)_i + 3(u_{ma})_i + 1(u_{mi})_i \}$
1/1	1	40	1	15	130	$D_1 = 9\left(\frac{1}{40}\right) + 3\left(\frac{15}{40}\right) + \left(\frac{130}{40}\right) = 4.6$
2/1	2	40	1	22	153	$D_2 = 5.7$
3/1	3	40	1	10	173	$D_3 = 5.3$
4/1	4	40	0	15	119	$D_4 = 4.1$
5/1	5	40	1	15	130	$D_5 = 4.6$
6/1	6	40	2	26	160	$D_6 = 9\left(\frac{2}{40}\right) + 3\left(\frac{26}{40}\right) + \left(\frac{160}{40}\right) = 6.4$

Figure 6.4 shows the chart.



**Fig. 6-4 Demerit control chart(D Chart) for Example 6.4**  
(Besterfiled,1990 page191)

### Example 6.5

Classifying defects into 4 classes, weights 100, 50, 10,1 are given to the classes A,B,C and D. 3 samples of size 1500 were drawn from the production process and the following table shows the results. If the standard indices for nonconformities per unit of Types A,B,C,& D are respectively  $u_{0A} = 0.001$ ,  $u_{0B} = 0.003$ ,  $u_{0C} = 0.008$  &  $u_{0D} = 0.009$ . Plot an attribute chart which monitor the all classes of defects simultaneously.

Sample No.(i)	Sample Size ( $n_i$ )	Count of Class A Defects ( $X_A$ ) <sub>i</sub>	Count of Class B Defects ( $X_B$ ) <sub>i</sub>	Count of Class C Defect ( $X_C$ ) <sub>i</sub>	Count of Class D Defects ( $X_D$ ) <sub>i</sub>
1	1500	2	5	12	13
2	1500	1	6	10	20
3	1500	3	2	8	24

### Solution

D chart is suitable. Using the above standards i.e.

$$u_{0A} = 0.001 \quad u_{0B} = 0.003 \quad u_{0C} = 0.008 \quad u_{0D} = 0.009$$

and the following demerits weights

$W_A = 100$ ,  $W_B = 50$ ,  $W_C = 10$ ,  $W_D = 1$ , we proceed as follows:

The central line would be:

$$D_0 = u_{0A}W_A + u_{0B}W_B + u_{0C}W_C + u_{0D}W_D \text{ or}$$

$$D_0 = 0.1 + 0.15 + 0.08 + 0.009 = 0.339$$

$$\begin{aligned}\sigma_{oD} &= \sqrt{\frac{W_A^2 u_{oA} + W_B^2 u_{oB} + W_C^2 u_{oC} + W_D^2 u_{oD}}{n}} \\ &= \sqrt{\frac{10000 \times \frac{1}{1000} + 2500 \times \frac{3}{1000} + 100 \times \frac{8}{1000} + \frac{9}{1000}}{1500}} \\ &= \sqrt{\frac{10 + 7.5 + 0.8 + 0.009}{1500}} = 0.11\end{aligned}$$

The limits:

$$UCL_D = D_0 + 3\sigma_{oD} = 0.339 + 3 \times 0.11 = 0.669$$

$$D_o = 0.339$$

$$UCL_D = D_0 - 3\sigma_{oD} = 0.339 - 3 \times 0.11 = 0.011$$

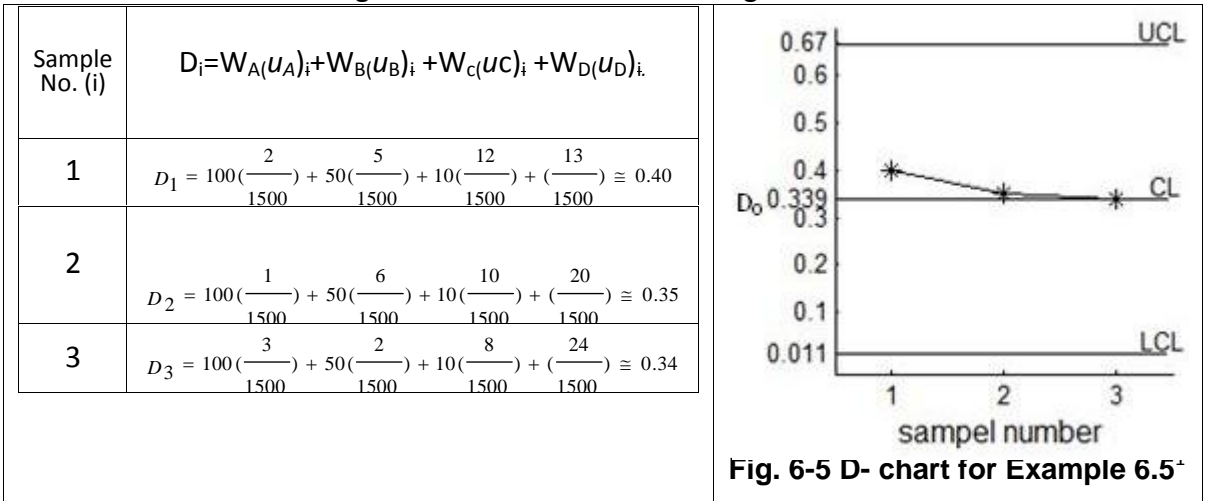
For each sample, the variable D has been calculated using

$$D_i = W_A(u_A)_i + W_B(u_B)_i + W_C(uc)_i + W_D(u_D)_i.$$

where

$$(u_A)_i = \frac{(x_A)_i}{n_i}, \quad (u_B)_i = \frac{(x_B)_i}{n_i}, \quad (uc)_i = \frac{(x_C)_i}{n_i}, \quad (u_D)_i = \frac{(x_D)_i}{n_i}.$$

The following table shows the 3  $D_i$ 's and Fig. 6.5 shows the chart.



<sup>1</sup> With thanks to Mr Amin Jafari former student of IE Dept ,college of Engineering of Shahid Bahonar University of Kerman, Iran

It should be pointed out that:

i)

Using  $D_0$  as central line, and in the calculations of D chart limits implies an intent to apply pressure to bring a process to no worse than the stipulated level. Using  $\bar{D} = \frac{\sum_{i=1}^k D_i}{k}$  as central line and in the calculation of limits implies an intent to illustrate how a process is performing (Grant & Leavenworth, 1988 page 295); whether defects are randomly distributed in the products or not.

ii)

Q control chart is another attribute control chart similar to D chart, discussed in references such as Grant & Leavenworth (1988).

iii)

Points indicating out of control status on D chart are harder to interpret than they are on the c chart and do not necessarily direct attention to the causes of a problem. It may prove more desirable to use several c charts for process analysis purposes, one for each class, reserving the Q and D chart for upper-level management reporting (Grant, Leavenworth, 1988, page 295).

## Exercises

### 6.1 ( 8.1 Grant & Leavenworth, 1988 page 296)

The following table gives the numbers of missing rivets noted at aircraft final inspection

Airplane No.	number of missing rivets	Airplane No.	number of missing rivets	Airplane No.	number of missing rivets
201	8	210	12	218	14
202	16	211	22	219	11
203	14	212	16	220	9
204	19	213	9	221	10
205	11	214	26	222	23
206	15	215	15	223	7
207	8	216	9	224	27
208	11	217	9	225	9
209	21				

Find  $\bar{C}$ , compute trial control limits, and plot a control chart for c. What aimed-at value of  $c'_0$  would you suggest for the central line for the subsequent period?

Answers:  $UCL_c = 25.28(25.5)$ ,  $LCL_c = 2.80(2.5)$ ,  $c'_0 = 12.96$

**6.2** Use Poisson distribution to find 0.995 and 0.005 probability limits for a c chart when  $c' = \mu_c = 4.2$  and also when  $c' = 9.5$ .

Answer: 1, 8 & 5, 15

**6.3** Find the trial control limits of a u chart for monitoring the surface finish of a certain product using the following data; plot the chart. Suppose the points falling outside the chart have assignable causes. Calculate the revised control limits.

Batch No.	Sample Size	Total Non-conformities	Batch No.	Sample Size	Total Non-conformities
1	10	45	15	10	48
2	10	51	16	11	35
3	10	36	17	10	39
4	9	48	18	10	29
5	10	42	19	10	37
6	10	5	20	10	33
7	10	33	21	10	15
8	8	27	22	10	33
9	8	31	23	11	27
10	8	22	24	10	23
11	12	25	25	10	25
12	12	35	26	10	41
13	12	32	27	9	37
14	10	43	28	10	28



**6.4** ( 8-10 Grant &Leaven worth ,1988 page 297)

A textile manufacturing initiates use of a c chart to monitor the number of imperfections found in bolts of cloth. Each bolt is the same length, width, weave, and fiber composition. A total of 145 imperfections were found in the last 25 bolts inspected. The four highest and lowest counts were:

Count of imperfections	
Highest	Lowest
20	4
16	4
10	5
9	5

- Calculate 3-sigma control limits of an appropriate chart for this process
- Is this process in control? If not, what aimed-at values for central line and control limits would you recommend for the next period?

**6.5** ( P.8-27 Grant &Leaven worth ,1988 page 301)

A u chart is used to control imperfections in the preparation of mats for advertising copy to be used in print media. The control statistic is number of flaws per 100cm<sup>2</sup> of mat area. A standard value of  $\mu_0$  of 1.5 flaws per 100cm<sup>2</sup> is used. A particular mat subjected to inspection is 18cm by26 cm and was found to have 16 flaws. a)Calculate control limits for this sample and test for conformance to the standard.

- At the time that this inspection was performed, the process actually was operating at a  $\mu_u$  of 3.0 flaws per 100cm<sup>2</sup>. What is the probability of not detecting this fact from the sample? Use Poisson's approximation ( where  $\mu c = nu$ ).

**6.6** ( P.8-30 Grant &Leaven worth ,1988 page 301)

A Producer of metal medallions and commemorative coins uses c charts to control imperfections on large orders for single items. All imperfections are recorded but not all cause and ultimate destruction of the item. 30 items constitute an inspection unit. After 20 inspection units have been inspected and the data recorded , the total count of imperfections is 35.

- Calculate the control limits for c chart for this process.
- What is the probability of type I error for this chart.
- Find the probability of type II error, should the process shift to a  $\mu_c$  of 4.0.

**6.7** Construct a control chart for the data in the given table for empty bottle inspections of a soft drink manufacturer. Assume assignable causes for any points that are out of control. What aimed-at values for central line and control limits would you recommend for the next period ?

NUMBER OF BOTTLES	CHIPS, SCRATCHES, OTHER	FOREIGN MATERIAL ON SIDES	FOREIGN MATERIAL ON BOTTOM	TOTAL NON-CONFORMITIES
40	9	9	27	45
40	10	1	29	40
40	8	0	25	33
40	8	2	33	43
40	10	6	46	62
52	12	16	51	79
52	15	2	43	60
52	13	2	35	50
52	12	2	59	73
52	11	1	42	54
52	15	15	25	55
52	12	5	57	74
52	14	2	27	43
52	12	7	42	61
40	11	2	30	43
40	9	4	19	32
40	5	6	34	45
40	8	11	14	33
40	3	9	38	50
40	9	9	10	28
52	13	8	37	58
52	11	5	30	46
52	14	10	47	71
52	12	3	41	56
52	12	2	28	42

**6.8** Assuming that  $W_c = 10$ ,  $W_{ma} = 5$ ,  $W_{mi} = 1$  are used for a three-class defect classification system, determine the central line, and control limits for a D control chart, when  $u_{0c} = 0.08$ ,  $u_{0ma} = 0.5$ ,  $u_{0mi} = 3$  and  $n = 50$ . If a sample of size 50 has 1 critical nonconformity, 35 major nonconformities and 110 minor defects, will this sample fall within the limit of the D control chart?

**A low percent of people live 90 years  
A great percent repeat one year 90 times**



# Chapter 7

## WARNING LIMITS & ADAPTATIONS OF CONTROL CHARTS

## Chapter 7 Warning Limits & Adaptations of Control Charts

### Aims

The objective of this chapter is to provide a description of occasionally used control charts. Control charts such as  $\bar{X}$  chart with unequal sample sizes, control chart for individual measurements, control charts for variables with probability limits, and  $\bar{X}$  chart with linear trend. The concept of moving range and its application to statistical quality control is also pointed out. The set of warning limits used in control charts is also introduced.

### 7.1 Warning Limits on Control Charts

Some authors of quality control have advocated the use of 2 sets of limits on certain control charts i.e. the outer and inner (warning) limits. The outer limits, sometimes called action limits, are the conventional limits, usually set at 3-sigma from the central line. The inner limits are recommended as warning limits and are usually at 2-sigma. Figure 7-1 shows a control chart with these two sets of limits i.e. UCL and LCL, upper and lower warning limits (UWL & LWL).

What follows in this section is some comments on warning limits of  $\bar{X}$  chart from Grant & Leavenworth (1988) page 307.

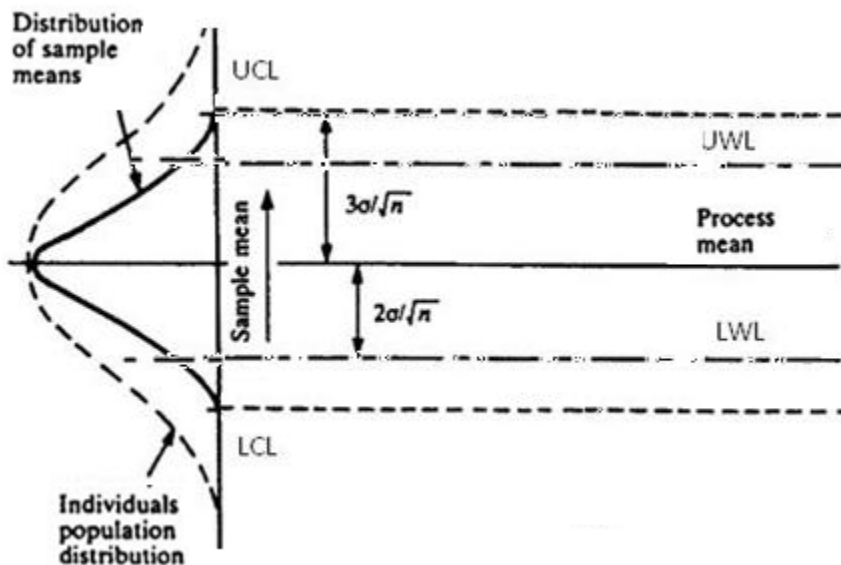


Fig. 7.1 A sample  $\bar{X}$  chart with UCL & LCL and UWL & LWL.

On the conventional  $\bar{X}$  chart with only one set of limits, the chart seems to give only two kinds of advice. It either says, "Look for trouble," or it says, "Leave the

process alone.” This has the virtue of definiteness. However, any such definite advice is sure to be wrong part of the time. Limits placed at 3-sigma are seldom wrong when they say, “Look for trouble,” but are much more often wrong when they say, “Leave the process alone.”

The inner limits or warning limits seem to add a third kind of advice. This might be phrased, “Start being suspicious that trouble is brewing.”

At first thought, the idea of having warning limits on  $\bar{X}$  charts may seem attractive. Nevertheless, there is a sound reason for the common practice of having only one set of limits and having these limits at or near 3-sigma. This reason is the greater definiteness of a single set of limits. Two sets of limits tend to be confusing with regard to the exact action to be taken when a point falls between the inner and the outer limits. This is particularly true if many of the people in a manufacturing plant who are using the  $\bar{X}$  and R control charts as a basis for action are not fully clear as to the principles underlying these charts. Inner limits will be exceeded at least 5% of the time as a matter of chance. If a single point just outside the inner limits is to be used as a basis for hunting for trouble there is bound to be unproductive hunting which may tend to destroy confidence in the control charts. Usually in a manufacturing plant so much trouble really exists that it does not pay to hunt for trouble without strong evidence that it is present.

Nevertheless, even though inner limits should not be drawn on most control charts, they can be extremely useful in the sophisticated interpretation of control charts by people who understand control-chart theory. Here the clue to action is given not by a single point outside either of the inner limits, but rather by two or more points, both outside the same inner limit. This is really a matter of sizing up extreme runs; it is somewhat comparable to the interpretation of extreme runs on the same side of the central line. For example, 2 points in succession outside the same inner limit on an  $\bar{X}$  chart give even stronger evidence of a shift in process average than a single point outside the outer limit. Two points out of 3 beyond one inner limit, 3 out of 7, or 4 out of 10, may all be considered as appropriate grounds for action.

However, this type of interpretation may be made by the quality control engineer or other qualified individual without confusing matters for his colleagues by having inner limit lines actually drawn on all  $\bar{X}$  charts. When a suspicious sequence of points is observed close to a conventional 3-sigma limit, the quality control engineer may imagine an inner limit two-thirds of the distance from the central line to the control limit. Or, if this is too much strain on the imagination, such a line may be drawn lightly on the portion of the chart to be studied( Grant & Leavenworth ,1988 page 307-8.)

## 7.2 Control charts adaptations

In this section the objective is to introduce some control charts that are occasionally used.

### 7.2.1 $\bar{X}$ control chart with unequal sample sizes

If the sample size is not constant, the control limits would not be straight line and each sample would have its own limits varying with its size, such as the dashed- lines on the chart shown in Fig. 7-1.

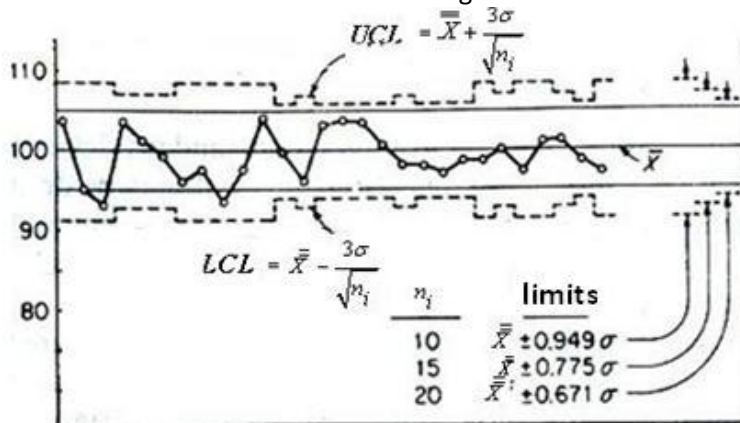


Fig. 7-1  $\bar{X}$  chart illustrating variation of control limits with sample size

(Extracted from Grant & Leavenworth, 1988 page 303)

What follows are some formulas for the limits of this case.

#### Case1: $\mu$ and $\sigma$ of the process are known

If the standard or aimed-at mean of the process is known, for  $i^{\text{th}}$  subgroup with sample size  $n_i$  the limits would be calculated as:

$$UCL_{\bar{X}} = \mu + \frac{3}{\sqrt{n_i}}\sigma$$

$$CL = \mu$$

$$LCL_{\bar{X}} = \mu - \frac{3}{\sqrt{n_i}}\sigma$$

#### Case 2: Process mean is known

In this case the limits for  $i^{\text{th}}$  subgroup with sample size  $n_i$  are calculated from:

$$UCL_{\bar{X}} = \bar{\bar{X}} + \frac{3}{\sqrt{n_i}}\sigma$$

$$CL = \bar{\bar{X}}$$

$$LCL_{\bar{X}} = \bar{\bar{X}} - \frac{3}{\sqrt{n_i}}\sigma$$

where

$\bar{\bar{X}}$ , the mean of all  $\bar{X}$ 's, is calculated from

$$\bar{\bar{X}} = \frac{\sum_{i=1}^k n_i \bar{X}_i}{\sum_{i=1}^k n_i},$$

$k$  is the number of subgroups,

$n_i$  is the size of  $i^{\text{th}}$  sample,

$A$  is equal to  $\frac{3}{\sqrt{n_i}}$

Needless to say, the samples with the same sample sizes have the same limits.

### Case3: $\sigma$ of the process is unknown

The standard deviation ( $\sigma$ ) of a normally distributed process, when not known, has to be estimated by either of the following ways:

-For each subgroup calculate its standard deviation  $s_i$  then, if the

measurements are normally distributed, calculate  $\hat{\sigma}_i = \frac{s_i}{c_4}$ . The process

standard deviation is estimated from  $\hat{\sigma} = \frac{\hat{\sigma}_1 + \dots + \hat{\sigma}_k}{k}$ ;  $c_4$  is read from Table U.

-For each subgroup with range  $R_i$ , if the measurements are normally

distributed, calculate  $\hat{\sigma}_i = \frac{R_i}{d_2}$  then  $\hat{\sigma} = \frac{\hat{\sigma}_1 + \dots + \hat{\sigma}_k}{k}$ ;  $d_2$  is read from Table U.

**When the sample sizes are large**, the following points worth knowing:

**i)**

For large samples, unless the largest subgroup is at least twice the smallest, it is satisfactory to compute un-weighted averages for the data (Grant and Leavenworth, 1988 page 305):

$$\bar{\bar{X}} = \frac{\sum_{i=1}^k \bar{X}_i}{k}, \quad \bar{s} = \frac{s_1 + \dots + s_k}{k}$$

instead of

$$\bar{\bar{X}} = \frac{\sum_{i=1}^k n_i \bar{X}_i}{\sum_{i=1}^k n_i} \text{ and } \bar{S} = S_p = \sqrt{\frac{(n_1-1)S_1^2 + \dots + (n_k-1)S_k^2}{n_1 + n_2 + \dots - k}}.$$



**ii)**

Table U shows that  $c_4$  approaches 1 as sample size increases to a largish number and in this case  $\bar{S}$  would estimate the process standard deviation:

$$\hat{\sigma} = \frac{\bar{S}}{c_4} = \bar{S} = \frac{s_1 + \dots + s_k}{k}$$

**iii)**

It could be verified that for large sample sizes the weighted average of the sample means ( $\bar{X}_i$  's) and the un-weighted  $\bar{X}$  i.e.  $\bar{X} = \frac{\sum_{i=1}^k n_i \bar{X}_i}{\sum_{i=1}^k n_i}$  and  $\bar{X} = \frac{\sum_{i=1}^k \bar{X}_i}{k}$  are not that different.

**iv)**

For large samples the following formulas could be used for  $\bar{X}$  and S charts( Grant & Leavenworth,1988 page 304):

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_3 \bar{S} \cong \bar{\bar{X}} + A \bar{S} \quad LCL_{\bar{X}} = \bar{\bar{X}} - A_3 \bar{S} \cong \bar{\bar{X}} - A \bar{S} \quad A = \frac{3}{\sqrt{n}}$$

$$UCL_S = B_4 \bar{S}, \quad LCL_S = B_3 \bar{S}, \quad B_4 \approx 1 + \frac{3}{\sqrt{2n}}, \quad B_3 \approx 1 - \frac{3}{\sqrt{2n}}.$$

One application of approximating  $B_3, A_3, B_4$  as above is when they are not available from Table U for large sample sizes.

**v)**

If the sample sizes are large or small but close to each other, an average  $n$

could be calculated from  $n = \bar{n} = \frac{\sum_{i=1}^k n_i}{k}$  and straight line limits be calculated

using this average sample size. This simplification is usually satisfactory for a start; separate limits for individual subgroup may be computed later for any doubtful cases ( Grant & Leavenworth,1988 page 305); for the samples we are not sure fall outside or not.

**Example 7.1** ( Grant and Leavenworth, 1988, page 305)

The shear strength of a spot welding process is under study. From 9 similar welding devices 9 random samples of different sizes are given in the following table:

Machine	Number of tests (n)	Shear strength (MPa)	
		Average	Standard deviation
A	128	743	63
B	127	695	47
C	126	711	67
D	114	668	51
E	126	736	80
F	126	791	58
G	126	686	50
H	111	801	92
J	119	604	64
Sum	1103	6435	572

The differences in sample size are not large and the largest subgroup is less than the twice the smallest. Therefore according to the rough rule in Sect.

(i) it is satisfactory to compute un-weighted averages:

$$\bar{\bar{X}} = \frac{\bar{X}_1 + \dots + \bar{X}_k}{k} = \frac{6435}{9} = 715 \text{ MPa}$$

$$\bar{S} = \frac{S_1 + \dots + S_k}{k} = \frac{572}{9} = 64 \text{ MPa}$$

For a start, the control limits are computed based on average subgroup size and later separate limits for any doubtful cases, if applicable. The average size is

$$\bar{n} = \frac{1103}{9} = 123$$

The factors  $A_3$ ,  $B_3$  &  $B_4$  are not available from Table U for this large sample size; but may be computed as follows:

$$A_3 \cong A = \frac{3}{\sqrt{123}}$$

$$B_4 \cong 1 + \frac{3}{\sqrt{2\bar{n}}} = 1.19$$

From these factors, the control limits may be computed for  $\bar{X}$  and S charts:

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_3 \bar{S} \cong 715 + \frac{3}{\sqrt{123}} \times 64 = 732$$

$$LCL_{\bar{X}} = \bar{\bar{X}} - A_3 \bar{S} \cong 715 - \frac{3}{\sqrt{123}} \times 64 = 698$$

$$UCL_S = B_4 \bar{S}, \quad LCL_S = B_3 \bar{S}, \quad B_3 = 1 - \frac{3}{\sqrt{2n}}, \quad B_4 = 1 + \frac{3}{\sqrt{2n}}$$

$$UCL_S = (1 + \frac{3}{\sqrt{2 \times 123}}) \times 64 = 76.24$$

$$LCL_S = (1 - \frac{3}{\sqrt{2 \times 123}}) \times 64 = 51.76$$

If you plot the control charts with these limits, you will notice that both charts show lack of control. That is, even though these spot welders are identical in design, they perform differently both with regard to average strength and uniformity of strength of welds (Grant and Leavenworth, 1988 page 306).

## 7.2.2 $\bar{X}$ charts with probability limits

It is worth remembering that  $\bar{X}$  values of the samples taken from a normally distributed universe follow the normal distribution and even from a non-normal universe, they approximately follow normal distribution when the sample size is 4 or more (Grant & Leavenworth, 1988, page 316). It could be easily shown that, without a change in the universe, if the limits of  $\bar{X}$  charts are computed according to the formulas given in the following sections the probability a point fall above UCL or under LCL is equal to  $\frac{\alpha}{2}$ .

### 7.2.2.1 probability limits for $\bar{X}$ charts $\mu$ & $\sigma$ known

If the standard  $\mu$  &  $\sigma$  are given or are obtainable from the design specifications of the quality characteristic, then:

$$UCL_{\bar{X}} = \mu + \frac{Z_{\frac{\alpha}{2}}}{3} A \sigma$$

$$CL = \mu$$

$$LCL_{\bar{X}} = \mu - \frac{Z_{\frac{\alpha}{2}}}{3} A \sigma$$

Where A is read from Table U and

$Z_{\frac{\alpha}{2}}$  from Table 6 at the end of the book.

### 7.2.2.2 Probability limits for $\bar{X}$ charts $\mu$ & $\sigma$ unknown

If  $\mu$  &  $\sigma$  are not given or cannot be obtained, then the central line is set as  $\bar{\bar{X}}$ , and for upper or lower control limits  $A_2$  and  $A_3$  are multiplied by  $\frac{Z_{\alpha}}{3}$ :

$$UCL_{\bar{X}} = \bar{\bar{X}} + \frac{Z_{\alpha}}{3} A_3 \bar{s} \quad LCL_{\bar{X}} = \bar{\bar{X}} - \frac{Z_{\alpha}}{3} A_3 \bar{s}$$

$$C.L. = \bar{\bar{X}}$$

$$UCL_{\bar{X}} = \bar{\bar{X}} + \frac{Z_{\alpha}}{3} A_2 \bar{R} \quad LCL_{\bar{X}} = \bar{\bar{X}} - \frac{Z_{\alpha}}{3} A_2 \bar{R}$$

$A_2$  and  $A_3$  are read from Table U.

For example if  $n=5$  and we would like the probability of the type I error in  $\bar{X}$  chart (falling a point under  $LCL_{\bar{X}}$  or above  $UCL_{\bar{X}}$  while no change has occurred) to be totally 0.05 then  $A_2 = 0.577$  and  $A_3 = 1.427$  are multiplied by  $\frac{Z_{0.025}}{3} = \frac{1.96}{3}$ .

### Example 7.2

Assume that probability limits rather than 3-sigma limits are to be used for  $\bar{X}$  chart using samples of size 4 with  $\bar{S}=2.1$   $\bar{\bar{X}}=10.2$ . If the probability of a point falling outside each limit, sampling from a normal universe were to be 0.001 without a change in the universe, find the upper and lower control limits of the  $\bar{X}$  chart.

**Solution:**

$$\frac{\alpha}{2} = 0.001. \quad C.L. = \bar{\bar{X}} \quad n = 4$$

$$UCL_{\bar{X}} = \bar{\bar{X}} + \frac{Z_{\alpha}}{3} A_3 \bar{s} \quad LCL_{\bar{X}} = \bar{\bar{X}} - \frac{Z_{\alpha}}{3} A_3 \bar{s}$$

From Table U  $A_3=1.64$ .

Using MATLAB  $Z_{0.001} = \text{normcdf}(.999) = 0.8411$

$$UCL_{\bar{X}} = 10.2 + \frac{Z_{0.001}}{3} \times 1.63 \times 2.1 \cong 11.16$$

$$LCL_{\bar{X}} = 10.2 - \frac{Z_{0.001}}{3} \times 1.63 \times 2.1 \cong 9.24$$

$$C.L. = \bar{\bar{X}} = 10.2$$

If subgroups data were given in the example the command

`xbarplot(data, 1 -  $\alpha$ )` in MATLAB would draw the  $\bar{X}$  chart.

### 7.2.3 S Control Chart with probability limits

Assuming the production process is normally distributed, non -3-sigma control limits for S control charts based on type I error ( $\alpha$ ) are given by:

$$UCL_S = B_{1-\frac{\alpha}{2},n} \sigma$$

$$CL = c_4 \sigma$$

$$LCL_S = B_{\frac{\alpha}{2},n} \sigma$$

where

$\sigma$  is the process standard deviation which may be estimated by  $\frac{\bar{s}}{c_4}$ ,

$\alpha$  is the probability of type I error i.e. the probability, that with no

change in the process or universe, a point will fall outside of the limits (Grant & Leavenworth, 1988 page 316)

$C_4$  is a ratio given by Table U

$B_{1-\frac{\alpha}{2},n}$  &  $B_{\frac{\alpha}{2},n}$  are factors whose values are given in Table 7-1 for 3  $\alpha$ 's.

**Table 7-1 Factors  $B_{1-\frac{\alpha}{2},n}$  &  $B_{\frac{\alpha}{2},n}$  for probability limits of S chart**  
(Grant & Leavenworth, 1988 page 317)

n	Lower limits			Upper limits		
	$B_{0.001}$	$B_{0.005}$	$B_{0.025}$	$B_{0.975}$	$B_{0.995}$	$B_{0.999}$
2	0.00	0.00	0.03	2.25	2.81	3.30
3	0.04	0.07	0.16	1.92	2.30	2.63
4	0.09	0.15	0.27	1.77	2.07	2.33
5	0.15	0.22	0.35	1.67	1.92	2.15
6	0.21	0.28	0.41	1.60	1.83	2.03
7	0.25	0.33	0.45	1.56	1.76	1.93
8	0.29	0.37	0.49	1.52	1.70	1.86
9	0.33	0.41	0.52	1.48	1.65	1.80
10	0.36	0.44	0.55	1.45	1.62	1.76

#### Example 7.3

The aimed-at standard deviation of a normally distributed process is  $\sigma = 0.2$ . Samples of size 5 is drawn from the process. Find the probability limits of the S-chart with  $\alpha = 0.002$ .

**Solution**

Since the distribution of the process is normal we could use the following relationship to calculate the probability limits

$$UCL_S = B_{0.999,5} \sigma = (2.15)(0.2) = 0.43$$

$$CL = c_4 \sigma = (0.94)(0.2) = 0.19$$

$$LCL_S = B_{0.001,5} \sigma = (0.15)(0.2) = 0.03$$

**7.2. 4 R chart with Probability limits**

Assuming the production process is normally distributed, non -3-sigma control limits for R control charts based on type one error are given by (Grant & Leavenworth, 1988 page 317):

$$UCL_R = D_{1-\frac{\alpha}{2},n} \sigma$$

$$C.L. = d_2 \sigma$$

$$LCL_R = D_{\frac{\alpha}{2},n} \sigma$$

Where

$\sigma$  is the process standard deviation,

$d_2$  is a ratio given by Table U,

$D_{1-\frac{\alpha}{2},n}$ ,  $D_{\frac{\alpha}{2},n}$  factors whose values are given in Table 7-2.

<b>Table 7-2<sup>1</sup> Factors <math>D_{1-\frac{\alpha}{2},n}</math>, <math>D_{\frac{\alpha}{2},n}</math> for probability limits of R chart</b> (Grant & Leavenworth, 1988 page 317):						
n	Lower limits factors			Upper limits factors		
	$D_{0.001}$	$D_{0.005}$	$D_{0.025}$	$D_{0.975}$	$D_{0.995}$	$D_{0.999}$
2	0.00	0.01	0.04	3.17	3.97	4.65
3	0.06	0.13	0.30	3.68	4.42	5.06
4	0.20	0.34	0.59	3.98	4.69	5.31
5	0.37	0.55	0.85	4.20	4.89	5.48
6	0.53	0.75	1.07	4.36	5.03	5.62
7	0.69	0.92	1.25	4.49	5.15	5.73
8	0.83	1.08	1.41	4.60	5.25	5.82
9	0.97	1.21	1.55	4.70	5.34	5.90
10	1.08	1.33	1.67	4.78	5.42	5.97

<sup>1</sup> Extensive tables Harter, 1960, Tables of Range....The Annals of Mathematical Statistics, vol. 31, p. 1122 [https://projecteuclid.org/download/pdf\\_1/euclid.aoms/1177705684](https://projecteuclid.org/download/pdf_1/euclid.aoms/1177705684)

### 7.2.5 $S^2$ chart with Probability limits

If an aimed-at or standard or actual value for the process variance ( $\sigma^2$ ) is given and the probability of type I error ( $\alpha$ ) is known, then the probability limits of  $S^2$  control chart for a normally distributed process is given by (Derman & Ross, 1997 page 97):

$$\Pr(S^2 > UCL_{S^2}) = \frac{\alpha}{2} \Rightarrow \Pr(\chi_{n-1}^2 > \frac{n-1}{\sigma^2} UCL_{S^2}) = \frac{n-1}{\sigma^2} UCL_{S^2} = \chi_{\alpha/2, n-1}^2$$

$$UCL_{S^2} = \frac{\sigma^2}{n-1} \times \chi_{\frac{\alpha}{2}, n-1}^2$$

$$C.L. = \sigma^2$$

$$LCL_{S^2} = \frac{\sigma^2}{n-1} \times \chi_{1-\frac{\alpha}{2}, n-1}^2$$

$\chi_{\frac{\alpha}{2}, n-1}^2$  and  $\chi_{1-\frac{\alpha}{2}, n-1}^2$  are read from Table 8 at the end of the book or could be calculated using MATLAB:

$$\chi_{\frac{\alpha}{2}, n-1}^2 = \text{chi2inv}\left(\frac{\alpha}{2}, n-1\right),$$

$$\chi_{1-\frac{\alpha}{2}, n-1}^2 = \text{chi2inv}\left(1 - \frac{\alpha}{2}, n-1\right).$$

$\sigma^2$ , if not given, may be estimated by the mean of sample variances i.e.  $\overline{S^2}$ .

**Example 7.4** (Derman & Ross, 1997)

Draw the  $S^2$  control chart having probability limits with  $\alpha = 0.27\%$  for monitoring a normally distributed process of which 20 samples of size 5 and the variances given in the following table.

i	1	2	3	4	5	6	7	8	9	10
$S^2$	17.6	19.3	6.2	9.8	7.5	21.4	11.1	26	14.4	38.4
i	11	12	13	14	15	16	17	18	19	20
$S^2$	16.4	15.6	9.9	16.2	12.2	67.2	63.3	15.9	17.4	12.6

#### Solution

The sample sizes are equal; the variance is estimated as follows:

$$\overline{S^2} = (17.6 + \dots + 12.6) / 20 = 20.92$$

The control limits are :

$$UCL_{S^2} = \frac{\overline{s^2}}{n-1} \times \chi_{\frac{\alpha}{2}, n-1}^2 = \frac{20.92}{5-1} \times 17.8 = 93.1$$

$$C.L. = \overline{s^2} = 20.92$$

$$LCL_{S^2} = \frac{\overline{s^2}}{n-1} \times \chi_{1-\frac{\alpha}{2}, n-1}^2 = \frac{20.92}{5-1} \times 0.106 = 0.554$$

$\chi_{\frac{\alpha}{2}, n-1}^2$  and  $\chi_{1-\frac{\alpha}{2}, n-1}^2$  were calculated using MATLAB:

$$\chi_{\frac{\alpha}{2}, n-1}^2 = \text{chi2inv}(0.00135, 4) = 0.106$$

$$\chi_{1-\frac{\alpha}{2}, n-1}^2 = \text{chi2inv}(1 - 0.00135, 4) = 17.8.$$

Figure 7-2 shows the control charts for  $S^2$  with the above control limits.

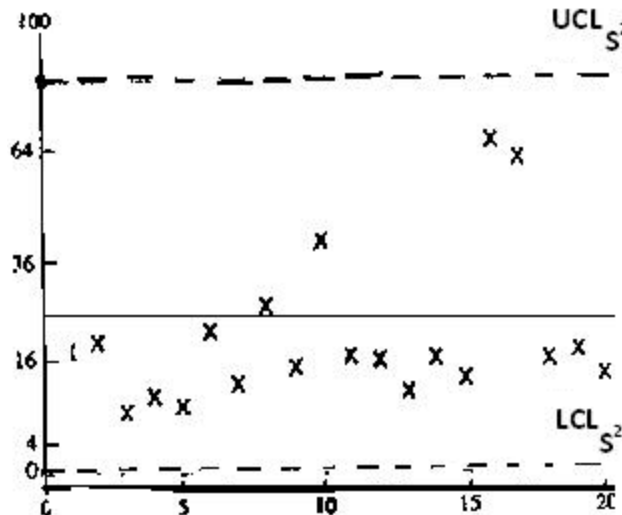


Fig. 7.2  $S^2$  Control Chart for Example 7.3.

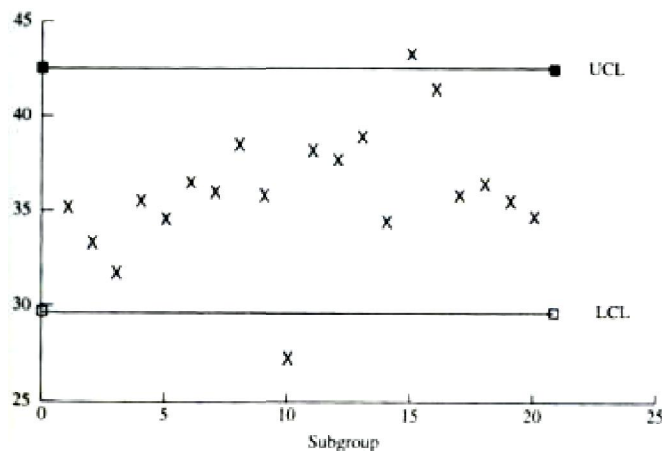


Fig. 7.3  $\bar{X}$  Control Chart for Example 7.3  
(Derman & Ross, 1977 page 97)



Another control chart related to this example i.e.  $\bar{X}$  control chart shown in Fig.7-3, as well as the following comment on the process from m Derman & Ross (1997), is worth paying attention:

" Note that  $X_{10}$  and  $X_{15}$  fall outside the  $\bar{X}$  control limits with the first value lying below  $LCL_{\bar{X}}$  and the second above  $UCL_{\bar{X}}$ . In addition,  $X_{16}$  is very close to its upper control limit. Hence, even though all of the sample variances lie within their control limits, it seems that the process has not yet established any type of statistical control for its average value—that is, it does not appear at present that the data values can be thought of as coming from some common normal population. We should thus delay for the moment any attempts to determine appropriate control charts for this system" (Derman & Ross, 1997 page 97).

## 7.2. 6 Chart for individual measurements

It is reminded that USL and LSL are upper and lower limits of the design specification. For example if the specification set by the designer is  $0.4037 \pm 0.0013$  we have  $USL=0.4050$ ,  $LSL=0.4024$ . Figure 7.4 shows, as a sample, a kind of chart that is not a control chart but sometimes are made from the individual measurements of each sample in Table 7-3. The 5 observations of every subgroup are plotted vertically on the chart with USL and LSL as chart limits

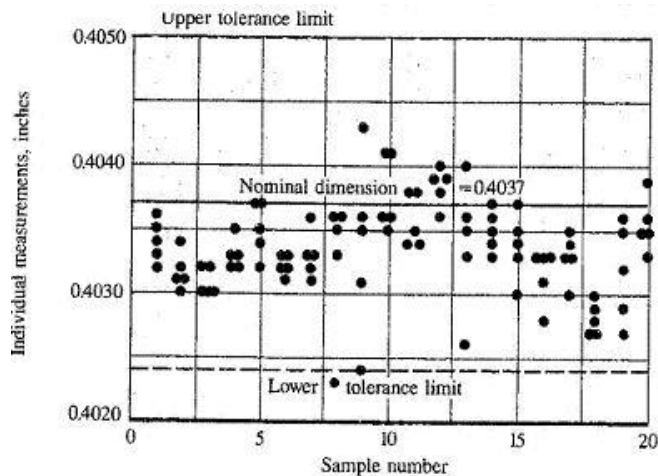


Fig. 7-4 Chart for individuals using LSL and USL

**Table7-3 Measurements of pitch diameter of threads on aircraft fittings**

Values are expressed in units of 0.0001 inch in excess of 0.4000 in. Dimension is specified as 0.4037±0.0013 inch

Sample No.	Measurement on each item of five items per hour					Average	Range
1	36	35	34	33	32	34.0	4
2	31	31	34	32	30	31.6	4
3	30	30	32	30	32	30.8	2
4	32	33	33	32	35	33.0	3
5	32	34	37	37	35	35.0	5
6	32	32	31	33	33	32.2	2
7	33	33	36	32	31	33.0	5
8	23	33	36	35	36	32.6	13
9	43	36	35	24	31	33.8	19
10	36	35	36	41	41	37.8	6
11	34	38	35	34	38	35.8	4
12	36	38	39	39	40	38.4	4
13	36	40	35	26	33	34.0	14
14	36	35	37	34	33	35.0	4
15	30	37	33	34	35	33.8	7
16	28	31	33	33	33	31.6	5
17	33	30	34	33	35	33.0	5
18	27	28	29	27	30	28.2	3
19	35	36	29	27	32	31.8	9
20	33	35	35	39	36	35.6	6
Total						671.0	124

The chart may be of interest to production supervision, but does not give the definite basis for action(Grant &Leavenworth, 1988 page 8).

## 7.2. 7

### Combination of Chart for individual X and Moving Range (I-MR chart)

In some processes it may be desired to plot X values in a kind of graph described here. In this case if process  $\sigma$  is not available, estimating  $\sigma$  from R is not possible, because each sample consists of one observation and no R is available; instead we use moving range(MR). Some examples of applications for individual X and Moving Range chart (I-MR chart) are certain chemical processes or slow industrial processes ; short production runs where data are scarce; destructive testing; special process test, and any process where individual measurements are necessary or expensive.

#### 7.2. 7.1 Definition of moving range(MR)

It is easier to illustrate moving rang than to define it. Suppose having a sample of size k, successive subgroups of size n i.e.

$\{x_j, x_{j+1}, \dots, x_{j+n-1}\}, j = 1, 2, \dots, k - n + 1$  are artificially constructed from the sample. The subgroups are

$$\{x_1, x_2, \dots, x_n\}$$

...

$$\{x_j, x_{j+1}, \dots, x_{j+n-1}\}$$

... and

$$\{x_{k-n+1}, \dots, x_k\}$$

Then the range of each subgroup, called moving range, is calculated:

$$MR_j = \max\{x_j, x_{j+1}, \dots, x_{j+n-1}\} - \min\{x_j, x_{j+1}, \dots, x_{j+n-1}\}, j = 1, 2, \dots, k - n + 1, n$$

is usually 2 or 3. If  $n=2$  then moving range is simply the difference between two consecutive data points.

X	MR(n=2)	MR(n=3)
$x_1$	$\rightarrow \{  x_2 - x_1 $ $\rightarrow \{  x_3 - x_2 $	$\rightarrow \{  x_{\max} - x_{\min} $
$x_2$		
$x_3$		
$\vdots$		
$x_j$	$\rightarrow \{  x_{j+1} - x_j $	$\rightarrow \{  x_{\max} - x_{\min} $
$x_{j+1}$		
$x_{j+2}$		
$\vdots$		
$x_k$	$\rightarrow x_k - x_{k-1}$	

### Example 7.5

Given the random sample  $(x_1 = 2.1, x_2 = 2, x_3 = 2.5, x_4 = 1.9, x_5 = 2.3, x_6 = 1.8)$  find MR's with  $n=3$ .

**Solution:**

$$MR_1 = \max\{2.1, 2, 2.5\} - \min\{2.1, 2, 2.5\} = 0.5$$

$$MR_2 = \max\{2, 2.5, 1.9\} - \min\{2, 2.5, 1.9\} = 0.6$$

$$MR_2 = 0.6, MR_3 = 0.6, MR_4 = 0.5$$

$$\overline{MR} = \frac{0.5 + 0.6 + 0.6 + 0.5}{4} = 0.55$$

**End of Example.**

### 7.2. 7.2 Chart for individual measurements- $\mu$ & $\sigma$ : known

If the standard or aimed-at values for process mean and standard deviation ( $\mu$  &  $\sigma$ ) are known, then:

$$UCL_X = \mu + 3\sigma$$

$$CL = \mu$$

$$LCL_X = \mu - 3\sigma$$

### 7.2. 7.3 Chart for individual measurements- $\mu$ & $\sigma$ : unknown

If  $\mu$  &  $\sigma$  are not known or cannot be determined, they are estimated as:

$$\hat{\mu} = \bar{X},$$

$$\hat{\sigma} = \frac{\overline{MR}}{d_2}, \text{ if the process is normally distributed.}$$

where

$\bar{X}$  is the mean of  $x_1, x_2, \dots, x_n$

$$\overline{MR} = \frac{MR_1 + MR_2 + \dots}{\text{the number of MR/s}} = \frac{\sum_{j=1}^{k-n-1} MR_j}{k-n-1}$$

$d_2$  a factor obtainable from Table U given the size of moving subgroups i.e. n.

The following example illustrates the control chart for individual observations using moving ranges<sup>1</sup>

#### Example 7.6

A new chemical process was studied to monitor its flow rate. The following 10 measurements have been reported

49.6   47.6   49.9   51.3   47.8   51.2   52.6   52.4   53.6   52.1

construct a control chart for the individual flow rates. Use moving range concept. Suppose the flow rate is normally distributed.

#### Solution

Column 3 of Table 7-4 shows the calculated moving ranges with n=2

Since

$$UCL_X = \mu + 3\sigma \quad C.L. = \mu \quad LCL_X = \mu - 3\sigma$$

Short term estimates for  $\mu$  &  $\sigma$  are as follows:

<sup>1</sup>(<http://www.itl.nist.gov/div898/handbook/pmc/section3/pmc322.htm>).

<b>Table7-4 Data on the flow-rate of a process</b>		
Measurement No. (j)	Flowrate( $x_j$ )	$MR_j(n=2)$
1	49.6	
2	47.6	49.6-47.6=2
3	49.9	49.9-47.6=2.3
4	51.3	1.4
5	47.8	3.5
6	51.2	3.4
7	52.6	1.4
8	52.4	.2
9	53.6	1.2
10	52.1	1.5

$$\hat{\mu} = \bar{X} = \frac{49.6 + \dots + 52.1}{10} = 50.81, \hat{\sigma} = \frac{\overline{MR}}{d_2} = \frac{\frac{2 + 2.3 + \dots + 1.5}{9}}{1.128} = 1.67$$

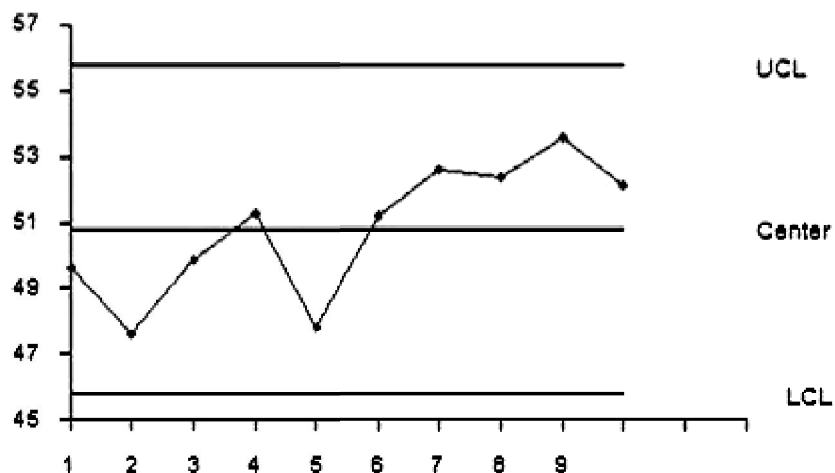
Then the control limits are given by:

$$UCL_X = \hat{\mu} + 3\hat{\sigma} = 50.81 + 3(1.67) = 55.80$$

$$C.L. = 50.81$$

$$LCL_X = \hat{\mu} - 3\hat{\sigma} = 50.81 - 3(1.67) = 45.82$$

Control Chart for individuals



**Fig. 7-5 Control chart for Example 7.5 using moving range**  
(<http://www.itl.nist.gov/div898/handbook/pmc/section3/pmc322.htm>)

Figure 7-5 shows the control chart. Since none of the plotted points fall outside either the UCL or LCL and no special pattern is observed. The chart indicates the process is stable and the flow rate is in control, It is worth mentioning another way to construct the individuals chart is by using the

sample standard deviation to estimate the process  $\sigma$ . Then, if the process is normally distributed, we can obtain the control limits of the chart from

$$UCL_X = \bar{X} + 3 \frac{S}{c_4}$$

$$CL = \bar{X}$$

$$LCL_X = \bar{X} - 3 \frac{S}{c_4}$$

where

$S$  is sample standard deviation

$c_4$  a factor obtainable from Table U given the size of the sample (individual measurements).

For example the standard deviation of the above individual measurements of flowrates i.e.  $x_1=49.6, \dots, x_{10}=52.1$  is  $S=2.0339$  and from Table U for size 10,  $c_4 = 0.9727$ ;

then

$$UCL_X = 50.81 + 3 \times \frac{2.0399}{0.9727} = 57.11$$

$$CL = 50.81$$

$$LCL_X = 50.81 - 3 \times \frac{2.0399}{0.9727} = 44.52$$

The plot of the individual chart with these limits is left as an exercise to the students.

## 7.2. 8 Moving Range Chart

The moving range (MR) chart is used to monitor the variability of a process for which samples are of size 1 is available. For this purpose moving ranges are calculated and the lower and upper control limits for chart are calculated using:

$$UCL_{MR} = D_4 \times \overline{MR},$$

$$CL = \overline{MR},$$

$$LCL_{MR} = D_3 \times \overline{MR};$$

$D_4$  &  $D_3$  are factors obtainable from Table U, given the size of moving subgroups i.e.  $n$ .

Of course if the aimed-at or standard process standard deviation ( $\sigma^2$ ) is given then

$$UCL_{MR} = D_2 \times \sigma,$$

$$CL = d_2 \sigma,$$

$$LCL_{MR} = D_1 \times \sigma;$$

$D_2$  &  $D_1$  are factors obtainable from Table U, given the size of moving subgroups i.e.  $n$ .

Then the MR's are inserted into the chart and ordinary interpretation takes place.

## 7.2. 9 $\bar{X}$ chart with a Linear Trend

(Grant & Leavenworth, 1988 page 323; Besterfield, 1990 page 112)

When the plotted points of a chart have an upward or downward trend, it can be attributed to an unnatural pattern of variation or to a normal pattern of variation such as tool wear. In some machining operations tool wear occurs at a uniform rate over the period of use of a tool. This wear, considered to be normal, may be one of the factors influencing the average value of some dimension of the product manufactured and may be responsible for a trend in this average. Where subgroups are selected in a way that spaces them uniformly with respect to this wear, control charts for  $\bar{X}$  often look something like Fig. 7-4 which reflect die wear. As the die wears, the measurement gradually increases until it reaches **the upper reject limit** (URL). The die is then replaced or reworked.

The  $R$  [ or  $s$  ] chart, however, is likely to remain in control yielding constant estimate of process  $\sigma$  (based on Grant and Leavenworth, 1988, page 324); however the dispersion may also be increasing.

The central line and control limits for  $\bar{X}$  chart in such a case should be sloping rather than horizontal. The equation for central line is,  
 $\bar{X} = a + bh$ ,

where

$\bar{X}$  subgroup average and represents the vertical axis

$h$  subgroup number ( $h=0, \dots, k$ ) and represents the horizontal axis

$a$  &  $b$  are calculated as follows:

$$b = \frac{k \sum_{h=0}^k \bar{X}_h - \sum_{h=0}^k h \sum_{h=0}^k \bar{X}_h}{k \sum_{h=0}^k h^2 - (\sum_{h=0}^k h)^2} \quad a = \frac{\sum_{h=0}^k \bar{X}_h - b \sum_{h=0}^k h}{k}$$

$\bar{X}_k$  is the mean of sample number  $h$ .

The control limits are sloping lines parallel to the central line;  $A_2 \bar{R}$  above and  $-A_2 \bar{R}$  below it.

**Example 7.7**

On certain operations in a machine shop it was a common experience for a definite steady trend in the average value of a dimension of the parts to be caused by rapid tool wear. Subgroup of size 2 were used in sampling which resulted in the following  $\bar{X}$  & R for the dimension. Construct the trended  $\bar{X}$  chart with URL=60 and LRL=10

Sample # (h)	0	1	2	3	4	5	6	7	8	9	10	11	12
$\bar{X}$	17	18	24	31	33	37	33	36	41	44	56	57	54
R	2	3	1	5	4	2	4	0	5	10	3	4	2

**Solution**

The following commands in MATLAB were used to draw Fig. 7.4

```
>>Xbar=[ 17 18 24 31 33 37 33 36 41 44 56 57 54]';
>>R=[2 3 1 5 4 2 4 0 5 10 3 4 2]';
>>h=(0:1:12)';
>>H=[ones(size(h)) h];regress(Xbar, H)
17.2537 3.2912
```

Then the central line has the equation  $\bar{X} = 17.2527 + 3.29h$ ;

```
>>plot(h, Xbar, '+');
```

Since for  $n=2$  from Table U is  $A_2 = 1.88$ ; the following commands

were used to plot  $UCL_{\bar{X}}, LCL_{\bar{X}} = CL \pm A_2 \bar{R}$ .

```
>>CL=17.25+3.29*h; UCL=CL+1.88*(mean(R)); LCL=CL-
1.88*(mean(R));
```

```
>>hold on;plot(h,CL);plot(h,UCL);plot(h,LCL)
```

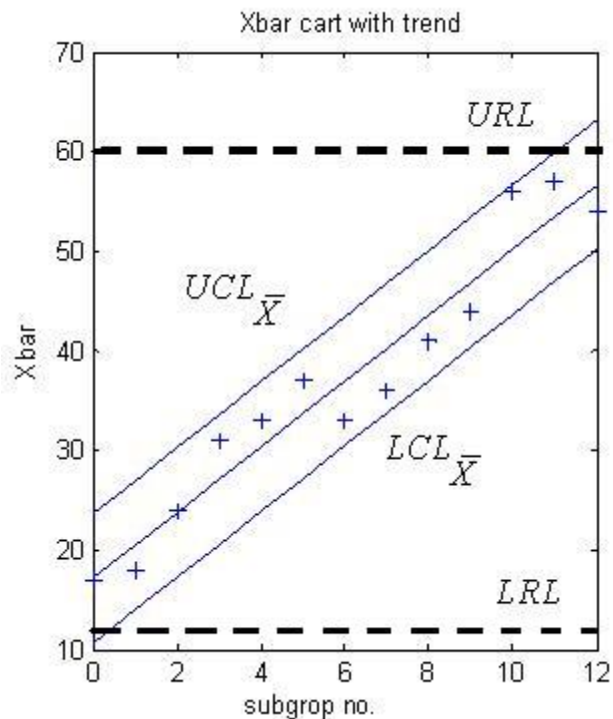
```
>>y=0*h+60;hold on; plot(h,y);% plots upper rejection limit
```

```
>>y=0*h+12;hold on; plot(h,y); %plots lower rejection limit
```

```
>>hold on; plot(h,Xbar, '+')
```

Fig.7-4 shows the trended chart.





**Fig. 7-4 Illustration of an  $\bar{X}$  chart with trend.**

the correlation coefficient between  $\bar{X}$  and sample number is  $r_{h\bar{X}} = 96.57\%$  as obtained from MATLAB:

```
>>corr=corrcoef(Xbar, h);r=corr(1,2)
r=0.9657
```

Those readers engaged in these type of charts, for a full discussion on the charts including interpreting the points falling outside the control limits (UCL & LCL) could refer to Grant and Leavenworth (1988) page 326.

There are other control charts for process mean including EWMA; Modified Shewhart – EWMA(Klein, 1997) and CUSUM for  $\bar{X}$ . EWMA control chart is more sensitive to some pattern of variations and less sensitive to some others with respect to  $\bar{X}$  chart. EWMA chart is dealt with in many references including Montgomery and Runger(1990) and MATLAB Help.

## Exercises

**7.1** (9-1 Grant and Leavenworth, 1988 page 336)

A certain company manufactures electronic components for television sets. One particular component is made to a critical length of 0.450 inch. On the basis of past production experiences, the standard deviation of this dimension is 0.010 in. Because of the critical nature of the dimension, the quality control group maintains warning limits on  $\bar{X}$  control chart as well as the normal 3-sigma control limits. The  $\bar{X}$  chart is based on subgroups of size four, and warning limits are maintained at two standard deviations from the mean. Compute the warning limits and the control limits for the  $\bar{X}$  chart.

Answer: 0.460, 0.440, 0.465, 0.435

**7.2** Problem 7.1 asked for the calculation of both warning limits and control limits for an  $\bar{X}$  chart. Using those data

(a) What is the probability that a subgroup average will exceed the UWL but not exceed the UCL when the process is correctly centered?

(b) Supervisor should be notified

(i) if 2 successive subgroup averages exceed one of the warning limits or

(ii) if one subgroup average exceeds either of the control limits. What is the probability that, when there has been no change in the process, the supervisor will have to be notified because of (i) or (ii)?

(c) If the process suddenly shifts to 0.460 with no change in the standard deviation, what is the probability that 2 successive points will exceed the upper warning limit but not exceed the upper control limit?

Answer: a) 0.0214 b) 0.0005 0.0014 c) 0.1165

**7.3** (9-7 Grant and Leavenworth, 1988 page 338)

Both 3-sigma control limits and 2-sigma warning limits are maintained on an  $\bar{X}$  chart for the internal diameter of a shock tube assembly. The aimed-at value is 35.5 mm and  $\sigma$  is 0.25 mm. The subgroup size is 4.

a) calculate UCL, LCL, UWL, LWL for an  $\bar{X}$  chart on this process.

b) Calculate UCL, LCL for an R chart to control dispersion

c) if the actual mean of the process is 35.75, find the probability that any given point would fall above the UCL and the probability that two points in succession would fall between UWL and LWL.

**7.4-**(9-20 Grant and Leavenworth, 1988 page 340)

Control charts for  $\bar{X}$  and  $s$  are maintained on the weight in ounces of the contents of a cereal container. The subgroup size is 8. After 18 subgroups,

$$\sum \bar{X} = 595.8 \text{ and } \sum s = 8.24$$

- Compute 3-sigma limits for the  $\bar{X}$  chart and estimate the value of  $\sigma$ .
- A decision is made to use 95% probability limits on the  $\bar{X}$  chart rather than 3-sigma limits. What would be the location of these limits?
- If the mean shifts to 33.4 what is the probability of detecting this shift on the first subgroup after the shift occurs, if the 3-sigma limits are used?
- What is the probability of detecting the shift described in (c) if 95% probability limits are used?

**7.5-**(9-21 Grant and Leavenworth, 1988 page 340)

A critical dimension in the production of a certain part specifies a length of  $60.00 \pm 0.07$  mm. It is proposed to use  $\bar{X}$  and  $R$  charts employing 99% probability limits to control this dimension. After 30 subgroups of 4 measurements each have been taken  $\sum \bar{X} = 1799.40$  and  $\sum R = 2.24$ .

- Calculate trial limits for this process.
- Calculate  $\sigma$  and estimate the proportion of product which does not meet specifications. Assume that the distribution of unit dimensions is approximately normal.

**7.6-**(Based on 9-24 Grant and Leavenworth, 1988 page 340)

Summarized below are daily analyses of  $\text{CO}_2$  as  $\text{CaO}$  at the intermediate stage of a chemical manufacturing process. Compute 3-day moving ranges for these data and establish a chart for individual  $X$  with limits  $\mu \pm 3\sigma$  and a chart for moving ranges to monitor this process basing the control limits on a standard process average of 0.660 and  $\bar{R}$  of 0.075.

Date	Per cent $\text{CO}_2$ as $\text{CaO}$	Date	Per cent $\text{CO}_2$ as $\text{CaO}$	Date	Per cent $\text{CO}_2$ as $\text{CaO}$
<b>5/ 1</b>	0.53	14	0.65	27	0.71
2	0.62	15	0.59	28	0.68
3	0.63	16	0.60	29	0.74
4	0.54	17	0.69	30	0.66
5	0.50	18	0.65	31	0.67
6	0.50	19	0.65	<b>6/ 1</b>	0.67
7	0.51	20	0.67	2	0.68
8	0.53	21	0.71	3	0.72
9	0.56	22	0.78	4	0.70
10	0.64	23	0.82	5	0.67
11	0.57	24	0.82	6	0.69
12	0.56	25	0.88	7	0.68
13	0.55	26	0.82		

**Thinking revives the alert heart and  
is a key to the doors of wisdom**



# Chapter 8

## Cumulative Sum Control Charts

## Chapter 8 CUSUM Control Chart

### Aims

Cumulative sum (CuSuM) control chart which could identify slight shifts in a process and are applicable to both variables and attributes. As a sample CUSUM chart for monitoring the process mean is introduced In this chapter.

### Cumulative sum control chart

A shortcoming of Shewhart's control charts is their low sensitivity to slight shifts of process parameter. An alternative to the charts is the cumulative sum control chart frequently called CuSum chart and sometimes abbreviated as CUSUM. The capability of detecting slight shifts by this chart is more than that of Shewhart's.

A CuSum control chart plots the cumulative sum of " the deviation of sample value( $\bar{X}$ ,  $R$ ,  $s$ ,  $p$ , ...) from a standard or aimed-at value". This kind of control chart has been used for monitoring process mean, standard deviation, percent defectives,....

### 8.1 $\bar{X}$ CuSum chart

$\bar{X}$  cumulative sum control chart used for monitoring the process mean. To illustrate how the chart is constructed suppose  $\bar{X}_i$  is the mean of  $i^{\text{th}}$  sample of size  $n$  ( $n \geq 1$ ) and  $\mu_0$  is the aimed-at process mean. This chart plots the values of cumulative sums ( $S_i$ )<sup>1</sup> i.e.

$$S_i = \sum_{j=1}^i (\bar{X}_j - \mu_0) = \sum_{j=1}^{i-1} (\bar{X}_j - \mu_0) + (\bar{X}_i - \mu_0)$$

or calculated as follows:

---

<sup>1</sup> Do not confuse  $S_i$  with the standard deviation of  $i^{\text{th}}$  sample; it is the cumulative sum of  $\bar{X} - \mu_0$  from the first to the  $i^{\text{th}}$  sample.

$$S_i = S_{i-1} + \bar{X}_i - \mu_0, \quad S_0 = 0.$$

Since the values of  $S_i$  are plotted on the CuSum charts and these values are gained from more than one sample, these charts are more sensitive to slight shifts in the process. An application of this control chart is to chemical industries where samples are usually of size one.

The values of  $S_i$  could be positive, negative or zero. If the production process is in-control about  $\mu_0$ , the values fluctuate around zero. Then if the actual mean of the process has shifted to a higher value, the CuSum chart shows an increasing trend; and if the actual process mean has shifted to a lower value, a decreasing trend on the chart will be expected. The actual mean is the value whose estimate is  $\bar{X}$  i.e. the mean of the sample means. Therefore a trend on the CuSum chart indicates a shift in the process mean. The cause of the shift has to be detected and the corrective action taken to eliminate or fix the causes of non-conformities or other undesirable situations.

### sample size

The sample size for the subgroups required to construct this control chart has been recommended as (Ewan, 1963 referenced by Juran et al, 1974 page 23-26):

$$n = \frac{2.25\sigma^2}{D},$$

where

$\sigma^2$  is the process variance

$D$  the actual value of shift magnitude, either plus or minus, that must be detected with virtual certainty (Grant and Leavenworth, 1988 page 380).

If  $\sigma^2$  is not known, it can be replaced by the variance of a tentative sample ( $S^2$ ) drawn from the process.

### Control limits of CuSum chart for sample average $\bar{X}$

There are two basic ways to present CUSUM control chart: tabular<sup>1</sup> or algorithmic CUMSUM and V-mask; the latter which will be described in this book. Remember that in Schewart's control chart :

**1-**The value of the shift (shift of the process mean from a standard value) to be detected is not given; instead a desired probability for detecting the shift is considered. This probability is increased by replacing the 3-sigma

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<sup>1</sup> See references such as Montgomeri & Rungers (1990) page 879

control limits with say 2.5-sigma or 2-sigma limits; or by increasing the sample size(n) ; because the more the sample size(n), the less  $A, A_2, A_3, \dots$  and the more tightened the control limits.

**2-** There is no relationship between the scales of the 2 axes of the Schewart's chart but in CuSum control chart:

-The amount of the shift ,D, is necessary for constructing the chart and

- A scaling factor, denoted by  $y$ , related to the geometry of the control chart and V-mask dimension needs to be given.  $y$  satisfies the following relationship:

$$a=y \times b,$$

where

$a$  is the distance between 2 successive subgroups no. on the horizontal axis in mm,

$b$  is the distance between 2 successive units on the vertical axis in mm.

$y$  is a scaling factor selected for the geometry of the chart.

Usually it is a value from the interval  $(\frac{\sigma}{\sqrt{n}} \text{ to } 2.5 \frac{\sigma}{\sqrt{n}})$

As an illustration, let  $y$  is selected to be  $y=4$  and suppose rectangular coordinate paper on which equal spacing between lines in both direction is provided and each 5 mm represent one sample no; i.e. the length between 2 successive sample no. is 5 mm. Then each 5 mm on the vertical axis represent 4 units of the axis; in other words the length between 2 successive units on the vertical axis 1.25 mm.

As the second illustration, if  $y=0.5$  and the sample numbers on the horizontal axis are separated by five-mm intervals; units on the vertical axis have to be separated by  $b = \frac{5}{0.5} = 10$  mm intervals.

The magnitude of scaling factor i.e.  $y$  affects the power of the control chart for detecting shifts.

### 8.1.1 V-Mask procedure for constructing $\bar{X}$ CuSum chart<sup>1</sup>

After plotting the  $S_i$  points, indicate the location of the last  $S_i$  by P.

Draw the line PO parallel to the horizontal axis, right hand side.

The distance of OP in units of horizontal axis(one unit=the distance

---

<sup>1</sup> From Grant & Leavenworth(1988) page350



between 2 successive sample numbers) is given by:

$$d = \frac{E(\alpha)\sigma^2}{nD^2} = \frac{E(\alpha)\sigma_{\bar{X}}^2}{D^2} = \frac{E(\alpha)}{\delta^2},$$

where

$D$  is the required shift to be detected, a positive or negative value or sometimes obtained from  $D = \delta \times \sigma_{\bar{X}}$ ;

The unit of  $D$  is the same as that of standard or aimed-at process mean,

$\sigma_{\bar{X}}$  is the standard deviation of samples' means;

$\delta$  is a negative or positive number.

$\alpha$  is the probability of Type I error accepted for the problem

$E(\alpha) = 2 \ln \left( \frac{2}{\alpha} \right)^{-1}$ ; for some values of  $\alpha$  given in Table 8-1:

<b>Table 8-1 Some values of function <math>E(\alpha)</math></b>								
$\alpha$	0.0027	0.002	0.005	0.01	0.02	0.025	0.05	0.1
$E(\alpha)$	13.2153	13.8155	11.9829	10.5966	9.2103	8.7641	7.3778	5.9915

The process variance is estimated by sample variance. If the process is normally distributed, its standard deviation could be estimated from  $\bar{S}/c_4$ .

If the process variance is not available or cannot be estimated,  $\hat{\sigma}_x^2 =$

$$\sqrt{\frac{\sum_{i=1}^k \bar{X}_i^2 - k\bar{\bar{X}}^2}{k-1}}$$

could be replaced in  $d = \frac{E(\alpha)\sigma_{\bar{X}}^2}{D^2}$

where  $\bar{X}_i$  is the mean of  $i^{\text{th}}$  sample  $i=1,2,\dots,k$

Note that if say  $d=5$  the length of PO is 5 times the distance between 2 successive sample numbers on the horizontal axis not 5 mm. Calculate the

angle in degree using  $\theta = \arctg\left(\frac{D}{2y}\right)$  or

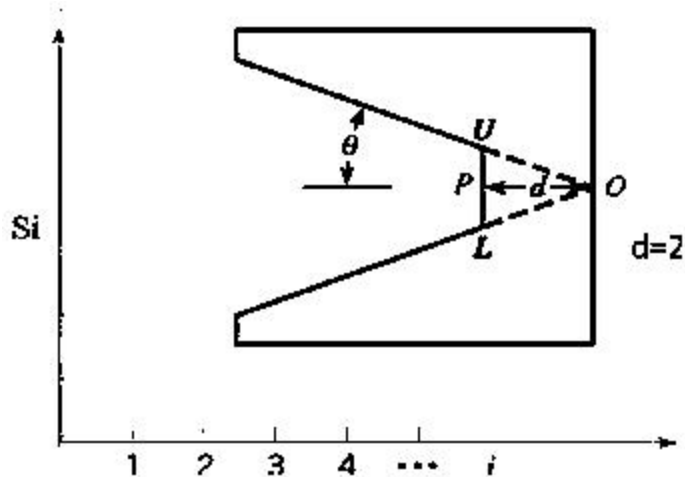
$$\theta = \arctg \frac{\delta \sigma}{2y\sqrt{n}}. \text{ If the unit is in radian, multiply it by } \frac{180}{\pi}$$

to convert into degree.

<sup>1</sup> If  $\alpha$  and  $\beta$  i.e. the type I & II errors probabilities of the CuSum chart is given then

$E(\alpha_{onesided}, \beta) = 2 \ln \frac{1-\beta}{\alpha/2}$

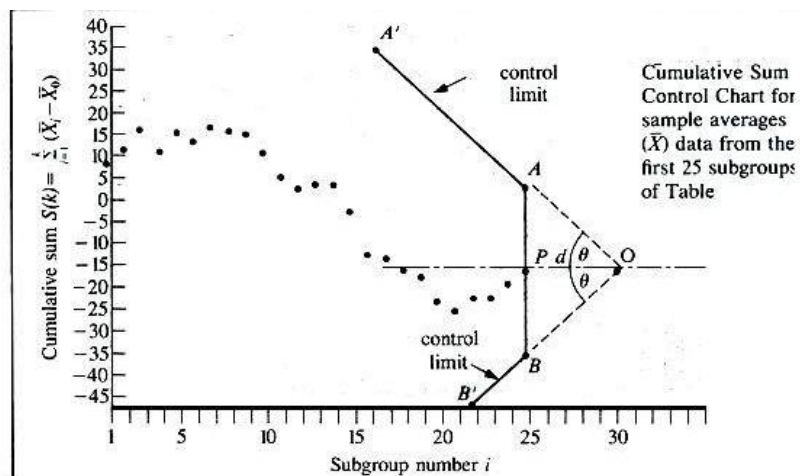
From point O draw the upper and lower control limits with angle  $\theta$  (see Fig. 8-1 as an illustration) .



**Fig. 8-1 A typical V-Mask**

([www2.fiu.edu/~kibriag/Stat5666/Handout/Chapter99.pdf](http://www2.fiu.edu/~kibriag/Stat5666/Handout/Chapter99.pdf))

Figure 8-2 shows a typical CUSUM control chart for monitoring process mean. The mean is said to be in control if the points on the chart are distributed randomly about zero and no points fall outside the limits.



**Fig. 8-2 A typical CUSUM chart for  $\bar{X}$**   
( Grant &Leavenworth,1988 page353)

CuSum control chart is useful especially for the cases where a sudden and sustained change(shift) is expected to occur in the process. As far as samples are distributed randomly around zero and none of fall outside the control chart, the chart shows the parameter is in control.

### Example 8.1

Twenty samples of size 1 ( $X_i$ ) in the following table shows the concentration of a special chemical solution;

Table 8-2 Calculations of Example 8.1										
I	1	2	3	4	5	6	7	8	9	10
$X_i$	102	98.4	98.3	98.4	102.0	98.5	99.0	97.7	100	98.1
$X_i - 99$	3	-4.2	-0.7	-0.6	3	-0.5	0	-1.3	1	-0.9
$S_i = S_{i-1} + X_i - 99$	0.3	-1.2	-1.9	-2.5	0.5	0.0	0.0	-1.3	-0.3	-1.2
I	11	12	13	14	15	16	17	18	19	20
$X_i$	101.3	98.8	101.1	98.4	97	96.7	100.3	101.4	97.2	101
$X_i - 99$	3.2	-3.0	1.2	-0.6	-2.0	-3.2	3.1	4.2	-1.8	0.2
$S_i = S_{i-1} + X_i - 99$	1.1	8.0	2.9	3.2	3.0	-2.0	-0.7	1.7	-0.1	1.9

Rows 4 and 8 of the above table shows the  $S_i$  values, computed from  $S_i = S_{i-1} + X_i - 99$  assuming  $\mu_0 = 99$ ,  $S_i$   $i = 1, \dots, 20$

where  $S_0 = 0$ .

If  $D = 2.5$ ,  $y = 2$  then

$$\theta = \arctan\left(\frac{2.5}{4}\right) = 0.5586 \text{ rad} = 0.5586 \times \frac{180}{\pi} \cong 32 \text{ deg}.$$

For computing  $d = \frac{E(\alpha) \times \sigma^2}{nD^2}$ ,  $\sigma$  has to be estimated. To do this the 20 observations are considered as one sample, and its standard deviation is obtained as  $s = 1.976$ . Since in this example  $\bar{X}_i = X_i$ ,  $\bar{\bar{X}} = \bar{X}$ . If the

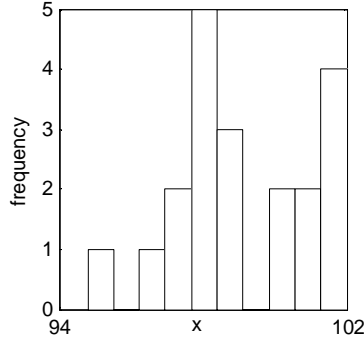
formula  $\sqrt{\frac{\sum_{i=1}^k \bar{X}_i^2 - k\bar{\bar{X}}^2}{k-1}}$ , is used the same value will be obtained for  $s$ .

Note that taking the 20 observations as a sample of size twenty and constructing the histogram as shown in Fig 8-3 does not indicate the observation are from a symmetric population. Therefore the estimates

$\hat{\sigma} = \frac{\overline{MR}}{d_2}$ ,  $\hat{\sigma} = \frac{s}{c_4}$  could not be used. Instead the process variance

$\sigma^2$  is estimated with  $s^2 = 1.976^2 \cong 3.91$ . Then

$d = \frac{E(1) \times \sigma^2}{nD^2} = \frac{5.9915 \times 3.91}{1 \times 2.5^2} = 3.74$  in terms of the horizontal axis units.



**Fig 8-3 The histogram of  $X_i$  values in Table 8-2**

Let the units of horizontal axis be 3 mm far from each other, then  $d = 3 \times 3.74 = 11.2$  mm.

Locate point P on the last  $S_i$  i.e. on  $S_{20}$ ,

Draw the line PO on the chart from point P to the right-hand side with the length 11.2 mm. Point O is obtained.

Draw limits from point O as lines of angle  $\theta = 32^\circ$  with respect to the horizon.

Figure 8-4 shows the  $\bar{X}$  CuSuM control chart for the data of Table 8-2 using Minitab<sup>1</sup>. The points are distributed randomly about zero in Fig 8-3 and no points has fallen outside the chart ; therefore the process mean is in control.

It is worth knowing that there are other control charts such as multivariate control charts. Hotelling's ( $T^2$ ) control chart is a well-known multivariate control chart. Multivariate control charts allow to monitor more than one quality characteristic simultaneously

<sup>1</sup> Path : Stat-control charts-time weighted charts –cusum-cusum options-type of cusum-two sided

in widow h give D value; in window k give y value

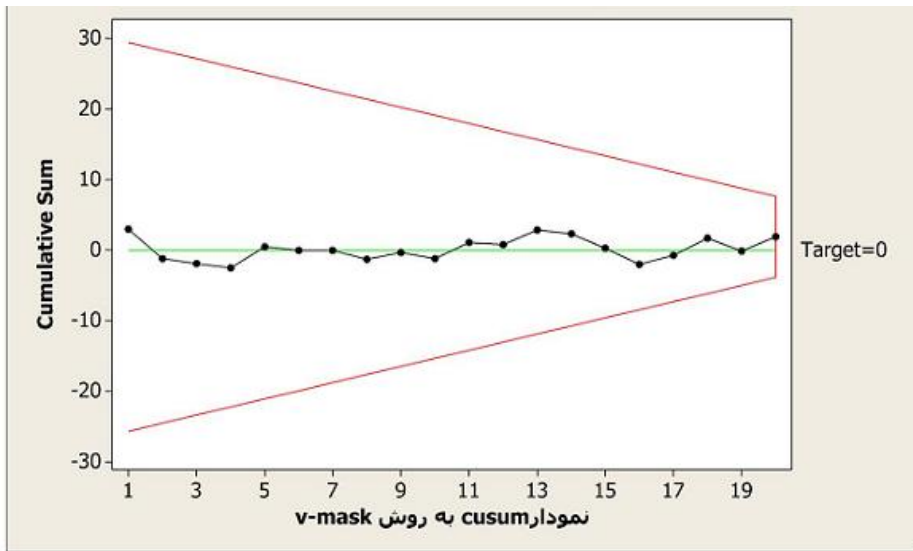


Fig. 8-4 The  $\bar{X}$  CUSUM chart of the data in Table 8-2

## Exercises

8.1. (Based on Grant and Leavenworth, 1988 page 359)

Prepare a cumulative sum control chart for  $\bar{X}$  using the following data:

Sample No.	Data			
1	200.5575	200.5655	200.9366	200.4809
2	200.5251	200.4925	200.3823	200.5119
3	200.5575	200.5655	200.9366	200.4809
4	200.2707	200.5349	200.4727	200.3335
5	200.7382	200.4627	200.5228	200.5589

Upper and lower specifications are: 201.5 and 199.5 in microseconds (ms). Use nominal value 200.5 and a risk level  $\alpha$  of 0.0027 (corresponding to 3-sigma limits on a Shewhart control charts). Taking  $D=7.8$  ms and  $y=5$ , plot the CuSum control chart. Is the process mean in control? Plot the  $\bar{X}$  Shewhart control chart for the data and compare its results with those obtained from the CuSum chart for  $\bar{X}$ .

### 8.2.

Find the procedure of plotting CuSum for R from internet or references such as Grant & Leanvenworth (1988) page 355, and plot the CuSum for R for the data of Table 8-2. Use  $\frac{\bar{R}}{d_2}$  as the standard value for process standard deviation ( $\sigma_0$ ). Shift of  $\sigma$  to  $1.6\sigma_0$  has to be detected. Draw the Shewhart R chart too, and compare the 2 charts results.

### 8.3.

Learn the tabular CuSuM chart procedure and construct an  $\bar{X}$  CuSuM tabular chart for the data given in Table 8-2 above.

**Planning with prudence  
safeguards you against regret**

# Chapter 9

## Some Aspects of Tolerance Interrelationship; Precision, Accuracy & Errors in Measurement

## Chapter 9 Some Aspects of Tolerance; Precision, Accuracy & Measurement Errors

### Aims

This chapter deals with the concepts of precision, accuracy and errors in measuring systems. Some aspects of tolerance interrelationship are also briefly discussed.

### 9.1 Relationship of assembled product tolerance & components tolerance

Suppose, using series configuration, n independent components with higher & lower specification limits  $U_i, L_i$  and mean  $\mu_i$  and standard deviation  $\sigma_i$  have been assembled into one product with higher & lower specification limits  $U_{sum}, L_{sum}$ , mean  $\mu_{sum}$  and standard deviation  $\sigma_{sum}$ :

$$\mu_{sum} = \sum \mu_i,$$

$$\sigma_{sum}^2 = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2.$$

Now if  $(U - L)_{sum} = k\sigma_{sum}$ ,  $U_i - L_i = (U - L)_i = k\sigma_i$

and k is the same for all n components, then

$$\sigma_{sum} = \sqrt{\sigma_1^2 + \dots + \sigma_n^2} \Rightarrow$$

$$\frac{(U - L)_{sum}}{k} = \sqrt{\frac{(U - L)_1^2}{k^2} + \dots + \frac{(U - L)_n^2}{k^2}} \Rightarrow (U - L)_{sum} = \sqrt{\sum_{i=1}^n (U - L)_i^2}.$$

Let  $\Delta = \frac{(U - L)_{sum}}{2}$  and  $t_i = \frac{(U - L)_i}{2}$  then:

$$\Delta = \sqrt{\sum_{i=1}^n t_i^2}.$$

Suppose the higher & lower specification limits of a dimension of a product are U & L and the dimension is normally distributed with mean  $\frac{U+L}{2}$  and the standard deviation  $\sigma$  is . It could be shown that if  $\alpha$  is the proportion of the distribution falling outside U, L and  $(U - L) = k\sigma$ , then k is given by  $k = 2 \times Z_{\frac{\alpha}{2}}$ .



### Example 9.1

$n$  similar independent components with specification  $\mu \pm t$  are to be assembled in series configuration. The assembled product must have the specification  $\mu_{sum} \pm \Delta$ . What is the relationship between  $\Delta$  and  $t$ ,  $\mu_{sum}$  and  $\mu$ ?

$$\mu_{sum} = n \times \mu$$

$$\text{Assume } \sigma_{part} = \frac{(U-L)_{part}}{k} \quad \text{and } \sigma_{sum} = \frac{(U-L)_{sum}}{k}, \Rightarrow$$

$$\Delta = \sqrt{\sum_{i=1}^n t_i^2} \quad \text{or}$$

$$\Delta = \sqrt{nt^2} \Rightarrow \quad \Delta = t\sqrt{n}$$

### Example 9.2

Three components with the specifications  $1.45 \pm 0.0040$ ,  $0.865 \pm 0.0032$ ,  $1.17 \pm 0.0028$  are to be assembled into one product in series configuration, what is the specification of the assembled product?

It is assumed  $(U - L)_{sum} = k\sigma_{sum}$ ,  $U_i - L_i = (U - L)_i = k\sigma_i$ .

### Solution

$$\Delta = \sqrt{0.0040^2 + 0.0032^2 + 0.0028^2} = 0.0060$$

$$\mu_{sum} = 1.45 + 0.865 + 1.17 = 3.4850$$

Then the specification of assembled product is :  $3.4850 \pm 0.0060$ .

### Example 9.3

Twenty five similar components, having specification  $20 \pm 0.1$ , have been assembled in series configuration, find the specification of the assembled product.

### Solution:

$$\mu_{sum} = n \times \mu = 25 \times 20 = 500 \quad t=0.1$$

$$\Delta = t\sqrt{n}$$

$$\begin{cases} U \\ L \end{cases} = 20 \times 25 \pm = 500 \pm 0.1\sqrt{n} = 500 \pm 0.5.$$

### Example 9.4

Suppose a part is composed of 3 components. The part is required to have specification  $90.000 \pm 0.250$ . What should be the specification of each component?

**Solution:**

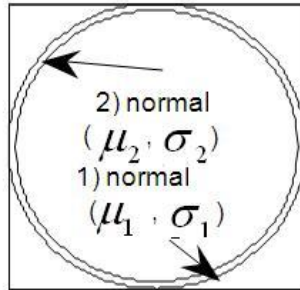
Assuming the same standard deviation for the 3 components:

For each component: since,  $t = \frac{\Delta}{\sqrt{n}} = \frac{0.250}{\sqrt{3}} 0.144$

the specification would be  $30 \pm 0.144$ .

## 9.2 On the difference between independent variables

Suppose a shaft with a cross section normally distributed  $(\mu_2, \sigma_2)$  is fitted into a bearing with internal diameter normally distributed  $(\mu_1, \sigma_1)$ . The difference between the two is enlarged in Fig. 9-1.



**Fig. 9-1 the enlarged difference between a shaft and a bearing diameters.**

The cross section of the space between the shaft and the bearing is a random variable normally distributed with mean  $\mu_{1-2} = \mu_1 - \mu_2$  and standard deviation of  $\sigma_{1-2} = \sqrt{\sigma_1^2 + \sigma_2^2}$ .

## 9.3 Error, precision and accuracy in measurement

When a measurement is made, the goal is to determine the true value of a quantity such as length, mass, time etc. In this regard the concepts of measurement error and precision & accuracy of measurement are dealt briefly here.

### 9.3.1 Measurement Error

To determine the size of a characteristic, we have to measure. Measurement is subject to error such that:

Measured value= true value+ measurement error.

Suppose a dimension of a unit of part, say, its internal diameter, is measured several times with the same(changeless) measuring procedure (consisting of changeless person or team, changeless device and changeless method). It is obvious that the true value of this unit is constant, therefore the standard deviation of the values obtained from the measuring method is

equal to the standard deviation of the method 's error. Therefore when several repeated measurements of a dimension of one unit of part is available, one could calculate the standard deviation of the measured values. The standard deviation of the measurement error is equal to standard deviation the measurements.

### 9.3.1.1 Error Components

The error of measurement has 2 important components: repeatability and reproducibility.

#### Repeatability

Repeatability refers to the variability of measurements of the same part by the same operator with the same equipment and the same method across all trials; thus, repeatability refers to the variation (measurement errors) in the measurement equipment. This component of the measurement variability is also referred to as equipment or gage variation. The less the variance of these measurements, the less the variance of the repeatability and the error.

If the operator has come up with a sample of n measurements which are normally distributed, then

$$\sigma_{\text{repeatability}} = \frac{S}{C_4} \text{ or } \frac{R}{d_2}$$

where

R is the range of the sample

S is the standard deviation of the sample

C<sub>4</sub> and d<sub>2</sub>, which depend on n, are coefficients available from Table U.

#### Reproducibility

Suppose there are several operators and each operator measures the same part with the same equipment and the same method several times. Reproducibility refers to the variability of the averages of measurements of the same part with the same equipment and the same method across different operators; thus, reproducibility is also sometimes referred to as appraiser variation. The less the variance of these averages, the less the variance of the reproducibility and the error.

It is worth knowing:

- 1) Getting more training by operators makes the measurements more smooth and reduces the error of reproducibility; but using exact measuring devices reduces the error of repeatability.
- 2) When the same method (including the same operator with the same device and the same procedure) measures a product item( not only one unit of it), the observed variations in the measurements are of two kinds:

Variations due to one unit to another, Variations due to the method; such that

$$\sigma_{mv} = \sqrt{\sigma_{tv}^2 + \sigma_e^2}$$

Where

$\sigma_{mv}$  is the standard deviation of the measured value,

$\sigma_{tv}^2$

is the variance of the measured values of one characteristic of different units of one product ; the variance of the product process or population,

$\sigma_e^2$

is the variance of measurement error.

### 9.3.1.2 Estimating the variance and the standard deviation of measurement error

To estimate the variance of a measuring system error ( $\sigma_e^2$ ), the following relationship could be used:

$$\sigma_e^2 = \sigma_{repeatability}^2 + \sigma_{reproducibility}^2.$$

What follows describes how to estimate each of the two components.

Suppose  $m$  operator, and a sample of  $k$  products are at hand. Each operator measures the  $i^{\text{th}}$  product ( $i = 1, 2, \dots, k$ )  $n$  times as indicated in Table 9-1. The number of such tables are as much as the number of the operators i.e.  $m$ ; in the table

$X_{ijl}$ :  $l^{\text{th}}$  measured value by  $j^{\text{th}}$  operator of the  $i^{\text{th}}$  unit of the product.

Table 9-1 measurements by operator No. J from sample of size k						
Product No.(i)	$k$ measurements obtained by $j^{\text{th}}$ operator from product No i			Variance ( $S_{ij}^2$ )	mean ( $\bar{X}_{ij}$ )	Range ( $R_{ij}$ )
1	$X_{1j1}$	...	$X_{1jk}$	$S_{1j}^2$		
.		...				
i	$X_{ij1}$	...	$X_{ijk}$	$S_{ij}^2$		
.		...				
n	$X_{kj1}$	...	$X_{kjk}$	$S_{nj}^2$	$\bar{X}_{nj}$	$R_{nj}$

The variances of repeatability and reproducibility could be estimated as follows using the  $m$  tables.

### Estimating $\sigma^2_{repeatability}$

To estimate the variance of repeatability, for each table calculate the mean of  $S_{ij}^2, i = 1, \dots, n$  and denote it by  $\overline{S_j^2}$ . Therefore we have as much as  $m \overline{S_j^2}$  calculated from

$$\overline{S_j^2} = \sum_{i=1}^n \frac{S_{ij}^2}{n}.$$

Then  $\sigma^2_{repeatability}$  is estimated as follows:

$$\hat{\sigma}^2_{repeatability} = \sum_{j=1}^m \frac{\overline{S_j^2}}{m}.$$

### Estimating $\sigma^2_{reproducibility}$

To estimate the variance of reproducibility, for each of the  $m$  table, calculate the mean of  $\bar{X}_{ij}, i = 1, \dots, n$  and denote it by  $\bar{\bar{X}}_j$ :

$$\bar{\bar{X}}_j = \sum_{i=1}^n \frac{\bar{X}_{ij}}{n}.$$

$\sigma^2_{reproducibility}$  is estimated as follows:

$$\hat{\sigma}^2_{reproducibility} = \sum_{j=1}^m \frac{(\bar{\bar{X}}_j - \bar{\bar{X}})^2}{m-1}$$

where

$\bar{\bar{X}}$  is the mean of  $\bar{\bar{X}}_j, j = 1, \dots, m$  i.e.

$$\bar{\bar{X}} = \sum_{j=1}^m \frac{\bar{\bar{X}}_j}{m} = \sum_i \sum_j \sum_{\ell} \frac{\bar{X}_{ij\ell}}{mnk}.$$

It is worth mentioning  $\bar{\bar{X}}$  is an estimate for the true value of the characteristic being measured.

### Estimating $\sigma_{reproducibility}$ & $\sigma_{repeatability}$ for normally distributed measurements

If the measurements are normally distributed, to estimate

$\sigma_{reproducibility}, \sigma_{repeatability}$  proceed as follows:

Suppose we have  $m$  operators and a sample of  $n$  products; each operator measures each of the products  $k$  times, as shown in Table 9-1.

For each operator there are  $n$  ranges(  $R_{ij}, i = 1, \dots, n$ ) and  $n$  standard deviations(  $S_{ij}, i = 1, \dots, n$ );

calculate the mean of ranges and standard deviations for each operator as follows:

$$\bar{R}_j = \sum_{i=1}^n \frac{R_{ij}}{n}, \quad \bar{S}_j = \sum_{i=1}^n \frac{S_{ij}}{n}$$

If the measurements are normally distributed, then

$$\hat{\sigma}_{\text{repeatability}} = \frac{\bar{R}}{d_2(k)} \quad \text{or} \quad \frac{\bar{S}}{C_4(k)}$$

where

$$\bar{R} = \sum_{j=1}^m \frac{\bar{R}_j}{m}, \quad \bar{S} = \sum_{j=1}^m \frac{\bar{S}_j}{m}.$$

$d_2(k)$  &  $C_4(k)$  are coefficients available from Table U at the end of the book; the coefficients depend on  $k$ , the number of trials each operator make to measure each unit of the sample.

If the measurements (all  $X_{ij\ell}$ ) are normally distributed, then

$$\hat{\sigma}_{\text{reproducibility}} = \frac{R_{\bar{X}}}{d_2(m)}$$

where

$$R_{\bar{X}} = \max(\bar{X}_j, j = 1, \dots, m) - \min(\bar{X}_j, j = 1, \dots, m)$$

$$\bar{X}_j = \sum_{i=1}^n \frac{\bar{X}_{ij}}{n}$$

$d_2(m)$  is a coefficient available from Table U.

If the measurements are normally distributed The estimate could also be calculated from:

$$\hat{\sigma}_{\text{reproducibility}} = \frac{\sqrt{\sum_{j=1}^m \frac{(\bar{X}_j - \bar{\bar{X}})^2}{m-1}}}{C_4(m)}.$$

$C_4(m)$  is a coefficient available from Table U.

### Example 9.5

A sample of size  $n=10$  has been taken from a production process. Three operators measure each of the units of the sample two times. The results are shown below:

Product No. (i)	Measured by operator 1 (j=1)					Measured by operator 2 (j=2)					Measured by operator 3 (j=3)				
	1	2	mean $\bar{X}_{ij}$	var $S_{ij}^2$	range $R_{ij}$	1	2	$\bar{X}_{ij}$	$S_{ij}^2$	$R_{ij}$		2	$\bar{X}_{ij}$	$S_{ij}^2$	$R_{ij}$
1	15	14	14.5	0.5	1	16	16	16.0	0	0	15	16	15.5	0.5	1
2	17	19	18.0	2.0	2	18	17	17.5	.5	1	18	18	18.0	0	0
3	16	16	16.0	0	0	17	15	16.0	2.	2	15	14	14.5	0.5	1
4	18	15	16.5	4.5	3	19	16	17.5	4.5	3	20	17	18.5	4.5	3
5	20	22	21.0	2.0	2	20	23	21.5	4.5	3	21	20	20.5	2.0	1
6	14	17	15.5	4.5	3	15	18	16.5	4.5	3	16	18	17.0	2.0	2
7	17	18	16.0	.5	1	17	18	17.5	.5	1	19	19	19.0	0	1
8	23	24	23.5	.5	1	21	24	22.5	4.5	3	22	23	22.5	.5	1
9	22	22	22.0	0	0	24	21	22.5	4.5	3	23	23	23.0	0	0
10=n	24	26	25	2.0	2	23	25	24	2.0	2	25	24	24.5	0.5	1
Mean			$\bar{X}_j = 18.8$	$\bar{S}_j^2 = 1.65$	$\bar{R}_j = 1.5$	Mean		$\bar{X}_j = 19.15$	$\bar{S}_j^2 = 2.75$	$\bar{R}_j = 2.1$	Mean		$\bar{X}_j = 19.25$	$\bar{S}_j^2 = 0.9$	$\bar{R}_j = 1.1$

To calculate error variance , first its components are calculated:

$$\hat{\sigma}_{repeatability}^2 = \sum_{j=1}^m \frac{\bar{S}_j^2}{m} = \frac{1.65+2.75+0.9}{3} = 1.7667,$$

$$\hat{\sigma}_{reproducibility}^2 = \sum_{j=1}^3 \frac{(\bar{X}_j - \bar{\bar{X}})^2}{3-1},$$

$$\bar{\bar{X}} = \sum_{j=1}^m \frac{\bar{X}_j}{m} = \frac{18.8+19.15+19.25}{3} = 19.0667$$

$$\widehat{\sigma_{reproducibility}^2} = \frac{(18.8^2+19.15^2+19.25^2)-3*19.0667^2}{2} = 0.0539$$

$$\sigma_{error|}^2 = \sigma_{reproducibility}^2 + \sigma_{repeatability}^2$$

$$\widehat{\sigma_{error|}^2} = 0.0539 + 1.7667 = 1.82.$$

If the measurements are **normally** distributed, the following formulae could be used to estimate the standard deviations ;

$$\hat{\sigma}_{repeatability} = \frac{\bar{R}}{d_2(k)} \text{ or } \frac{\bar{S}}{C_4(k)}, \quad \hat{\sigma}_{reproducibility} = \frac{R_{\bar{X}}}{d_2(m)}$$

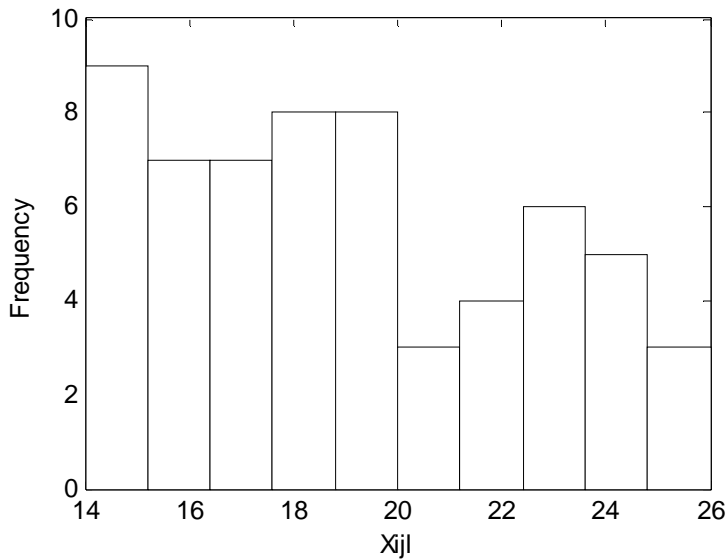
where

$$\bar{\bar{R}} = \sum_{j=1}^m \frac{\bar{R}_j}{m}$$

$$R_{\bar{\bar{X}}} = \max(\bar{\bar{X}}_j) - \min(\bar{\bar{X}}_j) \quad j=1,2,3=m.$$

$k=2$ , in our case.

As a quick check of normality, the histogram of the 60 measurements was built. The histogram, as shown below, is not symmetric, therefore the measurement are not normally distributed and we cannot use the above formulae for estimation. The histogram of the 60 measurements is shown below.



### 9.3.2 Precision and Accuracy of Methods of Measurement

Precision and accuracy are two important different measures regarding the ability of measurement methods in determining the value of any quality characteristic such as weight, dimensions, hardness, tensile strength, etc. A description of these two concepts follows.

#### Precision

In measuring a characteristic of a product, precision is the agreement of the individual values obtained in different measuring trials, with each other. The closer the values to each other, the the more precise the measuring method. In other words, since when measuring with a method the best value as the true value of the characteristic is the expected value



of the different measured values obtainable by the method, the closer the values to their mean, the more precise the method. Therefore precision is concerned with the amount of variability inherent in the data set of measurement; the less the variability, the more the precision of the method. Precision is sometimes defined as the reciprocal of the standard deviation of error ( $\frac{1}{\sigma_e}$ ) i.e.

$$\text{precision of measurement method} = \frac{1}{\sigma_e}.$$

Sometimes  $\bar{X} \pm S$  is given as the measure of precision, where  $\bar{X}$  and  $S$  are the mean and standard deviation of the different measured values given by the method from the same characteristic. The more the precision of the method, the closer the measured values and the less  $S$  and the width of  $\bar{X} \pm S$ . If the measured values are normally distributed, "we are reasonably sure (about 70% sure) that if the same measurement is repeated once more, that **next measurement** will be less than one standard deviation away from the average" (www2.southeastern.edu).

sometimes  $\bar{X} \pm \frac{S}{\sqrt{n}}$  is also computed in this subject. Suppose you have taken multiple measurements ( $n$  trials) with the mean  $\bar{X}$  and standard deviation  $s$ . If you do the entire experiment again with the same number of repetitions, be reasonably sure (about 70%, in normal distribution) the average value of the new experiment will be less than  $\frac{S}{\sqrt{n}}$  away from the average  $\bar{X}$  (extracted from www2.southeastern.edu)<sup>1</sup>.

## Accuracy

The accuracy of a measuring method is concerned with the agreement of the individual measured values by the method with the reference (true) value. Complete accuracy of a measuring method denotes the absence of *bias*, a deviation from the true value. The bias of a measurement method is defined as the difference between the expected value of the measurements obtainable from the method and the true value. If the mean of the measured values equals the true value or

$$E(\text{measured value}) = \text{the true value},$$

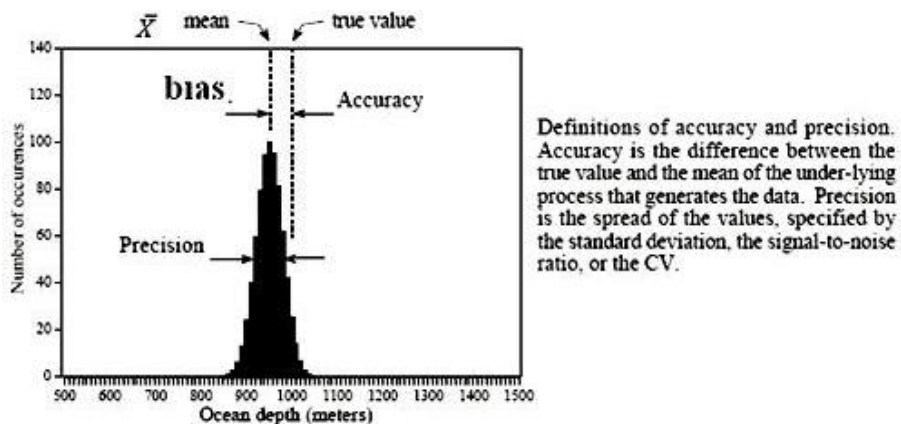
there is no bias in the method, and the method is completely accurate. The less the difference, the more accurate the method.

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<sup>1</sup>[https://www2.southeastern.edu/Academics/Faculty/rallain/plab193/labinfo/Error\\_Analysis/08\\_Multiple\\_Measurements.html](https://www2.southeastern.edu/Academics/Faculty/rallain/plab193/labinfo/Error_Analysis/08_Multiple_Measurements.html)

"In an accurate method, the average of value obtained from a set of measurements should differ from the true value by not more than would be expected a chance variation..."( Grant and Leavenworth, 1988 page 377). Figures 9-2 and 9-3 illustrate the concept of bias and help to the better understanding of precision and accuracy.

The problem here is how to find the true value in order to talk about the method accuracy . One way of knowing the true value is using an available highly precise method of measurement "that is believed to be without bias or using a part, in say a metrology laboratory, whose dimension is known.

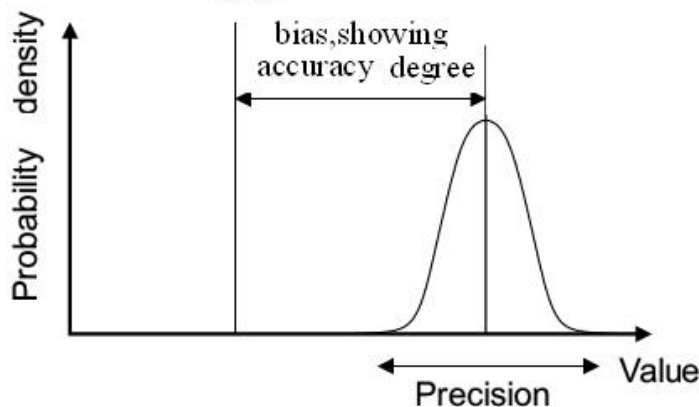


**Fig.9-2 Illustration of accuracy and precision**

(<http://www.dspguide.com/ch2/7.htm>)

Reference (true )

value



**Fig. 9-3. Precision and accuracy of a method with normally distributed measured values**

([https://en.wikipedia.org/wiki/Accuracy\\_and\\_precision](https://en.wikipedia.org/wiki/Accuracy_and_precision))

For more details about this subject and the application of control charts to it, see references such as Taylor(1997), Montgomeri(2005), Merrin(2017).

## Exercises

### 9.1

10 beginner students have measured a characteristic of a part. The results have been shown below. Trained students, using a better device and method, have obtained 421.58 as the mean of their measurements for the same part. Can you conclude anything about the precision and accuracy of the beginner measurers?

Student	Measurements		Discrepancy (Range)	Average value ( $\bar{X}$ )
A	421.59	421.37	0.22	421.48
B	421.73	421.65	0.08	421.69
C	421.70	422.28	0.58	421.99
D	421.79	421.72	0.07	421.76
E	421.64	421.67	0.03	421.66
F	421.58	421.56	0.02	421.57
G	421.53	421.69	0.16	421.61
H	421.52	421.66	0.14	421.59
I	421.69	421.68	0.01	421.68
J	421.76	421.69	0.07	421.72

### 9.2

The standard deviation of the measured values of a characteristic is 40. But the standard deviation of the measurement error of the characteristic is 15.

- Find the standard deviation of the true value of this characteristic.
- What amount of improvement in the standard deviation of the measured values of this technique is required to reduce the standard deviation of error to 2% of the standard deviation of the true value?

### 9.3

The weight of one specified part has been measured twenty times with one device and one method as recorded in the following table

i	$X_i$	$X_i - \bar{X}$	i	$X_i$	$X_i - \bar{X}$
1	0.8150	0.0002	11	0.8150	0.0002
2	0.8150	0.0002	12	0.8150	0.0002
3	0.8160	0.0012	13	0.8150	0.0002
4	0.8150	0.0002	14	0.8150	0.0002
5	0.8150	0.0002	15	0.8160	0.0002
6	0.8150	0.0002	16	0.8140	-0.0008
7	0.8140	-0.0008	17	0.8140	-0.0008
8	0.8140	-0.0008	18	0.8150	0.0002
9	0.8150	0.0002	19	0.8150	0.0002
10	0.8140	-0.0008	20	0.8140	-0.0008

What value do you recommend as an estimate for the true weight?

What is the standard deviation of the measurement error?

Specify the precision interval i.e.  $\bar{X} \pm S$  for this device

Answer  $\bar{X} \pm S = 0.8148 \pm 0.0006$ .

If the entire measurement trials are repeated and the mean of the measured values is calculated, give an interval which you are 70% confident the mean will fall in it.

#### 9.4

In figure 9-1, the inner part diameter is normally distributed with mean 0.5005 & standard deviation 0.0015 and the outer part has a normally distributed internal diameter with mean 0.5120 & standard deviation 0.0010. What is the probability that the inner part have a diameter greater than that of the internal part?

Ans.  $8.3548 \times 10^{-11}$ .

#### 9.5

Suppose a shaft with a cross section normally distributed  $(\mu_2, \sigma_2)$  is fitted into a bearing with internal diameter normally distributed  $(\mu_1, \sigma_1)$ .

What percent of the shafts cannot enter the bearings at all? Find the answer in terms of the parameters.

**Your best friend is your good mannerism**

# Chapter 10

## Acceptance Sampling:

Some Fundamental Concepts

&

Dodge-Romig Tables

## Chapter 10 Acceptance Sampling: Fundamental Concepts & Dodge-Romig Tables

### Aims

At the beginning of this chapter control methods used to accept or reject production lots are briefly introduced. Then the chapter deals with one of these control methods i.e. acceptance sampling in detail. The chapter also defines double and multiple sampling and introduces operating characteristic (OC) & AOQ curves which show the extent of the ability of a sampling procedure for distinguishing between good and bad lots. Important points of the OC curve and AOQ curves such as AQL and AOQL which help designing sampling plans, are also introduced here. Finally this chapter introduces Dodge-Romig tables to design sampling plans for accepting/rejecting a lot given (LTPD) LQL or AOQL.

### 10.1 Inspection Techniques for Accepting or Rejecting Production lots

To know the degree of the agreement of the quality of the input materials or produced goods with the given specifications or standards, several procedures and tests are used depending on the customer wants and needs and the possibility of implementing the procedure. Spot checking, 100% inspection, certification and acceptance sampling are 4 ways of checking product quality which are described below.

#### 10.1.1 Spot checking

Spot checking is a quality control system, in which often the quality of 10% of the product lot is checked (based on yes or no); the lot is accepted if the 10% -sample contains no defectives. No special arrangement is specified for taking the sample.

In a different method of spot checking, every say 50<sup>th</sup> or 20<sup>th</sup>... unit of the product is checked for acceptance. For example in a lot of 1000 units, every 50<sup>th</sup> is checked. i.e. 50<sup>th</sup>, 100<sup>th</sup>, 150<sup>th</sup>, ...totally 20 units are checked. If one defective was found, all units between this defective and the previous observed unit is 100% inspected; all defectives must be replaced with new ones. As an example if, say the 3<sup>rd</sup> one i.e. 150<sup>th</sup> unit is found defective, it is necessary to inspect all units from 101<sup>th</sup> to 149<sup>th</sup> individually. If none of these units is defective, only 150<sup>th</sup> unit will be replaced with a non-defective and the procedure of checking every 50<sup>th</sup> unit will continue; all the defectives between 101<sup>th</sup> and 149<sup>th</sup> units, if any, would be replaced by non-defectives.

Spot checking is costly and does not obey probability rules and has no scientific justification. Statistical sampling is preferred to spot checking.

### **10.1.2 100% inspection**

100% inspection at first glance might seem to be the best and most certain procedure to meet quality targets; however in practice this procedure has proved to be not that precise as it is expected. The monotony and repetitiveness of the work content might produce fatigue, dizziness and boredom. As a result all defectives are not detected. Another negative point of this procedure is the cost, which may cause excluding the inspection of some properties to reduce the cost. 100% inspection by automated means is likely but often costly.

### **10.1.3 Certification**

Supplier / vendor certification assures that a product is produced, packaged, and shipped under a controlled process resulting in consistent conformance to our requirements. The primary objective of the certification process is to assure consistent high quality and conformance to our requirements. If the supplier/vendor has good reputation for the quality of its product, certification eliminates or substantially reduce the need for final quality inspections by its customers.

### **10.1.4 Acceptance Sampling**

In this procedure, for accepting or rejecting lots random samples are taken from the lots and are inspected. The decision for accepting the lot will be made after inspecting the products of the sample based on an attribute characteristic or a variable characteristic. Accepting or rejecting lots on a variable basis will be discussed in chapter 13. The rest of this chapter and chapter 12 deal with acceptance sampling by attributes.

## **10.2 Acceptance Sampling by Attributes**

In this procedure, for accepting or rejecting lots random samples are taken from the lots and are inspected on an attribute basis. This procedure is called acceptance sampling by attributes. Sampling may be single, double and multiple.

Some examples of attribute inspection are:

- Checking whether a product is defective or not,
- Checking whether there is a special sign is on the part or not,
- Testing a diameter of product by a go/no-go gauge to see if it is sized correctly.

## List of symbols

AS	Acceptance Sampling
Ac=c	Acceptance number, allowable number of defectives in sample
$c_1$	Acceptance number for first sample in double sampling
$c_2$	allowable number of defectives in combined first and second samples
D	The number of defectives in the lot
n	Sample size
$n_1$	Size of 1 <sup>st</sup> sample in double sampling
$n_2$	Size of 2 <sup>nd</sup> sample in double sampling
N	Lot size
$\frac{D}{N}$	lot Proportion defectives
$p$	Proportion defective in sample $p = \frac{r}{n}$ .
$p'$	Proportion defective of production process
$P_a$	The probability of lot acceptance in a sampling plan
$P_{0.10},$	Proportion defectives having $P_a$ of .010, 0.95 and 0.99 etc under any given acceptance criteria
$P_{0.95},$	
$P_{0.99}, \dots$	
r	Number of defectives in a sample of size n
$\alpha$	Producer Risk, type 1 error, the probability of rejecting product of some stated desirable quality, $\alpha = 1 - P_a$ . Common $\alpha$ 's are 0.01, 0.05, 0.10.
$\beta$	Consumer Risk, Type 2 error, the probability of accepting a product of some stated desirable quality. Common $\beta$ 's are 0.01, 0.05, 0.10.

Acceptance sampling is a statistical technique in quality control. In this technique a random sample is taken from the lot, then using some well-known tables it is decided to accept or reject the lot. Acceptance Sampling is used not only in the final stage of a product, but also for the quality control of raw materials and semi-finished products.

Since no sampling plan gives 100% confidence, to determine an acceptance sampling plan the amount of risks  $\alpha$  and  $\beta$  which are acceptable to the concerned user have to be specified.

## Types of Acceptance Sampling Plans

Acceptance sampling plans can be *single, double or multiple*:



Type	Parameter symbol	Parameter Description
Single Sampling	$n, c$	$n$ : sample size, $c$ : allowable number of defectives in sample
Double Sampling	$n_1, c_1, n_2, c_2$	$n_1, n_2$ : first and 2 <sup>nd</sup> samples sizes, $c_1$ : allowable number of defectives in 1 <sup>st</sup> sample $c_2$ : allowable number of defectives in combined first and second samples
Multiple Sampling	$n_1, c_1, \dots, n_k, c_k$	$n_1, \dots, n_k$ : sample sizes, $c_1$ : allowable number of defectives in 1 <sup>st</sup> sample, $c_2$ : allowable number of defectives in combined first and second samples, $c_k$ : allowable number of defectives in combined samples 1 through k

In acceptance sampling, not that costly and time-consuming compared to 100% inspection, a product lot is accepted or rejected based on **simple** random samples taken from it. A simple random sample is one taken from a population in such a way that the members of the population have an equal chance of being chosen.

### 10.2.1 Single Acceptance Sampling

In this sampling procedure, one random sample of size  $n$  is taken from the lot; if the number of defectives in the sample exceeds a number called acceptance number and denoted by " $c$  or  $Ac$ ", the lot would be rejected. Let us assume that when a lot is rejected, it is 100% inspected and all the defectives are replaced by good items at the expense of the vendor or supplier before the lot is released. Rejection or acceptance number( $c$ ),

sample size(  $n$  ) and population size ( $N$ ) play key roles in single sampling plans. Let  $X$  denote the number of defectives in the sample, then the probability of accepting the lot with a single acceptance sampling plan would be:

$$P_a = \Pr(X \leq c) = \Pr(X = 0) + \Pr(X = 1) + \dots + \Pr(X = c)$$

### Example 10.1

A lot of size  $N=1000$  is inspected using the single sampling plan  $n=5$  &  $c=0$ ; what is the probability of accepting the lot by this plan for  $p' = 0.02\%$  &  $p' = 0.02$ .

### Solution

$N=100, n=5, c=0$ .

Since  $\frac{n}{N} = 0.05 < 0.10$  the distribution of  $X$  is considered binomial.

Let  $X$  denotes the number of defectives in the sample;

$$p' = 0.02: P_a = \Pr(X \leq c = 0) = \binom{5}{0} (0.02)^0 (0.98)^5 = 0.904$$

$$p' = 0.0002: P_a = \binom{5}{0} (0.0002)^0 (0.9998)^5 = 0.999$$

### 10.2.1.1 Operating Characteristic Curve (OC Curve)

In quality control an operating characteristic (OC) curve plots acceptance probability of a lot of items versus a quality level which is most commonly  $p'$  (lot fraction defective) or  $R$  (product reliability). The former is discussed in this chapter and the latter will be dealt in chapter 14. The graph shows how well the acceptance sampling plan discriminates between good and bad lots. The OC curve with its discriminatory power could be used to compare various sampling plans in order to choose an appropriate one.

The OC curve of an acceptance sampling plan described in this chapter depicts  $P_a$ , the probability of accepting a lot, versus its quality  $p'$ .  $p'$  is the process fraction defective with minimum zero and maximum 1. When for various  $p'$ , the corresponding  $P_a$ 's are calculated and plotted against each other on an X-Y plot, the resulted curve is the OC curve. On the horizontal axis,  $100p'$  could be shown instead of  $p'$ . In this chapter the emphasis will be on the OC curves for single sampling plans. What follows is a brief description of so-called type A and B OC curves.

#### 10.2.1.1.1 Type A and Type B OC curves

The fundamental tool for the analysis of sampling plans is the OC curve. OC curves are classified into 2 types:

- 1) type A OC curves and
- 2) type B OC curves.

**Type A** is for sampling from an individual (or isolated); it plots the probability that the lot will be accepted against lot proportion defective.

**Type B** is for sampling from a large lot or a process (such as the producer's process, which produced the lot). In this type shows proportion of lots which will be accepted ( $P_a$ ) is plotted against the process proportion defective.

Naturally, the probability distributions utilized in plotting these types of OC curves are inherently different (Schilling, Neubauer, 2008 chapter 4)

In Type A OC curves sampling is done from an individual finite lot, and  $P_a$ 's are computed and plotted versus lot qualities i.e. lot fraction defectives.

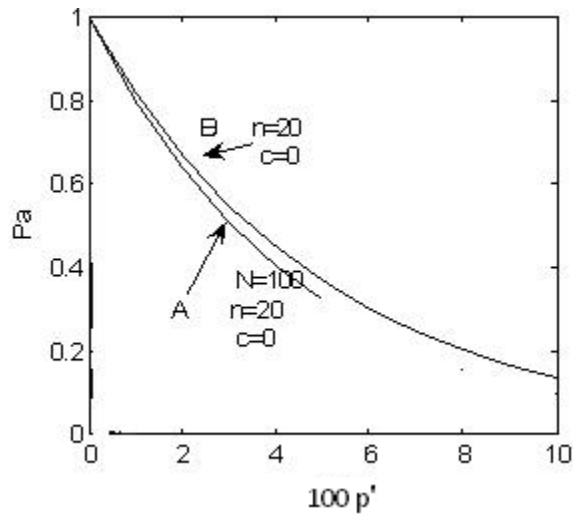
In Type B OC curves sampling is done from a process or lots of largish size as if the lot size were infinite.  $P_a$  is computed and plotted versus product or process quality i.e. process fraction defectives.

The statistical distributions utilized for plotting each type are naturally different.

"In principle, Type A curves should be computed using hyper-geometric distribution for which the binomial or Poisson distribution often give satisfactory approximations. Such curves are discontinuous; e.g. in a lot of 200 items there may be 0.5 or 1% defective but not 0.8% defective; however its common to draw this type as continuous" (extracted from Grant and Leavenworth, 1988 page 406).

In type B, probabilities are calculated as if the lot size is infinite and in principle, binomial distribution has to be utilized for this calculation; however Poisson distribution gives good results. Such curves are continuous. Figure 10-1 shows OC curve for both infinite and finite populations.

For the same sampling plan, type A OC curve is always below the type B OC curve i.e. type A curve always gives less probability of acceptance than type B curve. As the lot size increases, type A curve approaches type B curve.



**Fig. 10-1 OC curve for a finite population(A) & an infinite population(B).**

In practice, if the population is infinite or  $\frac{n}{N} < 0.1$ , the probabilities of single sampling plan is calculated using binomial distribution. The following MATLAB commands could be used to plot type B OC curve:

```
>> pprime= 0:.01:1;Pa = binocdf(c,n, pprime); plot(pprime,Pa)
```

Figure 10-2 shows the OC curves of 3 plans, which have been plotted using the following commands assuming  $\frac{n}{N} < 0.1$ ,

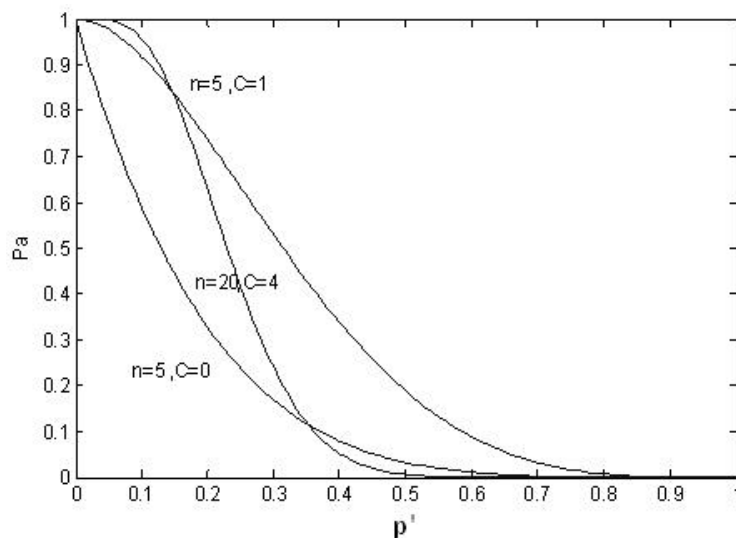
```
>> pprime=0:0.01:1;pa=binocdf(0,5, pprime);plot(pprime,pa) ;
hold on;
>> pprime=0:0.01:1;pa=binocdf(1,5, pprime);plot(pprime,pa);
hold on;
>> pprime=0:0.01:1;pa=binocdf(4,20, pprime);plot(pprime,pa);
```

Also, in practice if  $\frac{n}{N} \geq 0.1$ ,  $P_a$  (the probability of defectives in the sample not exceeding the specific number  $c$ ) is calculated using hyper-geometric distribution. The command for  $P_a = \Pr(X \leq c)$  in MATLAB is as follows:

```
Pa=hygecdf(c,N,D,n)
```

where

C	Acceptance number
N	Lot size
D	Number of defectives in the lot
n	Sample size



**Fig. 10-2 Comparison of 3 plans using OC Curve Type B<sup>1</sup>**

### Example 10.2

Plot the OC curves for the following plans. The samples are taken from a production process.

Plan I:  $n=5, c=0$

Plan II:  $n=5, c=1$

Plan III:  $n=20, c=4$

### Solution

Since the sampling is taken from the process, the population size is considered infinite and  $\frac{n}{N}$  assumed small, therefore the OC curve is of type B and the binomial distribution is utilized to plot the OC curve.

$$Pa = \Pr(X \leq c) = \sum_{r=0}^c \binom{n}{r} p^r (1-p')^{n-r}$$

For Plan 1:

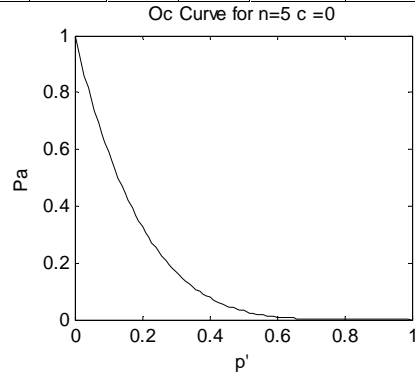
$$Pa = \binom{n}{0} p'^0 (1-p')^{n-0} = (1-p')^n$$

Therefore for constructing the OC curve of plan 1, compute  $Pa = (1-p')^n$  for several  $p'$  between 0 and 1, and plot the pairs in an X-Y coordinates. The following table shows the values of  $Pa$  calculated with binocdf

<sup>1</sup> With thanks to Mr Amir Hossein Ghassemi, a graduate of Shahid Bahonar University of Kerman, Iran

command in MATLAB and Fig. 10-3 shows the OC curve for Plan 1 ( $n=5, c=0$ ).

$P'$	0	0.02	0.05	0.1	0.2	0.4	0.6	0.8	0.9	0.95
$np'$	0	0.1	0.25	0.5	1	2	3	4	4.5	4.75
$Pa = \Pr(X \leq c)$	1	0.904	0.774	0.591	0.33	0.0778	0.010	0.000	0.000	0.000



**Fig. 10-3 The OC curve for plan 1 ( $n=5$  &  $c=0$ )**

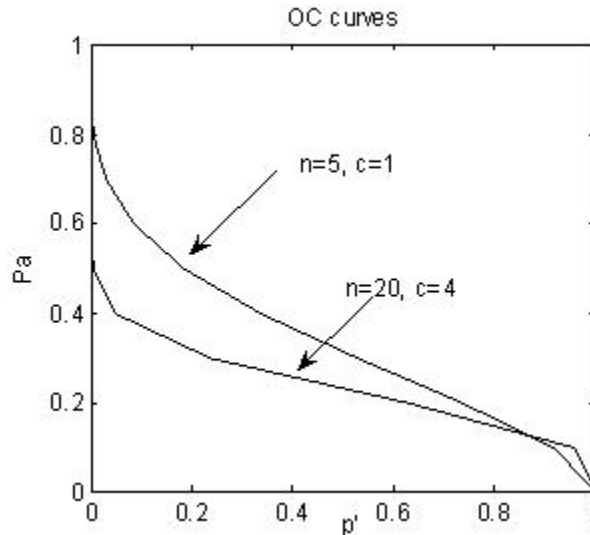
For Plan 2:

$$Pa = \Pr(X \leq c) = \sum_{r=0}^c \binom{n}{r} p^r (1-p)^{n-r}$$

$$n=5 \quad c=1$$

For plotting the OC curve of plan 2,  $Pa$  is computed for several  $p'$  between 0 and 1, and the points are shown in an X-Y coordinates. In a similar way the OC curve for plan 3 is plotted. The following table and Fig 10-4 shows

$P'$	0	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$Pa(\text{Plan2})$	1	0.974	0.9185	0.7373	0.5282	0.3370	0.19	0.087	0.308	0.0067	0.0005	0
$Pa(\text{Plan3})$	1	0.9974	0.9568	0.8246	0.2375	0.0510	0.0059	0.0003	0.0000	0.0000	0.0000	0



**Fig. 10-4 The OC curves for Plans (n=5,c=1) and (n=20,c=4).**

the acceptance probabilities and the OC curves for Plan II&III. For example if the fraction defectives of a production process is 0.023, the percent of the lots accepted using plans II &III are calculated as follows:

$$P_a = \Pr(X \leq 1) = \sum_{r=0}^1 \binom{5}{r} (0.023)^r (1-0.023)^{5-r}$$

$$\text{Plan II} \quad P_a = \text{binocdf}(1, 5, 0.023) = 0.9949 = 99.49\%$$

$$\text{Plan III} \quad P_a = \text{binocdf}(4, 2, 0.023) = 0.9999 = 99.99\%.$$

As the table shows the percent of accepted lots ( $P_a$ ) is high for low fraction defectives.

### Example 10.3

Plot the OC curve for the single sampling plan  $n=100, c=1$ . The lot size is largish.

#### Solution

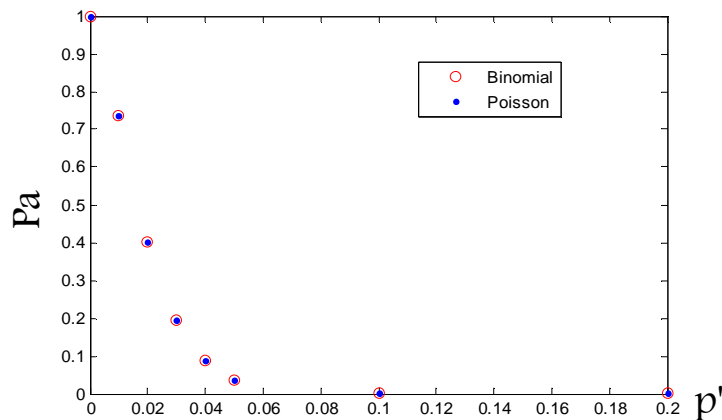
Since the sampling is taken from a large population and  $\frac{n}{N}$  is assumed small, the binomial distribution is used for calculating the acceptance probabilities:

$$P_a = P(X \leq c) \quad P_a = \Pr(x \leq 1) = \sum_{x=0}^1 \binom{100}{x} (p')^x (1-p')^{100-x}$$

however, as the following table shows the Poisson distribution approximates it well.

Comparison of $P_a$ for Plan $n=10, c=1$ computed with exact and approximate methods			
$p'$	$np' = \lambda$	$P_a$	
		Binomial(exact method)	Poisson (approximate method)
0	0	1	1
0.01	1	0.735	0.736
0.02	2	0.403	0.406
0.03	3	0.194	0.199
0.04	4	0.087	0.092
0.05	5	0.0371	0.04
0.1	10	$3.22 \times 10^{-4}$	$4.99 \times 10^{-4}$
0.2	20	$5.3 \times 10^{-9}$	$4.32 \times 10^{-8}$

Figure 10-5 compares the two set of probabilities. As the figure shows, the two sets of probability cannot be distinguished.

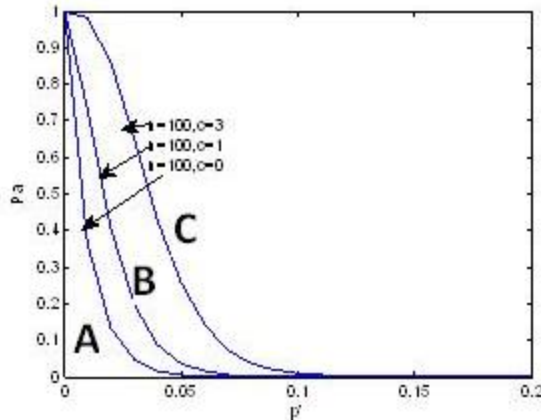


**Fig. 10-5  $P_a$  versus lot proportion defectives for Plan  $n=100, c=1$ .**

In quality control, OC curve could be used to select the plans that are effective in reducing risks. Consider Plans A,B and C in Fig. 10-6. The OC curve of each plan has been plotted using the following MATLAB command, given  $n$  and  $c$ :

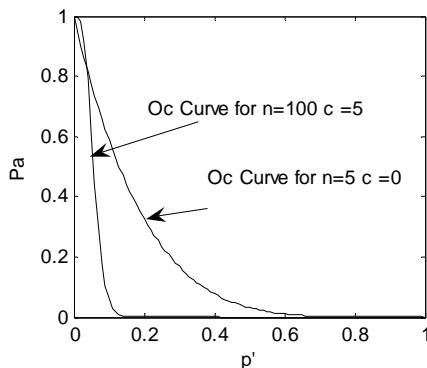
```
plot(0:.01:1,binocdf(c,n,0:.01:1)).
```



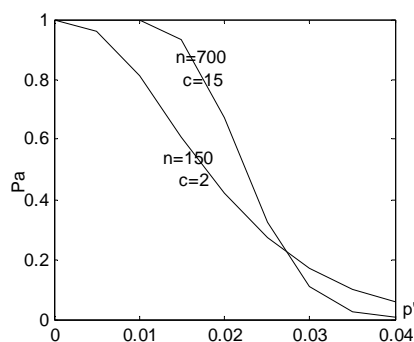


**Fig. 10-6 OC curves of 3 plans with the same  $c$  and different  $n$ .**

For a particular proportion defective  $p'$ , the acceptance probability in Plan C is greater than the corresponding values in Plan B and that of Plan B is greater than that in Plan A. The choice of right procedure from these 3 different plans depends on the lot or batch proportion defective. When the proportion is high, Plan A is appropriate; when the proportion is low Plan C is appropriate (why?). Figures 10-7 and 10-8 compare two OC curves; the interpretations are left to the readers.



**Fig. 10-7 OC curves for 2 plans**

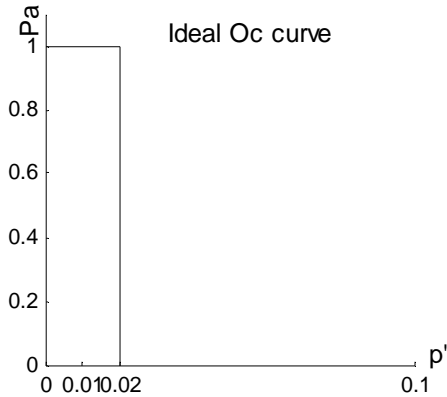


**Fig. 10-8 OC curves for 2 other plans**

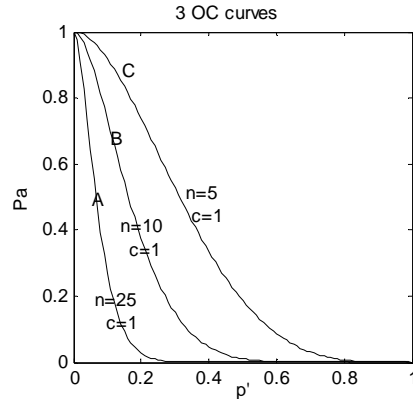
#### 10.2.1.1.2 Ideal OC curve

An ideal OC curve deals with the inspection procedure in which one might want to accept all lots whose proportion defectives are not greater than a desired proportion  $p'_0$  and rejects all lots with proportion defectives greater than  $p'_0$ . The only way to have the ideal OC curve is one hundred percent inspection. When the acceptance criterion (e.g. number  $c$  in single sampling plans "where  $c$  is the maximum number of defective items

that can be found in the sample and the lot still be accepted", Encyclopedia Britannica) is kept unchanged; the larger the sample size of the plan used for inspection, the nearer the OC curve approaches the ideal.



**Fig.10-9 An Ideal OC curve.**



**Fig.10-10 Effect of  $n$  on slope,  $c$  constant**

#### 10.2.1.1.3 Effect of sample size on OC curve

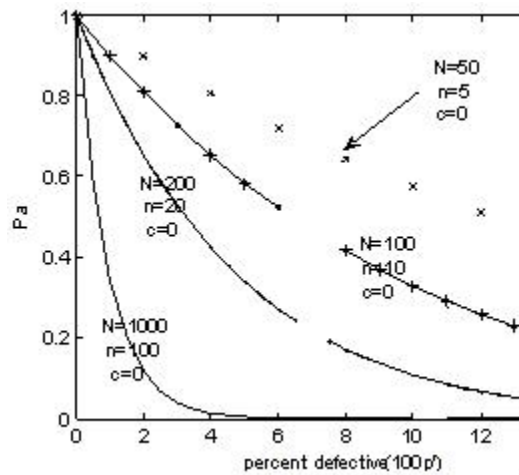
If  $n$  is kept constant, when the acceptance criterion  $c$  reduces, the OC curve tends to ideal and the discriminatory power of the plan in distinguishing between good and bad lots increases. See Fig. 10-6.

#### 10.2.1.1.4 Effect of acceptance number ( $c$ ) on OC curve

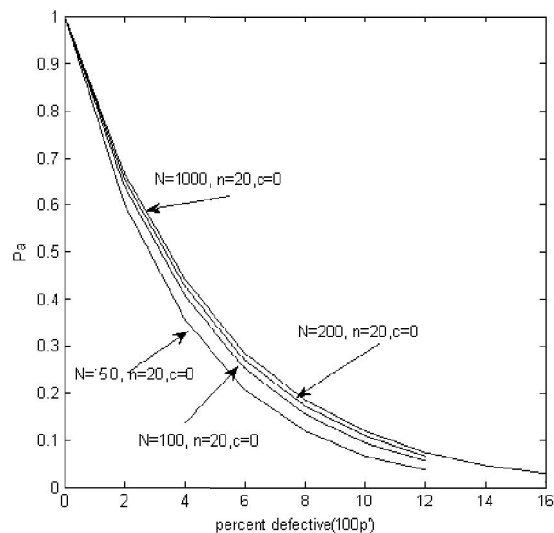
If the acceptance number ( $c$ ) is kept constant, as we increase the sample size, the slope of the OC curve and the discriminatory power of the plan in distinguishing between good and bad lots increases and the curve tends to ideal. See Fig10-10.

#### 10.2.1.1.4 Effect of constant ratio $\frac{n}{N}$ on OC curve

Before the widespread use of modern acceptance sampling systems, a common practice in industry was to specify that the sample inspected should be some fixed percentage of the lot such as 5, 10, 20%. Figure 10-11 shows how wrong this idea really is (Grant and Leavenworth, 1988 page 398-9). The absolute size of random sample is much more important than its relative size compared to the size of the lot; i.e. fixed sample size tends to constant quality protection (Grant and Leavenworth, 1988 page 401). Instead, fixed  $n$  and  $c$  for different lot size ( $N$ ) is much better than fixed  $\frac{n}{N}$  (Fig. 10-12).



**Fig. 10-11 Comparison of OC curves for 4 plans having the same  $c$  and  $\frac{n}{N}$ .**



**Fig 10-12, Fixed  $n$  &  $c$ , variable  $N$**   
(Grant and Leavenworth, 1988 page 401, re-plotted).

## Definitions

Associated with each sampling plan there is an OC curve on which two or three important points denoted by AQL, LQL(LTPD) and IP are of interest to the users of the plan. These points are defined as follows:

### 10.2.1.2 Acceptable Quality Level (AQL)

ANSI /ASQC Standard A2 (1987) defines AQL as

“The maximum percentage or proportion of variant units in a lot or batch that, for the purposes of acceptance sampling, can be considered satisfactory as process average”. AQL has a high probability of acceptance.

### 10.2.1.3 Limiting Quality Level (LQL)

ANSI /ASQC Standard A2 (1987) defines LQL as

“the percentage or proportion of variant units in a batch or lot for which, for the purposes of a acceptance sampling, the consumer wishes the probability of acceptance to be restricted to a specific low value”. LQL is a proportion defective with a low probability of acceptance.

### 10.2.1.4 Indifference Quality Level (IQL)

Indifference quality level (IQL) or control point for a sampling plan is defined as the proportions defective (quality level) having the acceptance probability of  $\frac{1}{2}$ . This point (IQL, 0.50) on the OC curve is also called as ‘point of control’. The lots with this quality level will accepted 50% of the time using the particular sampling plan.

### 10.2.1.5 Type I and Type II error; Consumer and producer risks

Type I and Type II error of an acceptance sampling plan which are useful in sample size determination are respectively false rejection and false acceptance of a lot by the plan. Type I error probability which in quality control is also called producer's risk is denoted by  $\alpha$  and type II error probability, called consumer's risk, is denoted by  $\beta$ .

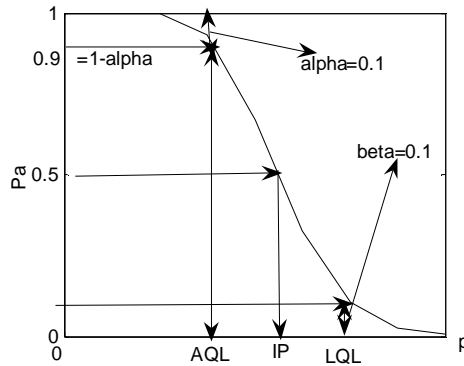
For a given sampling plan producer's risk ( $\alpha$ ) may be defined as follows:

The probability of not accepting a lot the quality of which fall within an acceptable range of values and is often desired to accept.

For a given sampling plan consumer's risk ( $\beta$ ) may be defined as follows:

The probability of acceptance of a lot the quality of which is rarely desired to accept and falls within a range of unacceptable values.

Fig.10-13 illustrates AQL, LQL, IQL,  $\alpha$  and  $\beta$ .



**Fig.10-13 AQL, LQL, IQL = IP,  $\alpha$  and  $\beta$  – an illustration**

Common values for  $\alpha$  and  $\beta$  are 1%, 5% and 10% in statistics.

The OC curve and the above points, which are 3 incoming quality levels, are not necessarily the unique indexes for the evaluation of sampling plans. Another such technique uses the AOQ curve and AOQL, defined below.

#### 10.2.1.7 Average Outgoing Quality curve (AOQ curve)

Suppose a lot is inspected with a single sampling plan  $n$  &  $c$ ; if the number of non-conforming products in the sample, denoted by  $r$ , is not greater than  $c$  ( $r \leq c$ ) the lot is accepted. Otherwise the lot is 100% inspected and all nonconforming products are replaced by good ones. In any case the nonconforming products of the sample are removed and replaced by good products. Now let

$N$  = Lot size

$n$  = Sample size

$p'$  = Proportion nonconforming of the process

$P_a$  = Lot acceptance probability by the plan

Then

The average number of nonconforming in the lot before sampling and inspection =  $Np'$ ,

The average number of nonconforming in the sample =  $np'$ ,

The average number of nonconforming in the lot after sampling =  $Np' - np' = (N - n)p'$ ,

The lot proportion of nonconforming after sampling =

$$\frac{(N - n)p'}{N} = \left(1 - \frac{n}{N}\right)p',$$

(Notice: after removing and replacing the defective items of the sample with good ones the sample returns to the lot).

The lot is accepted with the probability of  $P_a$  and is screened with probability  $1 - P_a$ . Assuming 100% inspection (screening) finds all the nonconforming products and they are replaced by good ones, on the average the quality of the outgoing lot (AOQ) with this single sampling plan would be:

$$AOQ = P_a \left(1 - \frac{n}{N}\right) p' + (1 - P_a)(0) \Rightarrow$$

$$AOQ = p' P_a \left(1 - \frac{n}{N}\right).$$

When  $\frac{n}{N}$  is very small or  $N$  is very large, then  $AOQ = p' P_a$ :

A plot of AOQ against  $p'$  is called **AOQ curve**, which, like OC curve, shows the ability of the plan to distinguish between good and bad lots.

#### Example 10.4

A single sampling plan  $n, c$  is used for inspecting coming lots of size  $N$ , the probability of accepting a lot with  $p'$  as its proportion of non conforming is  $P_a$ . What is

- the expected number of  $k$  lots with this plan.
- AOQ for an incoming lot inspected with a sample of size  $n$ .
- AOQ for all  $k$  lots with a sample of size  $kn$ .

#### Solution

Let

- $k$  = Total number of lots
- $kP_a$  = Number of lots expected to be accepted
- $Np'$  = Expected number of defective in a lot
- $kP_a (N-n)p'$  = Expected number of defectives in all accepted lots

The expected number of nonconforming products( defectives) per each accepted lot =  $\frac{kP_a(N-n)p'}{kN}$

Average outgoing quality for each lot equals:  $AOQ = p' P_a \left(1 - \frac{n}{N}\right)$ ,

If a sample of size  $kn$  is taken from the population of all lots which the size  $KN$ , on the average, AOQ would be the same:

$$AOQ = 0(1 - P_a) + \frac{kNp' - knp'}{kN} P_a = \frac{kP_a(N-n)p'}{kN} = p' P_a \left(1 - \frac{n}{N}\right).$$

### Example 10.5

Find AOQ for the following plans used for inspecting a lot of size  $N=500$  with 3% nonconforming:

Plan 1:  $n=49, c=1$ ;

Plan 2:  $n=100, c=2$ .

### Solution

Let  $X$  denotes number of non-conforming in the sample, then:

$$Pa = \Pr(X \leq c) = \text{binocdf}(c, n, p')$$

$$AOQ = (Pa)(p')(1 - \frac{n}{N})$$

$$\text{Plan 1 } n = 49 \quad c = 1$$

$$\frac{n}{N} < 0.1 \rightarrow \text{Binomial} \quad Pa = \Pr(X \leq 1) = \text{binocdf}(1, 49, .03) = 0.56$$

$$AOQ = 0.57 * 0.03(1 - 0.01) = 0.015$$

Plan 2:  $n=100 \quad C=2$

$$\frac{n}{N} > 0.1 \rightarrow \text{hypergeomet.}$$

$$n = 100, c = 2 \quad Pa = \text{hyge}(2, 500, 0.03 * 500, 100) = 0.395$$

$$AOQ = 0.03 \times 0.395(1 - \frac{100}{500}) = 0.00949.$$

### Example 10.6

Draw the AOQ curve for the plans of Example 10.5:

### Solution

Plan I $n=49 \quad c=1$											
$p'$	0	0.01	.02	.03	0.04	0.05	0.06	0.07	0.08	0.09	.1
$np'$	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
$P_a = \Pr(r \leq c)$	1	0.9106	0.736	0.555	0.400	0.28	0.190	0.126	0.827	0.053	0.039
A.O.Q (%)	0	0.82	1.32	1.5	1.44	1.26	1.03	0.80	0.60	0.43	0.30

Plan II $n=100$ $c=2$ $N=500$											
$p' = \frac{D}{N}$	0	0.01	.02	.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
D	0	5	10	15	20	25	30	35	40	45	50
$P_a = P_r(r \leq c)$	1	0.943	0.678	0.395	0.200	0.093	0.040	0.016	0.006	0.002	0.000
A.O.Q (%)	0	0.754	1.09	0.949	0.641	0.370	0.191	0.090	0.040	0.017	0.007

If the values of AOQ is plotted against  $p'$ , **AOQ curve** is obtained.

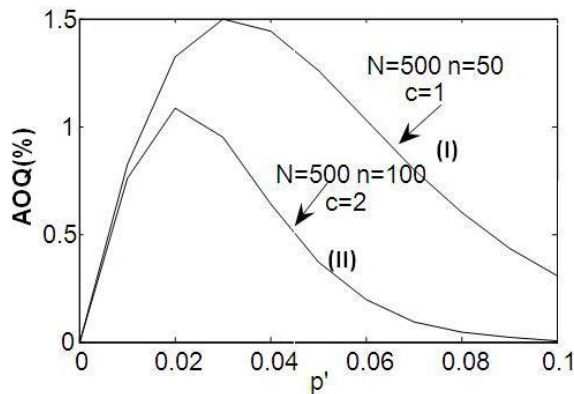
Figure 10-14 shows the AOQ curves of the two plans. These curves were plotted by the following MATLAB commands:

```
>>p=0:0.01:0.1;
```

```
Pa=binocdf(1,49,p);AOQ=Pa.*p*(1-49/500);plot(p,100*AOQ); hold on;
```

```
D=0:5:50;Pa=hygecdf(2,500,D,100);AOQ=Pa.*p*(1-100/500);plot(p,100*AOQ);
```

Figure 10-14 compares the 2 plans using their AOQ curves. Plan II, on average, results in better quality of outgoing lots (why?).



**Fig. 10-14 AOQ curves for 2 single sampling plans.**

#### 10.2.1.7.1 The Average Outgoing Quality Limit(AOQL)

When average outgoing quality (AOQ) is computed for all assumed percent nonconforming products in a lot submitted for inspection by a sampling plan; the maximum possible AOQ value in the outgoing lot is referred to as average outgoing quality limit (AOQL).

#### 10.2.1.8 The Average Total Inspection(ATI)

To describe ATI, consider a single acceptance sampling plan. If the lot is accepted immediately after inspecting the sample, only the products of the sample will be inspected and the defective ones will be replaced by good ones; with the probability of  $P_a$ . If it is not accepted immediately



(with probability of  $1 - P_a$ ), the lot will be screened i.e. all of the  $N$  products of the lot will be inspected and the defective units will be replaced by good ones. Therefore on average, as much as the following amount will be inspected :

$$ATI = np_a + N(1 - p_a) = n + (N - n)(1 - P_a).$$

ATI could be plotted against  $p'$ . It could be verified that Plan II in Fig. 10-14 has a greater ATI, which leads to greater inspection cost.

### 10.2.1.9 The Average Fraction Inspected(AFI)

As well as ATI i.e. the absolute measure of the amount of inspection in acceptance sampling, AFI is another measure which stands for the average fraction inspected. AFI denotes the long run fraction of items that are inspected. For a lot of size  $N$  being inspected by an acceptance sampling plan. AFI is defined as follows:

$$AFI = \frac{ATI}{N}.$$

#### 10.2.1.9.1 The Relationship between AFI and AOQ

Between AOQ and AFI of a lot with proportion nonconforming  $p'$  inspected by an acceptance sampling plan whether single, double and multiple, the following relationship holds:

$$AOQ = p'(1 - AFI) \quad \text{or} \quad AFI = 1 - \frac{AOQ}{p'}.$$

In a single sampling plan, it could be shown that AFI varies between  $\frac{n}{N}$  to 1.

#### Example 10.7

A sample of size  $n=20$  is taken from a lot of size  $N=500$ . If the proportion nonconforming of the production process is 5% and  $c=2$ , how many products on the average is inspected with this single sampling plan? What percent of  $N$  is this amount? Find average outgoing quality. Solve the problem with  $n=100$  again.

#### Solution:

The number of nonconforming products in the sample  $X$ , has a binomial distribution. The probability of accepting the lot would

$$P_a = \sum_{r=0}^{c=2} \binom{n}{r} p'^r (1 - p')^{n-r} = 0.9196 \text{ directly from the binomial}$$

distribution. Or by Poisson approximation From Table 4

For  $n=20$ ,

$$\lambda = np' = 20 \times .05 = 1 \quad Pa = \Pr(X \leq 2) = 0.92$$

$$ATI = nPa + N(1 - Pa) = (500)(0.08) + 2(0.92) = 58.4$$

$$AFI = \frac{58.4}{500} = 0.1168 \quad \text{or} \quad 11.68\%$$

$$AOQ = (p') \times (1 - AFI) = 0.05 \times (1 - 0.1168) = 0.04416$$

For  $n=100$

$$\lambda = 100 \times 0.05 = 5 \quad Pa = \Pr(X \leq 2) = 0.125$$

$$ATI = 100 \times 0.125 + 500 \times 0.875 = 450$$

$$AFI = \frac{ATI}{N} = \frac{450}{500} = 0.9 = 90\%$$

$$AOQ = p'Pa(1 - \frac{n}{N}) = \frac{5}{100}(0.125)(1 - \frac{100}{500}) = 0.005 = 0.5\%$$

$$AOQ = p'(1 - AFI) = 0.05 \times (1 - 0.9) = 0.5\%$$

As it is evident with  $n=100$ , AOQ would be much better.

### 10.2.2 Double sampling plan

The flow-chart shown in Fig. 10-15 shows how a double sampling plan, which is associated with 4 numbers  $n_1, c_1, n_2$  and  $c_2$ , works.

$r_1$  &  $r_2$  are number of nonconforming products in the first and second samples respectively.

#### Example 10.8

The double sampling  $n_1 = 50, c_1 = 0, n_2 = 100$  &  $c_2 = 2$  is to be used to inspect a lot of size 1000; suppose one nonconforming product was found in the first sample i.e.  $r_1 = 1$ . According to the above flowchart the lot could neither be accepted nor rejected. The second sample was taken. After inspecting the sample of size 100, suppose 2 nonconforming products were found in the second sample i.e.  $r_2 = 2$ . Therefore  $r_1 + r_2 = 3 > c_2 = 2$ ; the lot is rejected but all nonconforming products in the lot have to be replaced with good products.

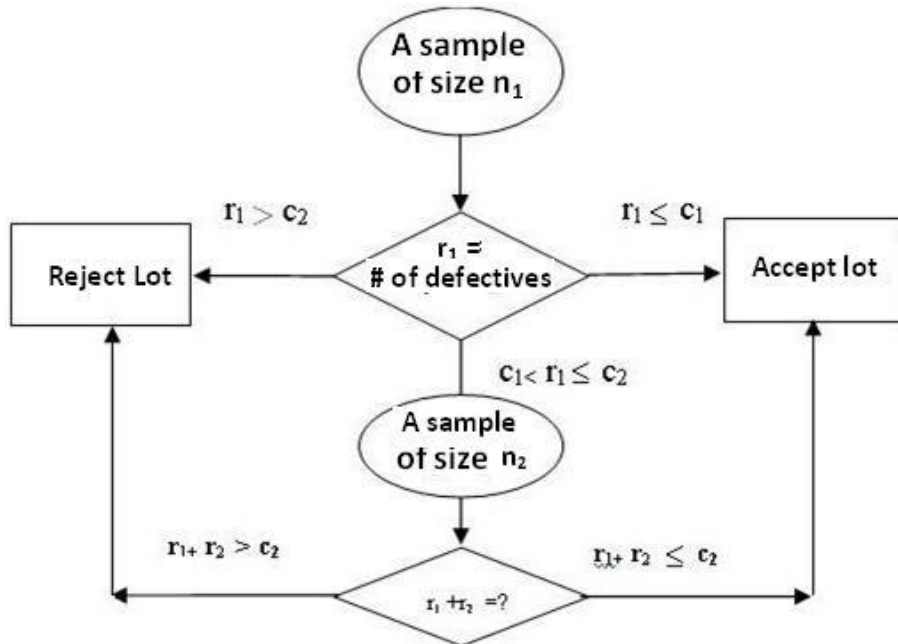


Fig. 10-15 The flow chart of a double sampling plan

### 10.2.2.1 A relationship for Plotting the OC curve of Double Sampling Plans

The relationship between lot proportion nonconforming ( $p'$ ) and the probability of accepting the lot using a double sampling plan is as follows (after Bowker & Lieberman, 1972, page 509):

$$P_a = \Pr(r_1 \leq c_1 | p') + \Pr(r_1 + r_2 \leq c_2 \text{ \& } c_1 < r_1 < c_2 \text{ given } p') =$$

$$\sum_{d_1=0}^{c_1} \binom{n_1}{d_1} p'^{d_1} (1-p')^{n_1-d_1} + \sum_{d_1=c_1+1}^{c_2-1} \binom{n_1}{d_1} p'^{d_1} (1-p')^{n_1-d_1} \left[ \sum_{d_2=0}^{c_2-d_1} \binom{n_2}{d_2} p'^{d_2} (1-p')^{n_2-d_2} \right]$$

The details could be studied in references such as Bowker & Lieberman (1972). The OC curve of a double sampling plan could be plotted using this relationship.

### 10.2.3 Multiple sampling plan

A multiple sampling plan is similar to the double sampling plan in that successive trials are made, if required. Associated with each of the trials there is a sample size, and acceptance number (lower limit) and a rejection one (upper limit) denoted by  $A_c$  and  $R_e$ .

As successive samples are taken, the number of defectives in the samples are added to get cumulative number of defects.

The plan acts as follows<sup>1</sup> :

If, for any sample, the cumulative number of nonconforming products (defectives) found exceeds the upper limit specified, the lot is rejected.

If, for any sample the cumulative number of nonconforming found is less than or equal to the lower limit, the lot is accepted.

If the number of defectives found is between the two limits, another sample is taken.

The process continues until the lot is accepted or rejected. Example 10.9 illustrates multiple acceptance sampling.

### Example 10.9

A multiple sample plan is given in the following table:

SampleNo.	1	2	3	4	5	6	7
<b>n</b>	<b>20</b>	<b>20</b>	<b>20</b>	<b>20</b>	<b>20</b>	<b>20</b>	<b>20</b>
<b>Ac</b>	<b>0</b>	<b>1</b>	<b>3</b>	<b>5</b>	<b>8</b>	<b>9</b>	<b>10</b>
<b>Re</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>8</b>	<b>10</b>	<b>11</b>	<b>11</b>

According to this plan a sample of size 20 is taken from the lot;

Let  $r_1$ =the number of nonconforming products(defectives) in the first sample

if  $r_1$  is zero the lot is accepted

if  $r_1$  is 4 or more it is rejected,

if  $r_1$  is between 0 and four the second sample is taken.

Let  $r_2$ =the number of defectives in the second sample

if  $r_1 + r_2 \leq 1$ , the lot is accepted;

if  $r_1 + r_2$  is 5 or more the lot is rejected;

otherwise, i.e. if  $r_1 + r_2$  is between 1 and 5, the third sample is taken, a similar decision is taken based on the cumulative number of defectives in the successive samples.

Note that

the Ac and Re numbers increase as we enter a new phase.

One advantage of double and multiple sampling over single sampling is giving a second chance to doubtful lots, and the other is having less ATI. From a psychical point of view, the producer whose lot has not been accepted yet in the first phase, expects the lot be accepted in the next phase(s). A disadvantage of double or multiple plans over single one is their need for more time and expenditure.

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<sup>1</sup> from [web.nchu.edu.tw/pweb/users/arborfish/lesson/12510.ppt](http://web.nchu.edu.tw/pweb/users/arborfish/lesson/12510.ppt)

## Exercises

**10.1-** A single sampling plan uses an acceptance number of 1 and samples of size 15 to inspect lots of size 50. Use hyper geometric distribution to calculate  $P_a$ , the probability of accepting the lot of 2,6,10,20 percent defectives.

Answer :

1, 0.789, 0.524, 0.121 (Bowker & Liberman 1988, p421).

**10.2-** (Bowker & Liberman 1988, page 421).

A single sampling plan has  $c=3$ ,  $n=110$ . Use Poisson distribution to approximate probability of acceptance lot 0.5,1.,2.,,3,4,5,6,8% defective.

Answer : 0.998, 0.974,0.819,0.580,0.359,0.202,0.105,0.025

**10.3-**(Bowker & Liberman 1988, page 421).

Plot the OC curve for the sampling plan in Problem 10.2 using MATLAB Software. What are the approximate values of lot percent defective for which probabilities of acceptance are 0.95,0.50 and 0.10 respectively?

Answer : 1.2, 3.3, 6.1

**10.4-**(Bowker & Liberman 1988, page 421).

A double sampling plan as follows:

a) Select a sample of 2 from a lot of 20. If both articles inspected are good, accept the lot. If both are defective, reject the lot. If 1 is good and 1 defective, take a second sample of one article.

b) If the article in the second sample is good, accept the lot. If it is defective, reject the lot. If a lot 25% defective is submitted, what is the probability of acceptance? Compute this by the method that is theoretically correct rather than by an approximate method.

Answer. 0.859

**10.5-** For single plan  $n=75, c=1$ , what are  $p_{0.95}$ ,  $p_{0.5}$ ,  $p_{0.10}$ ? i.e. what are proportion defectives with  $P_a=.95, .5, .1$ ?

Ans: 0.48% , 2.23% , 5.09%

Hint: Use MATLAB command fzero; for example

$p_{95} = \text{fzero}(@(p) \text{binocdf}(1,75,p) - .95, .05)$

$p_{95} = 0.0048$

**10.6-**(13-9, Bowker & Lieberman 1988, page 445).

Prepare an AOQ curve for single plan  $n=100, c=0$  from a largish lot.

What is AOQL?

Ans. 0.37%

**10.7-A** multiple sampling plan is as follows:

Sample No.	Individual sample size	Combined sample size	Acceptance number	Rejection number
1	5	5	*	2
2	5	10	0	2
3	5	15	0	3
4	5	20	1	3
5	5	25	2	3

\*Acceptance not permitted on first sample.

Assuming that lot size is large enough for Poisson distribution to be applicable, compute the probability of acceptance of a 10% defective lot in each of the 5 phases.

Answer. (The sum of probabilities is )0.586.

**10.8-** In Example 10.2 What proportion of defectives will result in accepting 95% of the lots using Plan II&III?

### 10.3 Design of plans in Acceptance Sampling

Given  $n$  and  $c$ , it is easy to plot the OC curve; now we would like to act vice versa; suppose one or two points from the OC curve are known and we would like to determine the specification of the single sampling plan i.e.  $n$  and  $c$ . It is conventional to design an acceptance sampling plan for the following cases:

1-Given an AQL

ISO2859-1 Tables introduced in chapter 11, or equivalent standard tables, could be used to develop the appropriate plan

2-Given LTPD,  $\beta$  and AQL or any other 2 points of the OC curve  
Chapter 11 deals with this case.

3-Given indifference point (control point)

When the control point of the OC curve, which is the quality level having a probability of acceptance of 0.50, Phillips Tables introduced in chapter 11 could be used to determine the plan

4-Given LTPD=LQL and  $\beta = 0.1$  and process average non conforming  
Dodge-Romig LTPD Tables are used for this case

5-Given AOQL and process average non conforming

At the end of this chapter, Dodge-Romig Tables for cases 4 and 5 are introduced; the first 3 cases are dealt in chapter 11.

#### 10.3.1 Dodge-Romig LTPD=LQL Tables (Tables A, B, C, C.1)

Dodge and Romig have developed an extensive and valuable set of tables, the plans of which have been designed to minimize the total inspection, including screening. Dodge-Romig LQL tables based on stated values of LQL are designed so that the probability of lot acceptance at the LTPD(=LQL) is 0.1 for all plans i.e.  $\beta = 0.1$ ; the tables have been provided for LTPD values 0.5%, 1%, 2%, 3%, 4%, 5%, 7% and 10%. Tables A & B at the end of the book are 2 examples of these kind of tables prepared for single sampling having LQL=5% and LQL=1%; and Table C and C.1 are for double sampling having LQL=1% and LQL=5%.

The plan is read from the table, given the lot size and the process mean defective percent ( $p'$ ); **if the process mean ( $p'$ ) is not known, the worst should be assumed.**

It is worth knowing in determining the parameters (sample sizes and acceptance numbers) for these tables, optimization has been utilized in such a way that given  $n$  and an estimate of the process average non-

conforming, the solution that minimizes  $AFI = \frac{ATI}{N}$  was selected. It is reminded that for single sampling  $ATI = nPa + N(1 - Pa)$ .

### Example 10.10

Determine a single "acceptance sampling plan " for inspecting lots of size 500.

a)

if  $LQL=1\%$  ,  $\beta = 0.1$  and the process average non-conforming  $p'=0.2\%$ .

b)

if  $LQL=5\%$  ,  $\beta = 0.1$  and the process average non-conforming is not known.

c)

Find AOQL for both cases

### Solution

a)

Table B  $\Rightarrow n = 180$   $c = 0$

b) For  $LQL=5\%$  refer to Table A. When process mean defectives ( $p'$ ) is not known it is advised

to use that column of the table which contains the largest  $p'$ . Therefore from Table A,  $n=150, c=4$  i.e. take a sample of size 150, if the number of nonconforming is at most 4, accept the lot.

c)

For case a AOQL=%0.13 from Table B,

For case b AOQL=%1.2 from Table A.

### 10.3.2 Dodge-Romig AOQL Tables (Tables I, J,K)

Another set of Dodge—Romig Tables are AOQL Tables, the plans of which have found wide acceptance are used to develop single, double or multiple plans given AOQL and the process average non conforming. Tables I & J are examples of such tables for single sampling having AOQL=2% and AOQL=3%. Table K is a Dodge-Romig Table for double sampling with AOQL=3%. The plan is read from the table, given the lot size and the process non-conforming average in percent; **if the process average is not known, use the worst average** (last column of the table on the right side )to read the plan.



**Example 10.11**

Determine a single sampling plan with AOQL=2% used to inspect a lot of size 1000 from a production process with average non conforming 0.03%.

**Solution**

From Table I  $n=18, c=0$ .

LQL for this plan with  $\beta = 0.1$  is LQL =12%= 0.12 read from the same table under  $100P_{0.10}$ .

**Example 10.12**

Determine the double sampling plan for an AOQL=3% used for inspecting a lot of size 1000.

**Solution**

The process average is not known; the worst is assumed.

-Table K: Under the largest process average

- $n_1=70$     $c_1=2$     $n_2=120$     $c_2=10$

-LQL=8.4%

## Exercises

**10.8-** (13-10 Grant & Leavenworth ,1988page 445)

The lot size  $N$  is 2000 in a certain AOQL inspection procedure. The desired AOQL of 2.0 % can be obtained with any one of three single sampling plans. These are  $n = 65, c = 2$ ;  $n = 41, c = 1$  &  $n = 18, c = 0$ . If a large number of lots 0.3% nonconforming are submitted for acceptance, what will be the average number of units inspected per lot (ATI) under each of these three sampling plans?

Ans. 67; 55; 123.

**10.9-** (13-33 Grant & Leavenworth ,1988page 448)

The following I , II and III Dodge-Romig single sampling plans are specified for a lot of 1500 items and a 2% AOQL .

I (  $n = 18, c = 0$  )

II (  $n = 65, c = 2$  ) and

III (  $n = 120, c = 4$  )

Assume the lots of product contain 1% nonconforming units in answering the following questions.

a) what is the probability of acceptance under each plan using Poisson distribution?

b) What is the AFI under each of the 3 plans?

c) What is AOQ for each plan?

**10.10-** (13-1 & 13-2 Grant and Leavenworth ,1988page 445)

a) What single sampling plan should be used for a lot size of 1,500 and an LQL ( LTPD ) of 5 % with consumer's risk of 0.10 if the process average is estimated as 0.6 % defective?

Answer  $n=130$   $c=3$

b) What double sampling plan should be used for the condition described in the previous part?

Answer  $n_1=55$   $c_1=0$ ,  $n_2=120$   $c_2=4$ .

**10.11-** Solve part a of problem 10.10 if LQL=1%.

**10.12-** What Dodge-Romig 2% AOQL single sampling plan should be used for lots of size 200, 1000, 5000. The average nonconforming products is 1%. Compute the Average Fraction Inspected of each case.

**10.13-** Compute  $P_a$ , AFI and AOQ of the plan in Problem 10.12 specified for a lot of size 1000, if the  $p'=2\%$ .

**The value of a man is what he does expertly**

# Chapter 11

ISO2859 for AQL

,

Cameron  
Tables

&

Phillips Tables for IQL

## Chapter 11 Design of Acceptance Sampling Plans using ISO2859, Cameron & Phillips Tables

### Aims

This chapter introduces a series of standard tables titled ISO 2859, which are used to set up acceptance sampling plans based on attributes with reference to an acceptable quality limit(AQL). Determining plans given 2 points of the OC curve is also dealt here; in this regard Cameron Tables are introduced. The chapter also introduces Philips Tables used to determine sampling plans given the indifference point (IQL).

### 11-1 ISO 2895 Standard

#### Introduction

International organization for standardization( ISO) is a non-governmental international organization with a membership of 161 national standards bodies<sup>1</sup>. Through its members, it brings together experts to share knowledge and develop international standards. ISO has developed many standards, several of which are in the field of quality control such as ISO 9000 series, ISO 2859, ISO3951.

**ISO 2859** is one of a class of sampling systems, consisting of sampling procedures for inspection by attributes together with rules for their operations. This standard originated from MIL STD -105 in 1974. New versions have been released after then. It is worth mentioning that (Luko &Neubauer,2012):

MIL-STD-105 or ABC-STD-105 came from a need for a sampling system that did not require 100% inspection for use in testing munitions and for other destructive tests. The result was the Army Service Forces inspection tables, which came out in 1942 and 1943. Improvement led to MIL-STD-105A, B, C, D, and E in subsequent years (1950, 1958, 1961, 1963, 1989).

MIL-STD-105 focused on plans indexed by AQL to protect manufacturers against rejecting appropriate lots; however the standard offered some plans indexed by LTPD with  $P_a=0.1$  &  $0.05$  to protect consumers against accepting bad lots.

The Army discontinued support for military statistical standards on February 27, 1995, proposing instead to use civilian standards. Meanwhile, other standards writing bodies, such as the American National Standards Institute (ANSI), the International Organization

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<sup>1</sup> See <https://www.iso.org/members.html>

for Standardization (ISO), and the International Electro-technical Commission, developed their own derivatives of 105 as civilian standards like ANSI/ASQZ1.4 & ISO2859 . An expert in acceptance sampling advises to use ANSI/ASQZ1.4 instead of MILSTD-105 inside USA and ISO2859 elsewhere.

ISO 2859 standard is a sampling system. The purpose of a sampling system is three-fold (Derman & Ross, 1997 page 160):

- i)  
to protect the customer from accepting bad lots,
- ii)  
to protect the producer from having good lots rejected,
- iii)  
to encourage the producer to make the necessary modifications needed to improve the quality of its items.

Sampling schemes and plans designated in the ISO 2859 series are applicable, but not limited, to inspection of :(ISO 28590:2017 manual)

- end items,
- components and raw materials,
- operations,
- materials in process,
- supplies in storage,
- maintenance operations,
- data or records, and
- administrative procedures

The sampling system of ISO 2859 series of standards, which is used for inspection by attributes under the general title "Sampling procedures for inspection by attributes", consists of the following five parts (From ISO 2859-3:2005 manual page IV):

**Part 1: ISO 2859-1,**

Sampling schemes indexed by acceptance quality limit (AQL) for lot-by-lot inspection

**Part 2: ISO 2859-2,**

Sampling plans indexed by limiting quality (LQ) for isolated lot inspection

**Part 3: ISO 2859-3,**

Part 3: Skip-lot sampling procedures

**Part 4: ISO 2859-4,**

Procedures for assessment of declared quality levels

**Part 5:ISO 2859-5,**

Part 5: System of sequential sampling plans indexed by AQL for lot-by-lot inspection.

It should be added that there is an introductory part i.e. ISO28590:2017 manual. This manual provides a general introduction to acceptance sampling by attributes and provides a brief summary of the attribute sampling schemes and plans used in ISO 2859-1, ISO 2859-2, ISO 2859-3, ISO 2859-4 and ISO 2859-5, which describe specific types of attribute sampling systems. It also provides guidance on the selection of the appropriate inspection system for use in a particular situation.

According to this manual, the sampling inspection schemes of ISO 2859-1, ISO 2859-2, ISO 2859-3 and ISO 2859-5 provide for quantification of the risk of accepting unsatisfactory product (known as the consumer's risk) and the risk of not accepting satisfactory product (known as the producer's risk) and for choosing a plan that allows no more risk than is acceptable.

In this chapter it is not intended to deal with ISO 2859 in details; but this acceptance sampling system for inspection by attributes will be briefly introduced.

**11.1.1 ISO 2859-1 Standard<sup>1</sup>**

ISO 2859-1 deals with determining acceptance sampling plans and procedures for inspection by attributes of discrete item. Sampling schemes designated in this part of ISO 2859, are used for lot-by-lot inspection. The schemes are applicable, but not limited, to the inspection of finished products ,components and raw materials, materials in process and supplies in storage. The plans and procedures of ISO 2859-1 are indexed in terms of the Acceptable Quality Limit (AQL). The manual of the standard contains some definitions, some of which are motioned below.

**Definitions**

A section of the manual of ISO 2859-1 has been devoted to definitions and terminology. Let us define sampling plan/ scheme , AQL and LQ:

**11.1.1.1 Sampling plan**

(from ISO 2859 manual):

A specific plan which indicates the number of units of product from each lot which are to be inspected (sample size or series of sample sizes) and the associated criteria for determining the acceptability of the lot (acceptance and rejection numbers).

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<sup>1</sup> International Organization for Standardization. 1999.  
ISO 2859-1: Sampling Procedures for Inspection by Attributes  
Part1: Sampling Schemes Indexed by AQL for Lot-by-lot Inspe.  
International Organization for Standardization, Geneva

#### 11.1.1.2 Sampling scheme

(from ISO 2859 manual):

A combination of sampling plans with switching rules.

#### 11.1.1.3 Limiting Quality (LQ)

(from ISO 2859 manual):

When a lot is considered in isolation, a quality level which, for the purposes of sampling inspection, is *limited* to a low probability of acceptance (from ISO 2859-1). LQ, which was formerly called lot tolerance percent defective (LTPD), is the worst level of quality that the consumer may tolerate. Plans indexed by LQ are dealt in ISO 2859-2

#### 11.1.1.4 Acceptable Quality Limit(AQL )

Acceptable Quality Limit(AQL) plays an important role in ISO 2859-1. It is the limit of a satisfactory process average for the purposes of sampling inspection, when a continuous series of lots is considered ([www.woodencrates.org/standards/MIL-STD-105.pdf](http://www.woodencrates.org/standards/MIL-STD-105.pdf))

Two definitions of AQL from 2 sources are followed:

##### Definition 1)

quality level that is the worst tolerable process average when a continuing series of lots is submitted for acceptance sampling(ISO 2859-1 manual).

##### Definition 2)

The **maximum** percent nonconforming ( or maximum number of nonconformities per 100 units) that for the purpose of sampling inspection can be considered satisfactory as a process average. The phrase " can be considered satisfactory" is interpreted as the producer risk (Besterfield,1990, page237) and denoted by  $\alpha$  which generally varies between 0.01 and 0.10.

The AQL is a parameter of the sampling scheme and should not be confused with the process average which describes the operating level of the manufacturing process. It is expected that the process average will be less than or equal to the AQL to avoid excessive rejections under this system( ISO2859-1 manual).

ISO 2859-1 standard also states:

"Although individual lots with quality as bad as the acceptance quality limit may be accepted with fairly high probability, the designation of an AQL does not suggest that this is a desirable quality level. Sampling schemes in International Standards such as this part of ISO 2859, with their rules for switching and for

discontinuation of sampling inspection are designed to encourage suppliers to have process averages consistently better than the AQL.”( ISO 2859:1,1991).

For example when the AQL is 1%, a 0.5% defective lot is best released. However, it would be better still to have produced a lot free of defects.

In ISO2859-1 Tables, AQL Ranges from 0.01 to 1000. It is worth remembering AQL might be specified as the maximum percent nonconforming or maximum number of nonconformities per 100 units.

Note that

-The AQL values between 0.01 through 10 could be either percent nonconforming or maximum number of nonconformities per 100 units.

-AQL values above 10.0 are to be interpreted as nonconformities per hundred units.

Then by AQL=150 it is meant that at most 150 nonconformities in one hundred units is acceptable.

-AQL in the tables are given in percent; i.e. , if AQL= 2.5% for example, we search for 2.5 in the tables not 0.025.

Since AQL is usually specified by and stated on a contract and the supplier or manufacturer guarantee that products are of acceptable quality when sold to the consumer, a caution is worth noting here from ISO2859-1 manual:

#### **CAUTION –**

"The designation of an AQL shall not imply that the supplier has the right knowingly to supply any nonconforming unit of product. In other words, the above phrase from ISO2859-1 states that AQL should not be regarded as a production goal or as permission to produce defects.

In using the plans indexed by AQL, the consumer is sure that the acceptance probability of the high-quality lots is high and the probability for bad lots is low."

It should be added that in certain literatures it is discussed that:

-LQ sampling *plans*, dealt with in the previous chapter and in ISO 2859-2, provide *protection* to the *consumer from accepting poor lots* , which is especially important when health and human welfare are involved.<sup>2</sup>

-AQL plans protects producers from having good lots rejected<sup>3</sup>.

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<sup>1</sup> From: [https:// qualityinspection.org/what-is-the-aql](https://qualityinspection.org/what-is-the-aql))

<sup>2</sup> [www.pharmaceuticalonline.com/doc/how-to-establish-sample-sizes-for-process-validation-using-ltpd-sampling-0001](http://www.pharmaceuticalonline.com/doc/how-to-establish-sample-sizes-for-process-validation-using-ltpd-sampling-0001)

<sup>3</sup> e.g, see [www.smartersolutions.com/resources/acceptable-quality-level-issues-and-resolution](http://www.smartersolutions.com/resources/acceptable-quality-level-issues-and-resolution) and Bowker & iberman ,1972 page522)



### 11.1.2 Acceptable Probabilities of lots if percent nonconforming= AQL:

-To calculate the probability of accepting the lots, use binomial distribution with AQLs less or equal 10, and Poisson distribution for AQL >10 i.e. 15 or more.

-When contracting, various AQL values might specified for a desired defect whose acceptance probability varies depending on the sample size and acceptance number ,almost from 88% to 99% (based on Grant and leaven worth ,1988 in Chapter 14 page 456 and Bowker &Lieberman,1972,page 523 on MILSTD-105-E which is essentially the source of ISO 2859).

-In this standard the decision to accept or reject the lots might be taken in one phase, two phases or several phases; in other words there are single sampling plans, double sampling plan s and multiple sampling plans in ISO 2859 standard.

#### 11.1.2.1 Break Even Point in Quality & Establishing AQL

Feigenbaum (1991) on page 504 gives the following definition for breakeven point(BEP) in quality:

Break even point for a given part of quality characteristic may be defined as the percentage ratio between two costs i.e.

"the cost of eliminating nonconforming units by inspection"  
and

"the cost for repairing [the product] when nonconforming units have been allowed to move onto the manufacturing floor"

The calculation of BEP value for a single part or quality characteristic follows:

1)Determine the cost of removing non conforming parts by inspection

2)Determine the average repair cost for units or assemblies made using these nonconforming parts.

3) calculate BEP from(Feigenbaum,1991,page 504)

$$BEP = \frac{\text{Cost of inspection per part}}{\text{Cost of repair per nonconforming assembly}}$$

Taylor(1992) gives the following relationship for calculating BEP:

$$\text{break event point } p' = BEP = \frac{U_h + \frac{S_h}{N}}{E \times (L + D - P)}$$

where

D	Discard value of unit
E	Efficiency of the 100% inspection
L	Loss incurred per defective unit sold
N	Lot size
P	Price unit sold for
$S_h$	Per lot setup cost of the 100% inspection
$U_h$	Per unit cost of the 100% inspection

The proof and illustration is given at the end of the chapter.

### 11.1.2.2 .1 Determining AQL from BEP

If product AQLs are viewed as break even qualities, then the BEP can be used to establish AQLs. Feigenbaum (1991) presents the following 2 simple approaches to select AQL using BEP:

#### First Approach

Select the AQL closest to or equal to the BEP from the AQL's used in standard Tables F, G and H (Feigenbaum,1991) i.e. from the following series of percents(%):

0.010, 0.015, 0.025, 0.040, 0.065, 0.10, 0.15, 0.25, 0.40, 0.65, 1, 1.5, 2.5, 4, 6.5, 10....

#### Example 11.1

In a factory the cost per unit of a kind product for complete inspection test is 0.01 dollar and cost of repair cost if nonconforming unit found in assembly is 1\$. What AQL do you suggest for this case if the lots are to be inspected using an AQL-indexed plan?

$$BEP = \frac{0.01}{1.1} = 0.0091 = 0.91\%$$

In Table F the closest AQL to 0.91 is 1 then AQL= %1.

This common approach is relatively inaccurate and in certain cases sacrifices some of the economy possible with sampling(Feigenbaum , 1991,page 505)

#### Second Approach

Relate the BEP to an AQL given in the following table(Feigenbaum , 1991):

BEP(%)	0.5-1	1-1.75	1.75-3	3-4	4-6	5.6-10.5	10.5-17
AQL(%)	0.25	0.65	1	2.5	4	6.5	10

### Example 11.2

Solve the previous example using the second approach.  
BEP=0.91% which lies in the interval 0.5-1. From the table AQL=0.25%; then a plan with this AQL is appropriate for inspecting the lots containing this product.

Feigenbaum(1991) adds that whether Approach 1 or 2 will be used in a given factory –or whether this kind of calculations is appropriate or not – depends on almost upon the circumstances in the factory...

In actual practice ,accurate details for BEP calculations may not be readily available in a given factory; and AQL selection is often dictated by past experience on a given part or quality characteristic (Feigenbaum,1991 page 507). The agreement between the manufacturer and the consumer is also important in establishing AQL (based on Eskandari,2000, page 14).

Concerning to use an AQL for each characteristic or an overall AQL for similar characteristics such as thickness, outer-diameter etc. or to use an AQL for each critical characteristic are dealt briefly in such references as Feigenbaum(1991) page 508. In this regard the following quotation is worth mentioning:

"Defects are commonly classified as critical, major, and minors. These categories correspond to differ severity's.... Within a category, the defects have similar consequences. For example, major defects render the product nonfunctional. The customer is little concerned with why it doesn't work. The impact is the same. Therefore it makes sense to group these defects together if their inspection costs are similar. In particular, visual defects should be grouped together. Functional defects with higher costs of testing should be inspected separately, not only using a separate sampling plan but also with a higher AQL. "

(<https://variation.com/classifying-defects-and-selecting-aqls/>).

To determine a plan for inspecting a lot given an AQL using ISO2859-1 some terms including inspection level and inspection severity are needed which are defined here.

#### *11.1.3 Inspection level*

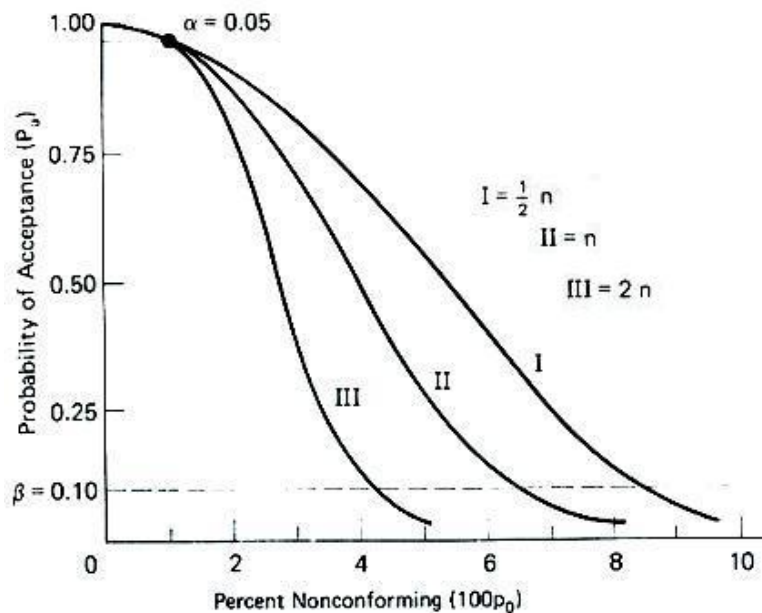
The inspection level of a plan gives a relationship between the lot size and the sample size. ISO 2859-1 specifies this relationship in English letters (A, B,...Q,R,..) given in Table E (reproduced) at the end of this book. As the table shows there are 7 levels grouped into two categories: General(I,I,III) and special (S-1,S-2,S-3,S-4).Three inspection levels, I, II, and III are for general use.

**Unless otherwise specified, level II shall be used.** Level I may be used when less discrimination is needed or level III when greater discrimination is required. Four additional special levels S-1, S-2, S-3 and S-4 are also given and may be used where relatively small sample sizes are necessary and large sampling risks can or shall be tolerated.

In the designation of inspection levels S-1 to S-4, care shall be exercised to avoid AOLs inconsistent with these inspection levels. In other words, the purpose of the special inspection levels is to keep samples small where necessary. For instance, the code letters under S-1 go no further than D, equivalent to a single sample of size 8, but it is of no use to specify S-1, if the AOL is 0,1 % for which the minimum sample size is 125."(Section 10.1, ISO2859-1 manual).

However, Inspection Level I may be specified when less discrimination is needed, or Level III may be specified for greater discrimination ( Grant & Leven worth ,1988,page 455).

Different levels of inspection provide approximately the same protection to the producer, but different protections to the consumer. Level III gives a steeper OC curve and increased inspection costs;... an application of it is for complex and expensive items (Besterfiled,1990 page 238)



**Fig. 11-1 Comparison of general inspection levels**  
(after Besterfiled,1990 page 239)

Levels S-1 through S-IV are useful where the cost of inspecting a product is high and therefore small samples are appropriate (Eskandari, 2000). An application of special levels is where destructive tests have to be used for inspection, which impose a largish expenditure.

Shifting from Level I to Level II and then to III increases the inspection volume (Eskandari,2000) and the discrimination power of bad lots from good ones. Level II has a medium power of discrimination between good and bad lots; whereas that of level III is as 1.6 times as for level II, because its sample is larger and that of level I is 60% of that of level II( Ebrahimi, 1992).

#### 11.1.4 inspection severity

The inspection severity is one of the criteria used as an input to determine the sample size. This criterion is classified into three groups:

##### a) Normal

A normal inspection severity results in a medium sample size.

Section 9. 1 ISO 2859-1 states :"**Normal** inspection shall be carried out at the start of inspection, **unless otherwise directed by the responsible authority.**" Table H deals with normal inspection.

It is advised If there is not a history of the qualities of lots supplied by the manufacturer or the vendor, acceptance sampling should be started at a normal severity (Eskandari,2000). The standard states some conditions for switching from one severity to another.

##### b) Tightened

More severe acceptance criteria designed to protect the customer must be used whenever the quality history is unsatisfactory. This gives the concept of tightened inspection as an alternative to normal inspection. A tightened inspection severity results in a larger sample size. Tightened inspection is intended when the submitted quality is worse than the AQL (Bowker& Lieberman,1972 page523). Table F Is concerned with this inspection.

##### c) Reduced

A reduced inspection severity results in a smaller sample size. Table G Is concerned with this inspection.

#### Example 11.3

For a batch of size  $N=600$  in normal inspection, determine the codes and the sample sizes for all inspection levels.

The codes (from Table E) and the sample sizes ( from Table H )for different levels are as follows :

Inspection level	Code	Sample size Normal inspection(AQL=10)
I	G	32
II	J	80
III	K	125
S-1	C	5
S-2	C	5
S-3	E	13
S-4	F	20
Notice that for certain other AQLs ,when encountered an arrow, the size at the arrowhead in the table has to be read, not the size for the original code		

It is worth explaining that in the ISO2859-1 Table for reduced inspection, there are some plans for which the difference of rejection acceptance numbers i.e. "Re-Ac" is greater than 1. What should be done if the difference between Ac and Re is more than 1?

In this case the standard ISO2859-1 states in Sec. 11.1.4:

"In reduced inspection, the sample may contain a number of nonconforming units or non conformities per 100 units between the acceptance and rejection numbers. In these circumstances, the lot shall be considered acceptable, but normal inspection shall be reinstated starting with the next lot".

Of course if the conditions changes, reduced inspection might be resumed. An advice is to use normal inspection when the quality standard is equal or better than the AQL; tightened, when the quality standard appears to have deteriorated; and reduced in case of the improvement of the quality level beyond AQL (Derman and Ross, 1997).

## 11.2 Selection an AQL-indexed plan from ISO 2859-1 Tables

After deciding on the AQL, the inspection level, inspection severity and type of sampling (single, double or multiple ) the following steps have to be followed . The following steps are for designing single plans whose tables are reprinted at the end of this book. Similar steps are followed for the other 2 types, but are not covered here.

### Step 1 :

Given the lot size and the inspection level, read a letter code from Table E (Table 1 in ISO 2859-1:1999). The default level is II. Not that if you

would like to inspect  $m$  similar lot with size  $N$  from the same supplier, you could take the lot size as  $m \times N$ .

**Step 2 :**

Given the AQL and the code of step 1, refer to Table H,F or G (Table 2-A&B&C in ISO 2859-1:1999) depending on the severity of the inspection and read the plan specifications (i.e. sample size, acceptance and rejection numbers (Ac, Re)). If encountered an arrow, ignore the sample size next to the letter code obtained in step 1 and read the plan corresponding to the letter code at the arrowhead.

The sample has to drawn at random from the lot, representing the whole lot.

If number of nonconformities is less or equal Ac, accept the lot, if it is greater or equal Re do not accept it.

If the sample size exceeds the lot size, the entire lot has to be taken as the sample.

The responsible authority shall decide how lots which are not accepted will be disposed of. Such lots may be scrapped, sorted (with or without nonconforming units being replaced), reworked, re-evaluated against more specific usability criteria, held for additional information, etc(ISO2859-1 section 7-1).

It should be emphasized that, although the absolute sample size( $n$ ) increases with lot size( $N$ ) in this standard system, the relative sample size ( $\frac{n}{N}$ ) decreases as the following illustrative table shows ( based on Grant and Leavenworth,1988 page 458):

N	Normal inspection Level II with AQL=1		$\frac{n}{N}$
	N	c	
51-90	13	0	$(\frac{13}{51}, \frac{13}{90})$
281-500	50	1	$(\frac{50}{281}, \frac{50}{500})$
1201-3200	125	3	$(\frac{125}{1201}, \frac{125}{3200})$
35001-150000	500	10	$(\frac{500}{35001}, \frac{500}{150000})$

**Example11.4**

Using ISO 2859 standard find a single acceptance sampling plan for normal inspection of lots of size 20000 with AQL=0.015%.

**Answer:**

For  $N=20000$  and default Level II from Table E gives code letter M. With this code and  $AQL = 0.015$  under normal inspection we encounter an arrow in Table H. The letter code at the arrowhead is P; then

Sample size  $n=800$

Acceptance number  $Ac=1$

Rejection number  $Re=0$ .

It is worth remembering if encountered with an arrow, the first sampling plan below or above the arrowhead has to be selected. For example in Table H for code E,  $n=13$ ; but for  $AQL=0.10\%$  we encounter an arrow; the plan  $n=125$  and  $c=0$  is given by the arrowhead and also for  $AQL=0.065\% \Rightarrow n=200$  &  $c=0$  from Table H.

**Example 11.5**

A lot of size 10000 is to be inspected using ISO 2859 standard with  $AQL = 0.25\%$ . Find

- a single acceptance plan for each of the 3 types of severity,
- What is the acceptance probability for the 3 plans of part a if the proportion nonconforming is equal to acceptable quality limit i.e.  $p' = 0.0025$ .

**Solution**

The code from Table E with default Level II is L. The plans read from Tables F, G, H with the code and the acceptance probability calculated using MATLAB command *normcdf* are shown below:

Severity	Table	Sample size	Ac	Re	Pa (For $p'=0.025$ )
Normal	H	200	1	2	$normcdf(Ac, n, p') = 0.8134$
Reduced	G	80	0	2	$normcdf(1, n, p') = 0.9826$
Tightened	F	315	1	2	$normcdf(Ac, n, p') = 0.8134$

**Example 11.6**

If  $N=400$ ,  $AQL=0.015\%$  what plan should be used with normal inspection?

**Answer**

With the default Level II and  $N=400$  from Table H the code is read "G". with  $AQL=0.015$  and Code G in Table H it is encountered with an arrow which yields  $n=800$ .

Since  $N=400$ , 100% inspection has to be done.



**Example 11.7**

For a lot of size 2500, find a normal and a tightened single sampling plan with AQL=4%.

**Solution**

From Table E the letter code is K,

For normal severity (Table H)

Sample size  $n = 125$

Acceptance no.  $Ac = 10$

Rejection no.,  $Re = 11$

For tightened severity (Table F)

Sample size  $n = 125$

Acceptance no.  $Ac = 8$

Rejection no.,  $Re = 9$

**Example 11.8**

In the previous example if the proportion nonconforming of the process is  $p' = 0.04$ , calculate the acceptance probability for the plan with normal and also tightened severity.

**Solution**

Normal:

$$MATLAB \text{ binocdf}(10, 125, 0.04); \quad Pa = \Pr(X \leq 10) = 0.9881$$

Tightened:

$$MATLAB \text{ binocdf}(8, 125, 0.04); \quad Pa = \Pr(X \leq 8) = 0.9359$$

Poisson Approximation:

$$Poisson \quad \lambda = np' = 5 \quad Pa = \Pr(X \leq 8) = 0.9319$$

**Example 11.9**

Solve Example 11.7 for all levels under normal severity.

Answer is given in the table below:

N=2500, AQL=4%, Normal severity								
Level		I	II	III	S1	S2	S3	S4
Code (from Table E)		H	K	L	C	D	E	F
From Table G	n	50	125	200	3	13	13	32
	Ac	5	10	14	0	1	1	3
	Re	6	11	15	1	2	2	4
$Pa(p' = AQL) = \text{binocdf}(Ac, n, 0.04)$		0.986	0.988	0.985	0.885	0.907	0.907	0.962

**Example 11.10** (from ISO 28590:2017)

A product has five dimensions to be checked. Dimensions 1 and 2 are in Class A with an AQL of 0,65 % and the other three dimensions are Class B with an AQL of 2,5 %. It was specified that general inspection level III should be used for all of the dimensions. The product is produced in lots of 900 items.

Table E gives code letter K for this situation.

Table H indicates the single sample size for normal inspection to be 125 and the acceptance numbers are 2 and 7 for AQL of 0,65 % and 2,5 %, respectively. Suppose a particular lot was inspected with this procedures and the results are

- one item nonconforming in dimension 1 only,
- one item nonconforming in dimensions 2 and 4,
- two items nonconforming in dimension 3 only, and
- three items nonconforming in dimensions 3 and 4.

There are two nonconforming items in Class A and five in Class B, therefore the lot is accepted.

**End of Example 11.10.**

### 11.3 Switching Rules of ISO 2859

The switching rules are an inherent part of acceptance sampling Standards such as ISO 2859-1 when a series of lots are being inspected with the standard. The rules constitute a regime to change between different levels of inspection (normal, tightened, reduced) according to previous decisions on lot acceptance. One purpose of the regime is to protect consumers against poor quality and pressure on the provider to submit satisfactory quality; the other is reducing the inspection cost in cases of proven good process quality (Balamurali et al. ,2008) i.e.to stop unnecessary inspections and therefore saving time and money.

The switching rules switch back and forth between three sampling plans that have different OC curves. That is, the normal plan, the tightened plan, and the reduced plan each has its own OC curve:



The rules are as follows(from ISO2859-1)

**The inspection often begins with normal severity ,unless otherwise stated.**

### **11.3.1 Normal to tightened**(from ISO2859-1)

When normal inspection is being carried out, tightened inspection shall be implemented as soon as 2 out of 5(or less than 5) consecutive lots have been non-acceptable on original inspection (that is, ignoring resubmitted lots or batches for this procedure).

### **11.3.2 Tightened to normal** (from ISO2859-1)

When tightened inspections being carried out, normal inspection shall be re-instated when five consecutive lots have been considered acceptable on original<sup>1</sup> inspection.

### **11.3.3 Reduced to Normal**(from ISO2859-1)

When reduced inspection is being carried out, normal inspection shall be re-instated if any of the following occur on original inspection:

- a) a lot is not accepted; or
- b) production becomes irregular or delayed; or
- c) other conditions warrant that normal inspection shall be re-instated.

### **11.3.4 Normal to reduced**

(from ISO2859-1)

When normal inspection is being carried out, reduced inspection shall be implemented provided that **all** of the following conditions are satisfied:

- a) the current value of an indicator named the "switching score" (see Sec. 11.3.4.1 below) is at least 30; and
- b) production is at a steady rate; and
- c) reduced inspection is considered desirable by the responsible authority.

In reduced inspection, the sample size becomes 40% of that in normal inspection. This increases consumer's risk ( $\beta$ ) and reduces producer's risk ( $\alpha$ ) slightly.

#### **11.3.4.1 Switching score**

Switching score is an indicator, used under normal inspection to determine whether the current inspection results are sufficient to allow for a switch to reduced inspection.

### **Calculation of the switching score(SC)**(from ISO2859-1)

ISO 2859-1 manual on page 12 presents the following procedure for calculating the switching score denoted by SC:

The calculation shall be initiated at the start of normal inspection unless otherwise specified by a responsible authority. The switching score shall be set

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<sup>1</sup> Original inspection means first inspection of a lot according to the provisions of ISO 2859-1

at zero at the start and updated following the inspection of each subsequent lot on original normal inspection.

<b>Table 11-1 Calculation of SC, for switching from normal to reduced</b>										
Lot No	Lot Size	AQL %	CODE	Sample size	$A_c$	defectives	STEP	SC	Future Action	Comments
Value of SC at the start								0		
1	180	0.40	G	32	0	0	II	2	Continue normal	Begin inspection with normal
2	200	1.5	G	32	1	1	II	4	Continue normal	
3	250	0.40	G	32	0	1	II	0	Continue normal	
4	450	1.0	H	50	1	1	II	2	Continue normal	
5	300	1.0	H	50	1	1	II	4	Continue normal	
6	80	1.5	E	8	0	1	II	0	Switch to tightened	2 lots rejected form 5 lots
7	800	1	J	80	1	1	--	--	Continue tightened	SC calculation Not applicable
8	300	0.4	H	50	0	0	--	--	Continue tightened	SC calculation Not applicable
9	100	1	F	20	0	0	--	--	Continue tightened	SC calculation Not applicable
10	600	1	J	80	1	0	--	--	Continue tightened	SC calculation Not applicable
11	200	2.5	G	32	1	1	--	--	Restore normal	
12	250	0.65	G	20	0	0	II	2	Continue normal	
13	600	1.5	J	80	3	1	I	5	Continue normal	Lot would be rejected if AQL were one degree tighter i.e.1%
14	80	1.5	E	8	0	0	II	7	Continue normal	
15	200	0.40	G	32	0	0	II	9	Continue normal	
16	500	1.0	H	50	1	0	II	11	Continue normal	
17	100	2.5	F	20	1	0	II	13	Continue normal	
18	120	2.5	F	20	1	0	II	15	Continue normal	
19	85	1.5	E	8	0	0	II	17	Continue normal	
20	300	1.0	H	50	1	1	II	19	Continue normal	
21	500	1.0	H	50	2	0	II	21	Continue normal	
22	700	1.5	J	80	3	1	I	24	Continue normal	Lot would be accepted if AQL were one degree smaller i.e. 1%
23	600	1.5	J	80	3	0	I	27	Continue normal	Lot would be accepted if AQL were one degree smaller i.e. 1%
24	550	1.5	J	80	3	0	I	30	Switch to reduced	
25	400	1.0	H	20	0	0	--	--	Continue reduced	

## Steps for SC calculations in single sampling plans:

Step 0-[Set SC= zero at the start];

**Step I** -when the acceptance number is 2 or more, add 3 to the switching score if the lot would have been accepted if the AQL had been one step tighter; otherwise reset the SC to zero;

**Step II**- when the acceptance number is 0 or 1, add 2 to the switching score if the lot is accepted; otherwise reset the SC to zero.

The application of SC is illustrated in the Table 11-1 in which 25 lots are inspected (Based on Annex A of ISO2859-1 manual).

### 11.3.5 Discontinuation of inspection

If the cumulative number of lots not accepted, in a sequence of consecutive lots on original tightened inspection, reaches 5, the acceptance procedures of this part of ISO 2859 shall not be resumed until action has been taken by the supplier to improve the quality of the submitted product or service, and the responsible authority has agreed that this action is likely to be effective. Tightened inspection shall then be used as if normal to tightened had been invoked.

#### Example 11.11

(Extracted from ISO 28590:2017 manual)

A kind of product is being supplied in lots of 4000. The AQL is 1.5% nonconforming. The general inspection level III is used with single sampling. Table E (Table 1 in ISO 2859-1:1999)for N=4000 gives code M, and Tables H ,F, G(Tables 2-A, 2-B and 2-C in ISO 2859-1:1999 ) give the following sampling plans .

Severity	Normal	Tightened	Reduced
Table	H	F	G
N	315	315	125
Ac	10	8	6
Re	11	9	7

Inspection starts with the normal plan. suppose lots 1 through9 were accepted; lots 10 and 12 were not accepted, requiring a switch to tightened inspection.

Normal inspection cannot be resumed until five successive lots have been accepted. If five lots are not accepted while on tightened inspection, the sampling inspection will be discontinued.

If the switching score calculated according to the ISO 2859-1 procedure reaches 30, reduced inspection may be used until a lot is not accepted.

## 11.4 Consumer's risk quality(CRQ=LQ=LTPD)<sup>1</sup> Tables in ISO2859-1

Although ISO2859-1 standard deals mainly with AQL, but there are some tables in it which are concerned with consumer's risk quality (CRQ=LQ=LTPD). Before introducing these tables let us read some quotations in the following paragraph:

There are two basic approaches to acceptance quality control, depending upon the nature of the lots to be inspected. A continuing supply of lots from the same producer is most effectively treated by a sampling scheme. A single lot, unique in itself, is treated by sampling plans designed for use with an **"isolated lot"**. This distinction is fundamental to acceptance sampling, and even the basic probability distributions used in these two cases are not the same(Schilling,Neubauer,2008).

If the series of lots is not long enough to allow the switching rules to be applied, it maybe desirable to limit the selection of sampling plans to those, associated with a designated AQL value, that give consumer's risk quality not more than a specified limiting quality protection. Sampling plans for this purpose can be selected by choosing a CRQ and a consumer's risk ( $\beta$  =probability of lot acceptance) to be associated with it (from:ISO2859-1 manual). In other words if a sampling plan is desirable for a lot or batch of an isolated nature, it should be chosen based on the CRQ and  $\beta$  (extracted from [homes.ieu.edu.tr/~ykazancoglu/BA410/.../0135033519\\_pp10a.ppt](http://homes.ieu.edu.tr/~ykazancoglu/BA410/.../0135033519_pp10a.ppt))

Tables 6-A,6-B,6-C and 7-A,7-B,7-C of ISO2859-1 give values of LTPD or LQ, changed to consumer's risk quality( CRQ) in the heading of these tables, for which the probability of lot acceptance under various sampling plans is  $\beta = 10\%$ . Tables 6(-A,B&C) apply when inspecting for nonconforming items and Tables 7(-A,B,C) apply ,when inspecting for number of nonconformities. For individual lots with quality levels less than or equal to the tabulated values of CRQ, the probabilities of lot acceptance are less than or equal to 10%. When there is reason for avoiding more than a limiting percentage of nonconforming units in a lot, Tables 6 and 7 may be useful for fixing minimum sample sizes to be associated with the AQL and the level specified for the inspection of the series of lots. ISO 2859-2 gives further details of the method of sampling for lots in isolation. In this book only one of the tables i.e.:

Table 6-A :CRQ for which  $\beta = 10\%$  has been reproduced in Table 11.2.

---

<sup>1</sup> Consumer 's risk quality is the percentage of nonconforming units (or nonconformities) in a batch or lot for which for purposes of acceptance sampling, the consumer wishes the probability of acceptance to be restricted to a specified low value(Elementary Statistical Quality Control, 2nd Edition edited by John T. Burr)

C O D E	n	<b>Table 11.2</b> (Table 6-A, ISO2859-1 page 38) <b>CRQ=LQL=LTPD (in percent nonconforming) &amp; <math>\beta = 0.10</math> for</b> <b>Single sampling, Normal inspection</b>															
		AQL(%) percent nonconforming items															
		0.010	0.015	0.025	0.040	0.065	0.10	0.15	0.25	0.40	0.65	1.0	1.5	2.5	4.0	6.5	10
A	2															68.4	69.0 <sup>*</sup>
B	3														53.6	54.1	57.6 <sup>*</sup>
C	5													36.9	37.3	39.8	58.4
D	8												25.0	25.2	27.0	40.6	53.8
E	13											16.2	16.4	17.5	26.8	36.0	44.4
F	20										10.9	11.0	11.8	18.1	24.5	30.4	41.5
G	32									6.94	7.01 <sup>*</sup>	7.50	11.6	15.8	19.7	27.1	34.0
H	50							4.50	4.54 <sup>*</sup>	4.87 <sup>*</sup>	7.56	10.3	12.9	17.8	22.4	29.1	
J	80						2.84	2.86 <sup>*</sup>	3.07 <sup>*</sup>	4.78	6.52	8.16	11.3	14.3	18.6	24.2	
K	125					1.83	1.84 <sup>*</sup>	1.97 <sup>*</sup>	3.08	4.20	5.27	7.29	9.24	12.1	15.7	21.9	
L	200				1.14	1.16 <sup>*</sup>	1.24 <sup>*</sup>	1.93	2.64	3.31	4.59	5.82	7.60	9.91	13.8		
M	315			0.728	0.735 <sup>*</sup>	0.887 <sup>*</sup>	1.23	1.68	2.11	2.92	3.71	4.85	6.33	8.84			
N	500		0.459	0.464 <sup>*</sup>	0.497	0.776	1.06	1.33	1.85	2.34	3.06	4.00	5.60				
P	800		0.287	0.290 <sup>*</sup>	0.311	0.485	0.664	0.833	1.16	1.47	1.92	2.51	3.51				
Q	1250	0.184	0.186 <sup>*</sup>	0.199 <sup>*</sup>	0.311	0.425	0.534	0.741	0.940	1.23	1.61	2.25					
R	2000	0.116 <sup>*</sup>	0.124 <sup>*</sup>	0.194	0.266	0.334	0.463	0.588	0.769	1.00	1.41						

**NOTES**

- 1 At the consumer's risk quality, 10% of lots will be accepted
- 2 All the values are based on binomial distribution
- 3 Superscript(\*) denotes that the value is for the original fractional acceptance number sampling plan(See Table 11-A of ISO 2859-1 standard manual on page 81)

Some applications of the table are illustrated here:

**Example 11.12****LTPD=CRQ with  $P_a=10$  and AQL are given to design a plan**

(based on example ISO 2859-1:1999 manual page 14)

Assume a CRQ of 5% nonconforming items with an associated probability of acceptance of 10 % or less is desired for individual lots. If an AQL of 1 % nonconforming item is designated for inspecting the series of lots, determine the corresponding sample size code.

Table 11-1 indicates that the minimum sample size shall be given by sample size letter code L.

**Example 11.13****Determining the CRQ=LQ with  $P_a=0.1$  of an AQL-indexed Plan**

A lot of size 400 is to be inspected with a single sampling plan of normal severity and AQL=4% .

- a) Find the consumer's risk quality (CRQ) for which the probability of acceptance is 10%.
- b) What are the specifications of the plan?
- c) What is the probability of acceptance if the proportion non-conforming is  $p'=4\%$ .

**Solution**

- a) The default level is II. The letter code from Table E for  $N=400$  is H. Given  $AQL=4\%$  and code H, from Table 11-1  $CRQ=LQ$  is read as 17.8%.
- b) Table H  $\Rightarrow n=50, Ac=5, Re=6$ .
- c)  $P_a$ , calculated by `binocdf(5,50,0.04)` in MATLAB, is 0.9856.

**Example 11.14**

A single sampling plan ( $n=315, c=21$ ) is used with normal inspection,  $\beta=10\%$  &  $CRQ=LQ=9\%$ . What AQL corresponds to this case?

**Answer**

From Table H, the code of the plan is M.

In Table 11-1 with this code, the nearest CRQ to 9 is 8.9 under the column 4; then  $AQL=4\%$ .

## 11-5 Designing single acceptance sampling plan with two points of an OC curve

In this section designing plans given any 2 points of OC curve or the 2 corresponding with  $\alpha$  &  $\beta$  is described

### 11.5.1 Constructing single sampling plan, given any 2 arbitrary points of OC curve

Suppose we have 2 points of an OC curve (each showing a proportion nonconforming and the corresponding  $P_a$ ) related to an unknown single sampling plan and would like to determine the specifications of the plan i.e. the sample size ( $n$ ) and the acceptance number ( $c$ ).

If the proportion non-conforming of a process is denoted with  $p'$ , the probability of accepting a lot from this process with a single sampling plan  $n$  &  $c$  is given by:

$$P_a = \Pr(X \leq c) =$$

$$\sum_{x=0}^c \binom{n}{x} p'^x (1-p')^{n-x} = \binom{n}{0} (1-p')^n + \binom{n}{1} p' (1-p')^{n-1} + \dots + \binom{n}{c} p'^c (1-p')^{n-c}$$

If the  $n$  and  $c$  is not known and 2 points of the curve i.e.

$$\begin{cases} p'_1 \\ Pa_1 \end{cases} \begin{cases} p'_2 \\ Pa_2 \end{cases} \quad p'_2 > p'_1$$



are known, the coordinates of the points are substituted in the following relationship

$$P_a = \binom{n}{0} (1-p')^n + \binom{n}{1} p' (1-p')^{n-1} + \dots + \binom{n}{c} p'^c (1-p')^{n-c}$$

and n & c be determined from the resulting simultaneous equations:

$$\begin{cases} P_{a_1} = \binom{n}{0} p_1'^0 (1-p_1')^n + \dots + \binom{n}{c} p_1'^c (1-p_1')^{n-c} \\ P_{a_2} = \binom{n}{0} p_2'^0 (1-p_2')^n + \dots + \binom{n}{c} p_2'^c (1-p_2')^{n-c} \end{cases} \quad \text{or} \quad \begin{cases} P_{a_1}' = \text{binocdf}(p_1', n, c) \\ P_{a_2}' = \text{binocdf}(p_2', n, c) \end{cases}$$

### Example 11.15

Design a single sampling plan such that if used with a lot from a process having 1% proportion nonconforming, the lot will be accepted 90% of the times and if used with a lot from a process having 10% proportion nonconforming, the lot will be accepted with the probability of 0.1.

### Solution

$$\begin{cases} 0.90 = \text{binocdf}(0.01, n, c) \\ 0.03 = \text{binocdf}(0.1, n, c) \end{cases}$$

The above system could be solved using the following code

```
clc;
d=0:.01:.21;
D1=1000000000;D2=1000000000;
i=1;
while abs(D1)>= d(i) & abs(D2)>= d(i);
    for n=1:60;
        for c=0:10
            D1= binocdf(c,n,.01)-.90;
            D2= binocdf(c,n,.1)-.03;
            if (abs(D1)<=d(i)&abs(D2)<=d(i));
                break;
            end;
        end;
    end;
    i=i+1;
end; disp(sprintf(' n= %3.0f, c= %2.0f', n, c));
Running the code results in n= 60 ,c= 1
```

**Table 11-3 Cameron's Table for constructing single-sampling plans given  $\alpha, \beta$** 

(Table T5-1, Schilling &amp; Neubauer, 2009 p 629; Table 13-13 Bowker &amp; Lieberman, 1972 p 540)

C	$p'_\beta/p'_{1-\alpha}$ for $\alpha = 0.05$			$np'_{1-\alpha}$	C	$p'_\beta/p'_{1-\alpha}$ for $\alpha = 0.01$			$np'_{1-\alpha}$
	$\beta = 0.10$	$\beta = 0.05$	$\beta = 0.01$			$\beta = 0.10$	$\beta = 0.05$	$\beta = 0.01$	
0	44.890	58.404	89.781	0.052	0	229.105	298.073	458.210	0.010
1	10.946	13.349	18.681	0.355	1	26.184	31.933	44.686	0.149
2	6.509	7.699	10.280	0.818	2	12.206	14.439	19.278	0.436
3	4.890	5.675	7.352	1.366	3	8.115	9.418	12.202	0.823
4	4.057	4.646	5.890	1.970	4	6.249	7.156	9.072	1.279
5	3.549	4.023	5.017	2.613	5	5.195	5.889	7.343	1.785
6	3.206	3.604	4.435	3.286	6	4.520	5.082	6.253	2.330
7	2.957	3.303	4.019	3.981	7	4.050	4.524	5.506	2.906
8	2.768	3.074	3.707	4.695	8	3.705	4.115	4.962	3.507
9	2.618	2.895	3.462	5.426	9	3.440	3.803	4.548	4.130
10	2.497	2.750	3.265	6.169	10	3.229	3.555	4.222	4.771
11	2.397	2.630	3.104	6.924	11	3.058	3.354	3.959	5.428
12	2.312	2.528	2.968	7.690	12	2.915	3.188	3.742	6.099
13	2.240	2.442	2.852	8.464	13	2.795	3.047	3.559	6.782
14	2.177	2.367	2.752	9.246	14	2.692	2.927	3.403	7.477
15	2.122	2.302	2.665	10.035	15	2.603	2.823	3.269	8.181
16	2.073	2.244	2.588	10.831	16	2.524	2.732	3.151	8.895
17	2.029	2.192	2.520	11.633	17	2.455	2.652	3.048	9.616
18	1.990	2.145	2.458	12.442	18	2.393	2.580	2.956	10.346
19	1.954	2.103	2.403	13.254	19	2.337	2.516	2.874	11.082
20	1.922	2.065	2.352	14.072	20	2.287	2.458	2.799	11.825
21	1.892	2.030	2.307	14.894	21	2.241	2.405	2.733	12.574
22	1.865	1.999	2.265	15.719	22	2.200	2.357	2.671	13.329
23	1.840	1.969	2.226	16.548	23	2.162	2.313	2.615	14.088
24	1.817	1.942	2.191	17.382	24	2.126	2.272	2.564	14.853
25	1.795	1.917	2.158	18.218	25	2.094	2.235	2.516	15.623
26	1.775	1.893	2.127	19.058	26	2.064	2.200	2.472	16.397
27	1.757	1.871	2.098	19.900	27	2.035	2.168	2.431	17.175
28	1.739	1.850	2.071	20.746	28	2.009	2.138	2.393	17.957
29	1.723	1.831	2.046	21.594	29	1.985	2.110	2.358	18.742
30	1.707	1.813	2.023	22.444	30	1.962	2.083	2.324	19.532
31	1.692	1.796	2.001	23.298	31	1.962	2.059	2.293	20.324
32	1.679	1.780	1.980	24.152	32	1.940	2.035	2.264	21.120
33	1.665	1.764	1.960	25.010	33	1.920	2.013	2.236	21.919
34	1.653	1.750	1.941	25.870	34	1.900	1.992	2.210	22.721
35	1.641	1.736	1.923	26.731	35	1.882	1.973	2.185	23.525
36	1.630	1.723	1.906	27.594	36	1.865	1.954	2.162	24.333
37	1.619	1.710	1.890	28.460	37	1.848	1.936	2.131	25.143
38	1.609	1.698	1.879	29.327	38	1.833	1.920	2.118	25.955
39	1.599	1.687	1.860	30.196	39	1.818	1.903	2.098	26.770
40	1.590	1.676	1.846	31.066	40	1.804	1.887	2.079	27.587
41	1.581	1.666	1.833	31.938	41	1.790	1.873	2.060	28.406
42	1.572	1.656	1.820	32.812	42	1.765	1.859	2.043	29.228
43	1.564	1.646	1.807	33.686	43	1.753	1.845	2.026	30.051
44	1.556	1.637	1.796	34.563	44	1.742	1.832	2.010	30.877
45	1.548	1.628	1.784	35.441	45	1.731	1.820	1.994	31.704
46	1.541	1.619	1.773	36.320	46	1.720	1.808	1.980	32.534
47	1.534	1.611	1.763	37.200	47	1.710	1.796	1.965	33.365
48	1.527	1.603	1.752	38.082	48	1.701	1.785	1.952	34.198
49	1.521	1.596	1.743	38.965	49	1.691	1.775	1.938	35.032

If ratio  $p'_\beta/p'_{1-\alpha}$  not found in the appropriate column, choose the value just less than the ratio.

### 11.5.2 Constructing sampling plan given $\alpha$ , $\beta$ using Cameron Table

Cameron's Table, reproduced in Table 11-3, is among the works developed in the field of acceptance sampling (Schilling, & Neubauer, 2008).

If a single sampling plan is to be designed given

$$\alpha = 5\% \text{ or } 1\% \text{ and } \beta = 10\% \text{ or } 5\% \text{ or } 1\%$$

then this table could be used to design the plan.

To construct a plan for a given  $p'_1, \alpha$  and  $p'_2, \beta$ .

Calculate the ratio  $(p'_2 = p'_\beta)/(p'_{1-\alpha} = p'_1)$ .

Refer to the appropriate  $\alpha$  and  $\beta$  column of Table 11-3,

Find the entry which is equal or just less than the desired ratio,

Read the acceptance number (c) directly and also  $np'_1$  from the table,

Determine the sample size by dividing  $np'_1$  by  $p'_1$ ,

(Bowker Lieberman, 1972 page 537).

#### Example 11.16

Find a single sampling plan for which if the lot is from a process with proportion nonconforming 0.02 then  $Pa=0.95$ ; and the lot is from a process with proportion nonconforming 0.04 then  $Pa=0.05$ .

**Solution**

$$p'_1 = 0.02, p'_2 = 0.04, \alpha = 0.5, \beta = 0.05$$

The greater proportion is divided by the smaller :

$$\frac{p'_2}{p'_1} = \frac{p'_\beta}{p'_{1-\alpha}} = \frac{0.04}{0.02} = 2$$

The value in the " $\alpha = 0.5, \beta = 0.05$ " column of Table 11-3 just less than this ratio is 1.999. Hence  $c=22$  and  $np'_1 = 15.719 \Rightarrow n=786$ .

#### Example 11.17

Find a single sampling plan in such a way that

$$\begin{cases} Pa_1 = 1 - \alpha = 0.95 \\ p'_1 = 0.1 \end{cases} \quad \begin{cases} Pa_2 = \beta = 0.1 \\ p'_2 = 0.25 \end{cases}$$

**Solution**

$$\frac{p'_\beta}{p'_{1-\alpha}} = \frac{0.25}{0.1} = 2.5 \quad \text{Table 11-3} \rightarrow \begin{cases} c = 10 \\ np'_{1-\alpha} = 6.169 \Rightarrow n = 62 \end{cases}$$

### *11-6 Designing single or double acceptance sampling plan given Indifference point of OC curve using Philips Standard Sampling system<sup>1</sup>*

How a sampling plan is designed when  $p'_{0.50}$ , the so-called indifference point or control point, of an OC curve is given?

Table D should be used for this purpose. " Table D is a reproduction of an attribute sampling system developed by ( Grant & Leavenworth, 1988, page 504).

Dr. Hamaker and his associates of N.V. Phillips' Gloeilampenfabrieken, Eindhoven, Holland.

Its plans which are either single or double are all based on the indifference point of the OC curve. For lots of size  $N \leq 1000$  the table gives single sampling plans and for  $N > 1000$  gives double sampling plans with  $n_2 = 2n_1$ .

#### **Example 11.18**

For inspecting lots of size  $N=800$ , find a sampling plan whose OC curve passes the indifference point 0.5%. Repeat case for  $N=3000$ .

#### **Answer**

From Table D for  $N=800$

Sample size  $n=225$  Acceptance no.  $C=1$

and also

for  $N=3000$

$n_1=200$   $C_1=0$   $C_2=0$   $n_2=2n_1=400$

The flowchart of Fig. 10-15 of chapter 10 shows how double sampling plans are implemented.

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<sup>1</sup> Developed by Hamaker, H.C., 1949. Lot inspection by Sampling, Philips Technical Review Vol11 pp 176 -182 Dec1949

## Appendix : More on the BEP

Dr Taylor , an expert in the field of acceptance sampling especially for medical purposes, introduces the breakeven point(BEP) in quality in the following way<sup>1</sup>:

Lots with proportion nonconforming up to the AQL are acceptable for release. Any lot [with quality] worse than the AQL should be rejected. This does not mean that all such lots are actually rejected. Often the sampling plan is only capable of rejecting lots significantly above the AQL. However, if a sampling plan rejects a 2% defective lot when the AQL is 1%, no one would consider that an error. This interpretation of the AQL closely coincides with what is called the breakeven point in quality.

The *break-even point* for a process is the point at which the total revenue curve intersects the total cost curve. The *break-even point* for 2 process is the point where their revenue curves or their cost curve intersect. Here the break even quality is that point where the "customer value"(VC) of 100% inspecting a lot is exactly equal to the "customer value" related to the case where no inspection is applied on the lot. Customer value(VC) is defined as(Taylor,1992 p61)

Customer value(VC) =

income from sale of units &discards – Production costs –Inspection Costs – Loss due to sale of defectives.

**Example** (Taylor,1992 page60 )

A switch used in television sets is currently 100% inspected. Quality improvement efforts have recently reduced the incidence of defective switches. As a result, the elimination of the 100% inspection is being considered. To address this issue, one needs to know the lot size, the quality of the lots, and the costs involved. The parameters<sup>2</sup> are as follows:

parameter	Description	value
p'	Lot percent defective	
	Mean Lot percent defective	1%
C	Per unit cost of manufacturing	\$ 2
D	Discard value of unit	\$ 0
E	Efficiency of the 100% inspection=The percentage of defective units detected by 100% inspection(Taylor,1992,page76)	90%
L	Loss incurred per defective unit sold	\$50
N	Lot size	1000

<sup>1</sup><http://www.variation.com/techlib/as-1.html>

<sup>2</sup> These parameters are covered in details in Sec. 4.3 &4.4 of Taylor(1992).

P	Price unit sold for	\$ 5
R	Per unit cost of rework	Not possible
Sh	Per lot setup cost of the 100% inspection	\$ 100
S <sub>p'</sub>	Standard deviation of lot percent defective	2 %
Ss	Per lot setup cost of acceptance sampling	\$ 0
Uh	Per unit cost of the 100% inspection	\$ 0.25
Us	Per unit cost of acceptance sampling	\$ 0.50

We would like to choose the action whose customer value(VC) per lot is more than that of the other action.

**Solution**

VC =

income from sale of units & discards – Production costs – Inspection Costs  
– Loss due to sale of defectives.

Based on the details mentioned in Sec 4.3 & 4.8 of Taylor(1992), to derive the relationship between the VC of 100% inspection and  $p'$  the following elements are included in VC:

Element Description	General Formula	Value for the example
Revenue from selling non defective products	$P \times N(1 - E \times p')$	$5 \times 1000(1 - 0.9p')$
Revenue from selling defective products	$D \times N \times E \times p'$	$0 \times 1000 \times 0.9(p') = 0$
Production cost	$N \times C$	$1000 \times 2$
Cost of 100 percent inspecting a lot of size N	$Sh + N \times Uh$	$(100 + 1000 \times 0.25) = 350$
Loss due to unintentional selling defectives	$L \times N(1 - E) \times p'$	$50000 \times (1 - 0.9)p'$

The VC of 100% inspection is as follows:

$$VC = 5000(1 - 0.9p') + 0 - 2000 - 350 - 50000 \times (1 - 0.9)p'$$

or

$$VC = 2650 - 9500p$$

The elements of applying no inspection:

Element Description	General	For the example
Revenue from selling non defective products	$P \times N$	$5 \times 1000$
Revenue from selling defective products	0	0
Production cost	$N \times C$	$1000 \times 2$
Cost of inspecting the lot N	0	0
Loss due to unintentional selling defectives of the lot of size N having $p'$ as (proportion nonconforming)	$L \times N \times p'$	$50 \times 1000p'$

The VC when applying no inspection would be:

$$VC = 5000 - 2000 - 0 - 50000(p') \quad \text{or}$$

$$VC = 3000 - 50000p'.$$

Figure 11-2 shows these cost lines plotted using the following MATLAB commands:

```
p=0:0.01:0.1; VC=3000-50000*p; plot(p, VC); hold on; VC=2650-9500*p;
plot(p, VC)
```

Here the intersection point is where the customer values of the actions (i.e. "100% inspection of the lot" and "not inspecting the lot and thus a number defectives in the lot are left") intersect. Somewhere between 0% defective and 100% defective, there is a point where it is a tie. This is the break even point of quality calculated as follows:

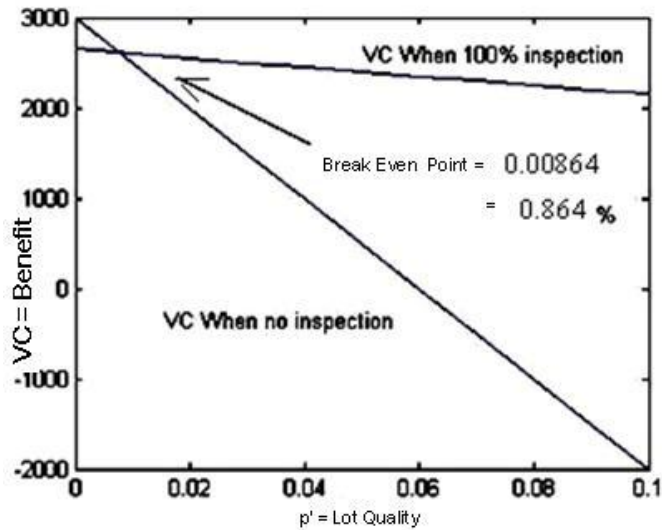
$$5000(1 - 0.9p') + 0 - 2000 - 350 - 50000 \times (1 - 0.9)p'$$

$$= 3000 - 50 \times 1000p'$$

$$\Rightarrow 5(-0.9p') + 50(0.9)p' = \frac{350}{1000}$$

$$\text{break even point } p' = \text{BEP} = \frac{\frac{350}{1000}}{0.9(50 - 5)} = 0.00864$$

or 0.864% nonconforming



**Fig 11-2 Break event Point of Quality, for the example.**

In general by equating the 2 relationships i.e.

$$VC = P \times N(1 - E \times p') + DNE \times p' - NC - Sh + N \times Uh - L \times N(1 - E) \times p'$$

and

$$VC = P \times N - NC - L \times N \times p' \Rightarrow$$

the  $p'$  as the BEP is calculated from the following relationship:

$$\begin{aligned} \text{break event point } p' &= BEP \\ &= \frac{Uh + \frac{Sh}{N}}{E \times (L + D - P)} \end{aligned}$$

where

- D Discard value of unit
- E Efficiency of the 100% inspection
- L Loss incurred per defective unit sold
- N Lot size
- P Price unit sold for
- Sh Per lot setup cost of the 100% inspection
- Uh Per unit cost of the 100% inspection

It is worth mentioning BEP could also help to decide whether to use 100% inspection or sampling inspection, as described below from Feigenbaum(1991) page 504.



First determine the actual percent non-conforming or process average nonconformity for the part or material in question. This value should be determined from the inspection results of several thousand parts. Then by comparing the BEP with average incoming percent nonconforming, it can be determined whether 100% inspection, no inspection or sample inspection is indicated. 4 typical situations may be indicated :

A)

The percent nonconforming of incoming material is ( $p'$ ) fairly close to the BEP, then sampling inspection may be the economical answer.

B)

$p'$  is some what higher than the BEP ,100% inspection will more than pay for itself.

C)

$p'$  is considerably lower than the BEP but erratic, sampling inspection may be indicated purely for the purpose of protection.

D)

$p'$  is considerably lower than the BEP but stable, then a case might be made for no inspection at all.

At the end of the chapter, a table summarizes the application of acceptable sampling Tables described in this chapter.

Quantity index	Single sampling	Double sampling
LQL=LQ=CRQ	Table A,B	C.1,C
IQL	Table D	D
AOQL	Tables I,J	K
AQL	ISO 2859 (Table E+F,G,H)	
2 points with $\alpha, \beta$	Cameron Table	

## Exercises

### 11.1

Full test of a part costs \$1 and \$0.18 has to be paid on the average to repair a product assembled using several parts including a defective one. Calculate the BEP. Determine the appropriate AQL based on this BEP.

### 11.2

A single sampling from ISO2589-1 is to be selected for inspecting lots of size 2500. Specify the plans for normal and tightened inspection of level II and AQL=4%.

Answer:  $n=125$   $c=10$ ;  $n=125$   $c=8$

### 11.3

In Problem 11.2 what is the probability of accepting a lot with each of the plans if the percent of nonconforming  $p'=AQL$ .

### 11.4

Suppose tightened inspection is performed in problem 2 and the percent nonconforming  $p'=4\%$ . What is the probability of shifting to normal severity after inspecting 5 such lots?

### 11.5

A single sampling from ISO2589-1 is to be selected for inspecting lots of size 3000. Specify the plans for reduced inspection of level II and AQL=4%.

Answer:  $n=50$   $Ac=5$  ;  $Re=8$

### 11.6

In problem 11.5, suppose the lot is of 8% nonconforming. Use Poisson approximation to calculate:

a) the lot is accepted and the reduced severity continues.

Ans 0.758

b) the lot is accepted and the severity be switched to normal.

Ans 0.164

c) the lot is rejected.

Ans 0.051

### 11.7

Write a MATLAB code to specify a single sampling plan in such a way that 6% of the lots are accepted if the proportion nonconforming is 0.005 94% of the lots are accepted if the proportion nonconforming is 0.1%.

### 11.8

Use Cameron Table to specify a single sampling plan, in such way that If the proportion nonconforming is 0.1% the acceptance probability will be 99% and

If the proportion nonconforming is 0.5% the acceptance probability will be 5% . Find the probability that a lot with  $p'=0.002$  is accepted with this plan?

### **11.9**

A company uses The Philips Standard System with a point of control at 1% nonconforming product. The lot size is 800.

What plan does the system require?

Find the proportions of nonconforming product that have 90 and 10% probabilities of acceptance.

What plan does the system require if the lots are size 1200 from a process with 5% nonconforming product?

What is the probability of rejecting lots having 2% defectives with Plan of part a ?

### **11.10**

(Based on Grant and Leavenworth,1988 page517)

A consumer has the choice of using an ISO 2851 standard single plan or a Philips standard single sampling plan for lots of size 500. The ISO plan would be used with an inspection level II, an AQL=1% . The Philips would be used with indifference point IQL=3%.

a)Find the acceptance criteria under both systems.

b) if the consumer primary concern is the assurance that lots 5% nonconforming are rejected, which plan should he choose? Explain.

### **11.11**

Could you design a single sampling plan that satisfies the following supplier and customer:

A supplier who wants 95% of the lots having 0.05% nonconforming products are accepted;

A consumer who 95% of the lots having 2% nonconforming products are not accepted.

**Thinking is a fine mirror**

# Chapter 12

## Dodge-Torrey AOQL plans for Continuous Production

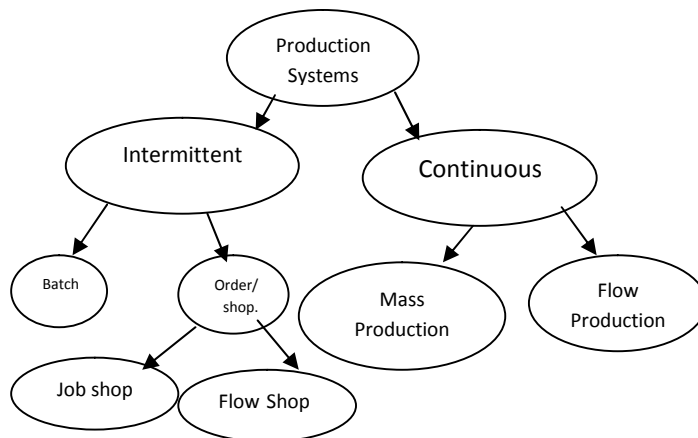
## Chapter 12 Continuous Sampling Plans

### Aims

This chapter deals with acceptance sampling of the products as they are being produced, not necessarily to accumulate a lot before making a decision on the quality of the product. The procedure used for this purpose is based on AOQL, a point on the AOQ curve. The chapter mainly deals with the so-called single level sampling plans CSP-1, CSP-2 & CSP-3, i.e. the plans which prescribe inspecting with a constant fraction  $f$  of the units. Multilevel continuous sampling plans are only defined here.

### 12-1 Classification of Production Systems

Production systems can be classified into continuous and intermittent, as Fig. 12-1 shows. Continuous production system is in turn divided into mass production and flow or process production. Intermittent production system is also divided into batch production and order/shop production. Order production could be either Job shop or flow shop. A Short descriptions of the production systems follows.



**Fig. 12-1 A Classification of production systems**

**Continuous production system** Is the one in which the materials, are **continuously** in motion undergoing chemical reactions or subject to mechanical or heat treatment.

**Intermittent production system** is a manufacturing method of producing several low volume different products. The process is interrupted in order to modify the setups to produce different products or to execute different specifications of the same basic design.

**Flow / process production system** involves a continuous movement of materials through chemical reactions.

**Mass production system** manufactures standardized goods on a large scale, often done using an automated assembly line.

**Batch or large lot production** is a production technique that produces multiple units in a series of steps.

The degree of repetitiveness increases from Job production to batch production, from batch production to mass production and from mass production to process production (Sinha, 2008).

**Order/Shop system**, divided into flow shop and Job shop, does jobs according to the orders from the customers.

**flow shop** scheduling processes jobs on machines in a set order. *It is assumed that all jobs* in a flow shop problem are to be processed on the same machines with identical precedence ordering.

In contrast, a

**job shop** places no restrictions on the processing order; each job may have a different routing; the precedence depends on each job.

The purpose of this chapter is to introduce plans for acceptance sampling of the products as they are being produced in a mass production process to make a decision on the quality of the product.

## 12.2 Acceptance Sampling plans for continuous (mass production) process

When the production is continuous, the formation of inspection lots for lot-by-lot acceptance is artificial and may be impractical. The first sampling plan proposed for inspection of consecutively produced units was that of Dodge. Dodge described his procedure, namely, CSP-1 as follows (Grant & Leavenworth, 1988 page 521):

### 12.2.1 CSP-1

The steps of CSP-1 are as follows:

(a) At the outset, inspect 100% of the units consecutively as produced and continue such inspection until  $i$  units in succession are found clear of defects.

(b) When  $i$  units in succession are found clear of defects, discontinue 100% inspection, and inspect only a fraction  $f$  of the units, selecting individual sample units one at a time from the flow of product, in such a manner as to assure an unbiased sample.

(c) If a sample unit is found defective, revert immediately to a 100% inspection of succeeding units and continue until again  $i$  units in succession are found clear of defects, as in paragraph (a).

(d) Correct or replace, with good units, all defective units found.

Figure 12-2 or Table 12-1 gives the necessary information for the selection of such a plan from sixteen values of AOQL.

### Example 12.1

Suppose to inspect a mass production, an AOQL of 2% and inspection of 1 out of 20 pieces from the conveyor belt is desired, then  $f = 0.05$  or 5%. From Fig. 16.1 the value of  $i$  is found to be  $i = 76$ . the acceptance plan is then as follows:

(a) Inspect all the units consecutively as produced until 76 units in succession are free from defects.

(b) As soon as 76 successive units are free from defects, inspect a sample consisting of only one unit out of every twenty. Accept all the product as long as the sample is free from defects.

(c) Whenever one of these sample units is found defective, resume 100% inspection until 76 units in succession have again been found free from defects. Then resume sampling inspection.

Since 1943 various modification of CSP-1 have been devised including CSP-1, CSP-2, CSP-3, CSP-4, CSP-5, CSP-F,...

### 12.2.2 CSP-2

The CSP-2 plan was proposed by Dodge and Torrey as modification over CSP-1. CSP-2 delays 100% inspection when a sample unit is found for the first time, it is invoked only if a second defect occurs in the next  $k$  or less sample units.

They describe CSP-2 as follows:

Plan CSP-2 differs from Plan CSP-1 in that, once sampling inspection is started, 100% inspection is not invoked when each defect is found but is

invoked only if a second defect occurs in the next  $k$  or less sample units. In other words, if two defects observed during sampling are separated by  $k$  or less good inspected units, 100% inspection is invoked. Otherwise sampling is continued. Although the factor  $k$  might conceivably be assigned any value, the only CSP-2 plans prepared for use have been those in which  $k = i$ . Table 12-1 gives values of  $i$  in CSP-1 and CSP-2 using 5% and 10% samples and various AOQL values. These values for CSP-2 were obtained from a graph similar to Fig 12.2, given by the Dodge-Torrey article.

<b>Table 12.1 Values of <math>i</math> &amp; <math>k</math> for <math>f= 5\%</math> and <math>10\%</math> and some AOQL in CSP-1 , CSP-2</b> (from Grant & Leaven worth, 1988,p.523)				
AOQL %	$f=5\%$		$f=10\%$	
	$i$ CSP-1	$i(=k)$ CSP-2	$i$ CSP-1	$i(=k)$ CSP-2
0.3	510	650	370	400
0.5	305	390	220	290
1	150	195	108	147
2	76	96	55	72
3	49	64	36	48
4	37	48	27	36
5	29	38	21	29
6	24	31	17	23
8	18	23	13	17

### Example 12.2

Describe a CSP-2 plan for a continuous production if  $AOQL=2\%$  ,  $f=5\%$ .

#### Solution

From Table 12-1 the value of  $i=k$  is read 96. Then the CSP-2 plan is:

1. Inspect all the units consecutively as produced until 96 units in succession are free from defects.
2. As soon as 96 successive units are free from defects, inspect a sample consisting of only 1 out of every 20. Accept all the product as long as the sample is free from defects.
3. If one defective is found in this sampling, continue sampling inspection for the time being. However, if a second sample defective is found within the next 96 samples, resume 100% inspection immediately. Continue 100% inspection until 96 units in succession have been found free from defects. Then resume sampling under the foregoing rules with ,  $f=5\%$  i.e. 1 piece in random out of every 20 pieces.



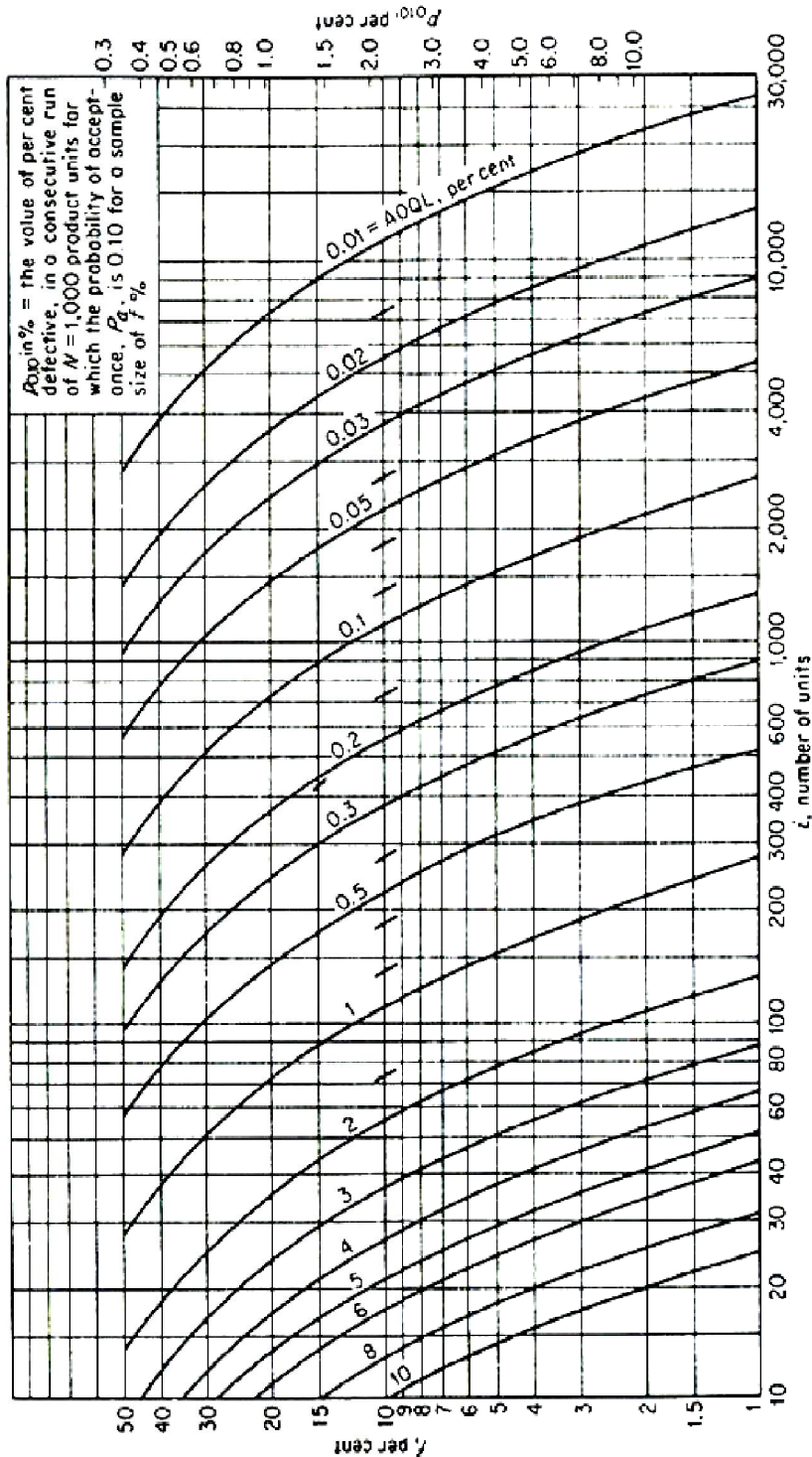


Fig 12-2 Values of  $f$  (in percent), AOQL(in percent) and  $i$

Figure 16-1 Curves for determining values of  $f$  and  $i$  for a given value of AOQL in Dodge's plan for continuous production CSP-1. (Reproduced by permission from H. F. Dodge, *A Sampling Inspection Plan for Continuous Production*, "The Annals of Mathematical Statistics," vol. 14, pp. 264-729, September, 1943.)

**AOQ Formulas for CSP-1 and CSP-2**

(Grant &amp; Leavenworth, 1988 page 532)

$$CSP - 1: AOQ = \frac{p(1-f)q^{i-1}}{f + (1-f)q^{i-1}}$$

 $p$  = fraction nonconforming

$$q = 1 - p$$

$$CSP - 2: AOQ = \frac{p(1-f)(2q^i - q^{2i})}{f \times q + (1-f)(2q^i - q^{2i})}$$

**12.2.3 CSP-3**

Plan CSP-3 is a refinement of CSP-2 to provide greater protection against a sudden run of bad quality. When one sample defective is found, the next four units from the production line are inspected. If none of these are defective, the sampling procedure is continued as in CSP-2. If one of the four units is defective, 100% inspection is resumed at once and continued under the rules of CSP-2. In CSP-3, the value of  $i$  used for a given  $f$  and AOQL is the same as in CSP-2 (Grant & Leavenworth, 1988 page 523).

**12.3 Multilevel Continuous sampling plans (CSP-M)**

To secure economies and obtaining certain other advantages, sometimes it is desired to start sampling with a relatively large fraction sampled such as  $f = \frac{2}{3}, \frac{1}{3}, \frac{1}{2}$  and permit a subsequent change to a smaller sampling fraction. In one particular of such a plans, developed by Lieberman and Solomon (1955), the plans start with 100% inspection, as in CSP's, and continues until  $i$  good units found in succession. Then sampling inspection is started with a fraction  $f$ ; if acceptable units found in succession, the sampling inspection continues with fraction  $f^2$ . Another  $i$  conforming units in a row permit for sampling inspection with fraction  $f^3$ , etc. When a unit is rejected, inspection is shifted back to a next lower level. The number of levels range from 2 to an infinite number.

**12.3.1 A Source for Multilevel Plans**

Several years ago Dept. of Industrial Engineering of Stanford University prepared a pamphlet (AMC Manual 74). Later with some modification it was published as

**Inspection and Quality Control Handbook (Interim) H 106,**

**Multilevel Continuous Sampling Procedures & Tables for Inspection by Attrib.**

Table 12-2 shows the plans discussed in the above documents.

It should be added that there are other plans for continuous inspection such as CSP-F, CSP-V, Wald-Wolfowitz Type Plans and that of Girshick who provided a modification of the Wald and Wolfowitz approach. These methods are described in the quality books and in the internet. One final thing worth mentioning is that Mil-STD-1235 is an existing standard on continuous sampling which combines Handbooks H-106 & H107. This standard is dealt in the internet and references such as Bowker and Lieberman book on statistics and quality control.

Table 12-2 Values of <i>i</i> for multilevel sampling plans (Grant&Leavenworth, 1988 p526)													
AOQL %	$f = \frac{1}{2}$					$f = \frac{1}{3}$					$f = \frac{2}{3}$		
	Number of levels (k)												
	k = 1	k = 2	k =3	k = 4	k =5	k =1	k = 2	k =3	k = 4	k =5	k =1	k =2	k = 3
20.0	^	^	^	^	^	^	^	4	5	6	^	^	^
15.0	^	^	^	4	5	^	4	6	7	8	^	^	^
10.0	^	4	6	8	9	^	7	10	12	13	^	^	^
7.5	^	6	9	11	13	6	11	14	16	18	^	^	^
5.0	5	11	15	18	20	10	18	22	25	27	^	5	8
4.0	7	14	19	22	25	14	23	29	32	34	^	7	11
3.0	11	20	26	31	34	19	32	39	43	46	5	11	16
2.0	18	31	40	47	51	31	48	59	66	71	9	18	25
1.5	25	43	55	63	69	42	66	80	89	95	13	25	34
1.0	39	65	83	95	104	64	100	120	134	142	21	39	52
0.75	54	88	112	128	140	87	134	161	179	191	30	53	71
0.50	82	132	168	193	210	133	202	243	269	287	46	80	107
0.35	119	197	241	275	302	190	290	349	386	408			
0.25	167	269	337	386	422	269	406	488	540	576	96	164	217
0.15	218	446	564	636	706	450	680	815	903	960			
0.10	421	675	847	969	1059	677	1022	1224	1354	1443			

^ Sampling plans are not available for values of  $i$  less than 4.

## Exercises

### 12.1 (Grant and Leavenworth, 1988 page536)

In CSP-2 plan it is desired to apply sampling inspection to 1 item in every 20 and to maintain an AOQL of 2%.

a)What are the values of  $i$  &  $k$

b)Calculate AOQ if incoming lots contain 4% nonconforming units .

### 12.2 (Grant and Leavenworth, 1988 page534)

It is desired to use CSP1 and inspect 1 piece out of every 15 and to maintain an AOQL of 2%. Determine the value of  $i$  . if 1 piece from every 8 pieces is selected , what should be the value of  $i$ ?

Answer : 68.

### 12.3 (Grant and Leavenworth, 1988 page534)

In Dodge-Torrey CSP-2 it is desired to apply sampling inspection to 1 piece out of every 10 and to maintain an AOQL of 3% . What should be the value of  $i$ ? Assume  $i=k$ .

Answer: 48

### 12.4 (Grant & Leavenworth, 1988, page 535)

It is desired to apply continuous sampling inspection to 1 piece out of every 10 and to maintain an AOQL = 2%.

Find the value of  $i$  for CSP1. Answer :55

Find the values of  $i$  &  $k$  for CSP-2. Answer: 72

Prepare flowcharts of the operations of these plans.

**The ignorant never realizes  
his  
mistake nor accepts advice**

# Chapter 13

## Acceptance Sampling by Variables using ISO-3951

## Chapter 13 Acceptance Sampling by Variables using ISO-3951

### Aims

This chapter introduces the international standard ISO-3951. This standard is used for sampling inspection of a product characteristics that are variable. Given a value for AQL and LSL or USL or both for a measurable characteristic of a product, the standard gives a plan for accepting /rejecting a lot of the product.

### 13.1 Acceptance Sampling by Variables

If accepting/ rejecting a lot is not to take place based on pure observation and the characteristic to be inspected is measurable with a scale, then the plans of acceptance sampling inspection by variables would be helpful. Examples of measurable characteristics are output voltage of an adaptor, the leak (water, air, gas,...) from an isolation and the fuel consumption of an engine. For inspecting these kind of characteristics, ISO 3951 applicable. It is worth mentioning that every plan of this system of acceptance sampling for variables could be used for one normally-distributed variable characteristic, while a plan of acceptance sampling by attributes could deal with several similar attribute characteristics.

### 13.2 ISO 3951 Standard

ISO 3951 is an international standard for sampling inspection by variables which in essence is similar to Mil-STD-414(1957-1999) & ANSI/ASQCZ1.9. This standard consists of the following five parts, under the general title "sampling procedures for inspection by variables" (ISO3951-1-2013F page vi):

**Part 1:**

Specification for single sampling plans indexed by acceptance quality limit (AQL) for lot-by-lot inspection for a single quality characteristic and a single AQL ,

**Part 2:**

General specification for single sampling plans indexed by AQL for lot by lot inspection of independent quality characteristics,

**Part 3:**

Double sampling schemes indexed by AQL for lot-by-lot inspection

**Part 4:**

Procedures for assessment of declared quality levels,

**Part 5:**

Sequential sampling plans indexed by AQL for inspection by variables  
(known standard deviation)

It is worth knowing that ISO 3951 standard introduces several single sampling and double sampling procedures. It is worth mentioning that an advantage of double sampling is smaller ATI and saving time and money in the long run. However, this chapter introduces some single sampling plans for variables from Parts 1 and 2 of the standard i.e. ISO 3951-1 & ISO 3951-2.

### **13.2.1 ISO 3951-1**

(from ISO 3951-1:2013Epage vii)

Part 1 of the standard i.e. ISO 3951-1 introduces a system of single sampling plans for inspection by variables in order to be used for accepting or rejecting lots. This part of ISO 3951 is indexed in terms of the acceptance quality limit (AQL) and is designed for users who have simple requirements. A more comprehensive and technical treatment is given in ISO 3951-2 which is complementary to Part 1.

#### **13.2.1.1 Assumptions**

The assumptions of this part of the standard are:

a)

The lots submitted for inspection are a continuing series of lots of discrete products all supplied by one producer using one production process,

b)

A single quality characteristic, measurable on a continuous scale, is to be inspected. Measurement error is negligible; [The measurements are done independently],

c)

Lots have not been screened for nonconforming items. The reason for this assumption is the fact that the procedures in this part of ISO 3951 are not suitable for application to already screened lots,

Moreover (from ISO 3951-1:2013Epage 1):

d)

where production is stable (under statistical control) and

**the characteristic,  $X$ , is distributed according to a normal distribution or close to it,**

e)

where a contract or standard defines a lower specification limit( $L$ ), an upper specification limit( $U$ ) or both; an item is qualified as conforming if and only if its measured quality characteristic,  $X$ , satisfies the appropriate one of the following inequalities:

- 1)  $X \geq L$  (i.e. the lower specification limit is not violated);
- 2)  $X \leq U$  (i.e. the upper specification limit is not violated);
- 3)  $X \geq L$  and  $X \leq U$  (i.e. neither the lower nor the upper specification limit is violated).

### 13.2.1.2 Definitions

#### 13.2.1.2.1 Acceptable Quality Limit(AQL)

AQL is the worst tolerable fraction non conforming when a continuing series of lots is submitted for acceptance sampling.

#### 13.2.1.2.2 Inspection Level

The level of inspection is actually a relationship between the sample size and the lot size given in letter codes (A, B,H,...). The relationship is an empirical one rather than a mathematical relationship based on probability. There are 7 levels in this standard :

3 general levels i.e. I,II,III

4 special level i.e. S-1,S-2,S-3,S-4.

#### Default level is II

**-If no level has been specified, inspection level II shall be used** (ISO3951-1-2013 page12).

-Special levels are for the cases where the test is destructive or the cost of testing is high and small sample sizes are necessary and larger risks can be tolerated.

-level S-I gives the smallest sample size and Level III gives the greatest size.

-level I with respect to level II gives smaller size and greater risk.

-Level III with respect to level II gives greater size and smaller risk (Balamurali, et al, 2014).

#### 13.2.1.2.3Inspection Severity

The degree of tightness or severity of inspection in this standard is stated as normal, tightened or reduced. Inspection severity affects sample size and acceptance criteria. The standard advises to use normal inspection at the start unless otherwise designated (ISO3951-2013 E page 24) with the possibility of shifting to tightened or reduced.

### 13.3 Switching Rules of ISO 3951

The switching rules are an inherent part of an acceptance sampling standard such as ISO 3951-1 when a series of lots are being inspected with it. The rules say how to shift from one severity to other or when to



discontinue the inspection. The rules are as follows (from ISO3951-1 page24):

### **13.3.1 Normal to tightened**

Normal inspection starts at the start of inspection unless otherwise stated. Tightened inspection shall be instituted when two lots on original normal inspection are not accepted within any five or fewer successive lots.

### **13.3.2 Tightened to normal**

When tightened inspections is being carried out, normal inspection shall be re-instated when five consecutive lots have been considered acceptable on original<sup>1</sup> inspection.

### **13.3.3 Normal to reduced**

Reduced inspection may be instituted after 10 successive lots have been accepted under normal inspection, provided that either of the following happens

a) These lots would have been accepted if the AQL had been one step tighter (e.g. 0,65 % instead of 1,0%),

NOTE :If a value of acceptability constant  $k$  for this tighter AQL is not given in Table T ( $s$ -method) or Table Q.1 ( $\sigma$ -method), refer to the supplementary acceptance constants provided in Table V.

b) Production is in statistical control, and reduced inspection is considered desirable by the responsible authority.

### **13.3.4 Reduced to Normal**

Reduced inspection shall cease and normal inspection be reinstated if any of the following occurs on original inspection:

a) a lot is not accepted;

b) production becomes irregular or delayed;

c) reduced inspection is no longer considered desirable by the responsible authority.

### **13.3.5 Discontinuation of inspection**

If the cumulative number of lots not accepted in a sequence of consecutive lots on original tightened inspection reaches five, the acceptance procedures of this part of ISO 3951 shall be discontinued.

---

<sup>1</sup> Original inspection means first inspection of a lot according to the provisions of ISO 2859-1

Inspection under the provisions of this part of ISO 3951 shall not be resumed until action has been taken by the supplier to improve the quality of the submitted product or service and the responsible authority has agreed that this action is likely to be effective. Tightened inspection shall then be used as if Sec 13.3.1 in the above has been invoked.

### 13.4 Designing Acceptance sampling Plans by ISO 3951

#### Symbols

betacdf	MATLAB command for beta cumulative distribution
$f_{\sigma}$	Factor relating maximum process standard deviation to U-L
$f_s$	Factor relating maximum sample standard deviation to U-L
k	Acceptability constant
L	Lower specification limit
N	Lot size
n	Sample size
p*	Max. acceptable value for the estimate of the process fraction non conforming
$P_L$	Process fraction non-conforming below lower specification limit
$P_U$	Process fraction non-conforming above lower specification limit
$\hat{p}$	Estimate of the process fraction nonconforming
$\hat{P}_L$	Estimate of $P_L$
$\hat{P}_U$	Estimate of $P_U$
$Q_U \& Q_L$	$Q_L = \frac{L - \bar{X}}{s}, Q_U = \frac{U - \bar{X}}{s}$
S	Sample standard deviation
$S_{\max}$	$S_{\max} = (U - L)f_s$ , Maximum sample standard deviation
U	Upper specification limit
X	measured quality characteristic
$\bar{X}$	Sample mean
$\phi_Z(a)$	CDF of standard normal at point a
$\mu$	Process mean, E(X)
$\sigma$	Process standard deviation, the standard deviation of measured quality characteristic
$\sigma_{\max}$	Maximum process standard deviation: $\sigma_{\max} = (U - L)f_{\sigma}$

When sampling –by- variable system of ISO 3951 is to be used for inspecting a lot, there are two kind of methods available in this standard namely:

s-method and  $\sigma$  -method. The former is applicable to the cases where the variability(the standard deviation) of the production process or the lot is not known and the latter is for the case where the standard deviation of the process is a known and constant. When the variability is unknown, in the cases where a single specification is given there are 2 types of acceptance criteria:

Form K & Form p\*<sup>1</sup>.

The steps of some of the procedures from ISO3951-1 and ISO3951-2 are summarized in a Table at the end of this chapter and described and illustrated Below. But before describing the plans and procedures, let us show how the proportion nonconforming of a production process is estimated in ISO 3951 plans.

### Estimation of the process proportion nonconforming

If the measured quality characteristic is denoted by X, the proportion non conforming of a production process or a is that values of which satisfies one of the following inequalities:

- 1)  $X < L$  (i.e. the lower specification limit is violated);
- 2)  $X > U$  (i.e. the upper specification limit is violated);

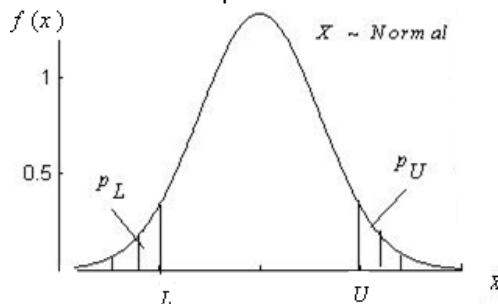
Since X is assumed to be normally distributed, if the process average and standard deviation i.e.  $\mu$  &  $\sigma$  were known, the proportions  $\Pr(X > U)$ ,  $\Pr(X < L)$  in Fig. 13.1, would be calculated as follows

$$P_L = \Pr(X < L) = \Phi_Z\left(\frac{L - \mu}{\sigma}\right),$$

$$P_U = \Pr(X > U) = 1 - \Phi_Z\left(\frac{U - \mu}{\sigma}\right) = \Phi_Z\left(\frac{\mu - U}{\sigma}\right),$$

$$p = \Pr(X < L) + \Pr(X > U) = \Phi_Z\left(\frac{L - \mu}{\sigma}\right) + \Phi_Z\left(\frac{\mu - U}{\sigma}\right).$$

$\Phi_Z(a)$  is the cdf of standard normal at point a.



**Fig 13.1 proportion non conforming if  $\mu$  &  $\sigma$  were known**

<sup>1</sup> In MILSTD414: Form 1 and 2

However the  $\mu$  &  $\sigma$  are often unknown and the best estimator of  $p$  is not obtained by just plugging the sample average and standard deviation estimators into the above relationships.

It is shown in the standard 3951 manual that if a random sample of size  $n$  is taken from the process or the lot, the estimate of the proportion non conforming ( $\hat{p}_L$  &  $\hat{p}_U$ ) would be calculated as indicated in the following Table. A description is in (ISO 3951-2:2013E: page 18 or Annex L). In the table  $\varphi_Z(a)$  represents the cumulative distribution function of standard normal at point  $a$ .

standard deviation	Specification Limits	estimate of proportion nonconforming
$\sigma$ Unknown	Lower only	$\hat{P}_L = \text{betacdf}(.5*(1-Q_L*(n^*.5)/(n-1)), (n-2)/2, (n-2)/2)$
	Upper only	$\hat{P}_U = \text{betacdf}(.5*(1-Q_U*(n^*.5)/(n-1)), (n-2)/2, (n-2)/2)$
	2-Sided	$\hat{p} = \hat{P}_U + \hat{P}_L$
$\sigma$ Known	Lower only	$\hat{P}_L = 1 - \varphi_Z\left(\frac{\bar{X} - LSL}{\sigma} \sqrt{\frac{n}{n-1}}\right)$
	Upper only	$\hat{P}_U = 1 - \varphi_Z\left(\frac{USL - \bar{X}}{\sigma} \sqrt{\frac{n}{n-1}}\right)$
	2-Sided	$\hat{p} = \hat{P}_L + \hat{P}_U$

The proof is in references such as chapter 4 of a book to be published by Dr Wayne Taylor<sup>1</sup>, a specialist in acceptance sampling; and in the following manual:

Office of the Assistant Secretary of Defense (15 October 1958).  
Technical Report: Mathematics and Statistical Principles Underlying  
Military Standard 414.

As mentioned earlier there are 2 types of methods in international standard ISO 3951 to inspect a lot :

s-method and  $\sigma$  -method, depending whether the process variability is known or not, i.e. the standard deviation of the measured quality characteristic ( $\sigma$ ) is unknown or known. Some of the procedures of s- and  $\sigma$  -methods are summarized in a table at the end of this chapter and described and illustrated below.

<sup>1</sup> See [www.variation.com](http://www.variation.com), Taylor enterprises

### 13.4.1 Case I : process variability( $\sigma$ ) unknown

When the distribution of the characteristic being measured is roughly normal with unknown  $\sigma$ , the method used to inspect the lot is generally called s-method. Some of the procedures for this kind, given lower specification or upper specification limits or both(U&L) are described and illustrated here.

#### 13.4.1.1 S-method, Single Specification , form k

In this method, given (L or U), one takes a random sample of size n ( $x_1, \dots, x_n$ ) from the lot, accepts the lot if

$$\bar{X} \geq L + Ks \text{ or } \frac{\bar{X} - L}{s} \geq k$$

Or

$$\bar{X} \leq U - ks \text{ or } \frac{U - \bar{X}}{s} \geq k$$

where

$\bar{X}$  is the sample mean

S is the sample variance

k is an acceptability constant given by the standard.

The procedure of the s-method for this situation is:

After deciding on

i)AQL ( if necessary , using converting Table L),

choose one of the AQLs available in ISO 3951 Tables (Tables such as N or P at the end of this book), a value between 0.01% to 10%

ii)Inspection level

and

iii)Inspection severity (i.e. normal , tightened and reduced)

follow the steps below:

-Given the lot size(N) and inspection level , read letter code from Table M,

-Given AQL, the code and severity of inspection, read sample size(n) and parameter k from:

Table	S-method, Type of inspection
T	Normal
T.1	Tightened
T.2	Reduced

-Take a random size of size  $n$  from the lot and measure the desired quality characteristic of each product of the sample,

-Calculate  $\bar{X} = \frac{\sum x_i}{n}$ ,  $S = \frac{\sum x_i^2 - n\bar{X}^2}{n-1}$ .

-Calculate  $Q_L = \frac{L - \bar{X}}{S}$  if  $L$  is given, or  $Q_U = \frac{U - \bar{X}}{S}$  if  $U$  is given

-Accept the lot if  $Q_L$  or  $Q_U \geq k$ . When  $Q_U$  or  $Q_L$  is negative or ( $Q_U$  or  $Q_L < k$ ) conclude the lot does not satisfy your desired quality.

### Example 13.1

A thermostat manufacturer packs its product in lots of size 35, the maximum temperature that the thermostat is allowed to open is 100 degree centigrade. Specify a plan from Standard ISO 3951 to inspect the lot with normal severity and AQL=1.3%. The temperature at which the thermostats open is roughly normally distributed.

### Solution

From Table L, AQL = 1.5% and by default Level II is chosen.

The steps are:

-From Table M given  $N=35$ , the letter code D is read

-From Table T with Code D, AQL = 1.5% and normal inspection  $\Rightarrow k=1.476$ ,  $n=6$

-Take a random sample of size 6 :  $x_1, \dots, x_6$ ,

-suppose  $\bar{X} = 97.8$   $s = \sqrt{\frac{\sum x_i^2 - n\bar{X}^2}{n-1}} = 1.926$ ,

-Calculate  $Q_U = \frac{U - \bar{X}}{S} = \frac{100 - 97.8}{1.926} = 1.14$

-The lot is rejected since  $1.14 = Q_U < k = 1.47$ .

### 13.4.1.2 S-method, Single Specification, form $p^*$

After deciding on

i) AQL ; use Table L if necessary,

ii) Inspection level and

iii) Inspection severity (i.e. normal, tightened and reduced),

follow the steps below:

-Given the lot size( $N$ ) and inspection level, read letter code from Table M,

-Given AQL, the code and the severity of inspection, read  $n$  and the maximum allowable proportion nonconforming in the lot ( $p^*$ ) from :

Table	S-method, Type of inspection
N	Normal
N.1	Tightened
N.2	Reduced

-Take a random size of size n from the lot and measure the desired, quality characteristic of each product of the sample,

-Calculate s and  $\bar{X}$  .

-Calculate  $Q_L = \frac{\bar{X}-L}{s}$  if L is given, or  $Q_U = \frac{U-\bar{X}}{s}$  if U is given, if negative reject the lot, otherwise go the next step

-Calculate the proportion non conforming beyond L or U ( $\hat{p}_L$  or  $\hat{p}_U$ )<sup>1</sup>

Note that  $\hat{P}_L$  or  $\hat{P}_U$  could be computed by the following MATLAB commands, given  $Q_L$  &  $Q_U$ :

PLhat=betacdf(.5\*(1-QL\*(n^.5)/(n-1)),(n-2)/2,(n-2)/2)

Puhat= betacdf(.5\*(1-Qu\*(n^.5)/(n-1)),(n-2)/2,(n-2)/2)

The values of  $\hat{P}_L$  &  $\hat{P}_U$  for some n & Q are also available from Table O.

-Accept the lot if  $\hat{p}_L$  or  $\hat{p}_U \leq p^*$ , reject it if  $\hat{p}_L$  or  $\hat{p}_U > p^*$ .

### Example 13.2

Solve the previous example with S-method- Form p\*.

From Table L, AQL =1.5% and by default Level II is chosen. The steps are

1) Given N=35, from Table M, the letter code D is read

2) Code D, AQL =1.5% and normal inspection, From Table N  $\Rightarrow p^* = 5.22\%$ , n=6

3)Take a random sample of size 6

4)suppose  $\bar{X} = 97.8$ , s=1.926 ,

5)Calculate  $Q_U = \frac{100-97.8}{1.926} = 1.14$

Using MATLAB:

6)n=6;Qu=1.14;Puhat=betacdf(.5\*(1-Qu\*(n^.5)/(n-1)),(n-2)/2,(n-2)/2)

<sup>1</sup> These 2 commands use a beta distribution with both parameters equal to  $\frac{n-2}{2}$  ( based on ISO 3951-1-2013E page 18) :

$$\hat{p}_L = G_{(n-2)/2} \left[ \frac{1}{2} \left( 1 - Q_L \frac{\sqrt{n}}{n-1} \right) \right] \quad \hat{p}_U = G_{(n-2)/2} \left[ \frac{1}{2} \left( 1 - Q_U \frac{\sqrt{n}}{n-1} \right) \right]$$

where

$G_{\frac{n-2}{2}(a)}$  represents the value of beta cumulative distribution function with both parameters at point

a.

$$\hat{P}_U = \text{Puhat} = 12.47\%.$$

Or From Table O, with  $n=5$ ,  $Q_U = 1.14 : \hat{P}_U \cong 13$

7) The lot is rejected because  $\hat{P}_U > p^* = 5.22$ .

### 13.4.1.3 S-method, Double Specifications

(ISO3951-1-2013 page 16)

Although this case could be treated as 2 single specification limits separately, there are some procedures for considering the 2 limits simultaneously, one of which is as follows:

After deciding on

- i) AQL, if necessary, using converting Table L,
- ii) Inspection level and
- iii) Inspection severity (i.e. normal, tightened and reduced)

follow the steps below:

- 1) Given the lot size ( $N$ ) & inspection level, read letter code from Table M,
- 2) Given AQL, the code and severity of inspection, read sample size ( $n$ ) and parameter  $p^*$  i.e. the max. allowable proportion nonconforming from :

Table	Type of inspection
N	Normal
N.1	Tightened
N.2	Reduced

- 3) Take a random size of size  $n$  from the lot and measure the desired quality characteristic of each product in the sample,

- 4) Calculate  $s\bar{X}$ . Read parameter  $f_s$  from:

Table	Type of inspection
P.1	Normal
P.2	Tightened
P.3	Reduced

If  $\bar{X}$  is outside  $[L, U]$  or  $S > s_{\max} = (U - L)f_s$ , reject the lot, otherwise go to step 5

- 5) Calculate  $Q_L = \frac{\bar{X} - L}{S}$  and  $Q_U = \frac{U - \bar{X}}{S}$ ; if either of them is negative, reject the lot, otherwise go to step 6.

- 6) If  $S \leq S_{\max}$  calculate  $\hat{p}_L$  &  $\hat{p}_U$  using the following MATLAB commands

$$PL\text{ hat} = \text{betacdf}(.5 * (1 - Q_L * (n^{.5}) / (n - 1)), (n - 2) / 2, (n - 2) / 2)$$

$$Pu\text{ hat} = \text{betacdf}(.5 * (1 - Q_U * (n^{.5}) / (n - 1)), (n - 2) / 2, (n - 2) / 2)$$



The values of  $\hat{P}_L$  &  $\hat{P}_U$  for some  $n$  and  $Q$  are also available from Table O and S.

Estimate  $\hat{P}$ , i.e. the total proportion nonconforming beyond the specification limits, from  $\hat{p} = \hat{p}_L + \hat{p}_U$ ,

Accept the submitted lot if  $\hat{p} \leq p^*$ , reject it  $\hat{p} > p^*$ .

**Example 13.2** (ISO 3951-2:2013(E) page 19)

A kind of weapon (Torpedoes) supplied in batches of 100 is to be inspected for accuracy in the horizontal plane. Positive or negative angular errors are equally unacceptable, so combined control of the double specification limits is appropriate. The specification limits are set at 10 m either side of the point of aim at a distance of 1 km, with an AQL of 4 %. Because testing is destructive and very costly, it has been agreed between the producer and the responsible authority on special inspection level S-2. Determine a plan for the inspection.

**Answer**

The AQL is available in ISO3951 Tables, therefore need not be converted using Table L. Normal inspection is selected and the steps listed below are followed:

Step 1: With  $N=100$  & Level S-2, Table M gives code B,

Step 2: With code B and AQL of 4 Table N gives

$$n=3 \text{ and } p^* = 19.25\% = 0.1925.$$

Step 3: Suppose the result of the random sample of size 3 are 8.8, 6.7 and -5.0,

Step 4:  $\bar{X} = 3.5\text{m}$ ,  $s = 7.437\text{m}$ , Table P.1 gives  $f_s = 0.475$

$$s_{\max} = (U - L)f_s = [10 - (-10)] \times 0.475 = 9.5 \quad \text{we continue since } s < s_{\max},$$

Step 5:  $Q_U = \frac{U - \bar{X}}{s} = 0.8741$  &  $Q_L = \frac{\bar{X} - L}{s} = 1.815$  since neither  $Q_U$  nor  $Q_L$  are negative, the lot is not rejected and we continue,

Step 6: calculation of  $\hat{p}_L$  &  $\hat{p}_U$ ,

with MATLAB:

$$n=3; QL=1.815; PLhat=betacdf(.5*(1-QL*(n^.5))/(n-1)), (n-2)/2, (n-2)/2)$$

$$PLhat=0$$

$$n=3; QU=0.8741; PUhat=betacdf(.5*(1-QU*(n^.5))/(n-1)), (n-2)/2, (n-2)/2)$$

$$PUhat= 0.2266$$

$$\hat{p} = \hat{p}_L + \hat{p}_U = 0 + .227 = 0.227$$

Step 7: since  $\hat{p} = .227 > .1925 = p^*$ , the lot is rejected although all inspected items in the sample within the specification limits

### Example 13.3

Lots of 15-Watt amplifier are to be inspected to see if their power is in the range [12 16]. Determine an acceptance sampling plan with AQL =2% for this case.

### Answer

From Table L.1 AQL =2.5, the default level in inspection is II and normal inspection is applied.

Step 1: Table M gives code D for N=40 and level II

Step 2, with AQL=2.5 and normal inspection Table N gives n=9 and  $100p^* = 9.717$  or  $p^* = 0.097 = 9.7\%$

Step3 &4: suppose  $\bar{X} = 14.6$  and  $s = 1.14$

Step 5:  $Q_U = \frac{U - \bar{X}}{S} = \frac{16 - 14.6}{1.14} = 1.23$ ,  $Q_L = 2.28$ .

Step 6: calculation of  $\hat{P}_L$ ,  $\hat{P}_U$  &  $\hat{p}$  :

$n=9; Q_L=2.28; P_{Lhat} = \text{betacdf}(.5 * (1 - Q_L * (n^{.5}) / (n-1)), (n-2)/2, (n-2)/2)$

$P_{Lhat} = 0.0017 = 0.17\%$

$n=9; Q_U=1.23; P_{Uhat} = \text{betacdf}(.5 * (1 - Q_U * (n^{.5}) / (n-1)), (n-2)/2, (n-2)/2)$

$P_{Uhat} = 0.1057 = 10.57\%$

$\hat{p} = \hat{p}_L + \hat{p}_U = .17 + 10.57 = 10.74\%$

The lot is rejected since it  $\hat{p} > p^*$ .

### 13.4.2 Case II: Process variability( $\sigma$ ) known

In ISO 3951, there are some procedures, called  $\sigma$ -method for inspecting variable characteristics of the products of lots by acceptance sampling when the characteristic is normally distributed and variability of the process i.e.  $\sigma$  is known. The  $\sigma$ -method shall only be used when there is valid evidence that the standard deviation of the process( $\sigma$ ) can be considered constant with a known value(ISO3951-1:2013E page 21). Below a procedure for dealing with single specification limit and one for double specification limits are introduced.

### 13.4.2.1 $\sigma$ -method, Single Specification, Form k

After deciding on AQL, Level and the severity of inspection,

Step 1: Read the code from Table M,

Step2: Read sample size  $n$ , and acceptability constant  $K$  from :

$\sigma$ -method Type of inspection	Table in this book	Table in ISO3951-2:2013 Manual
Normal	Q.1	C.1
Tightened	Q.2	C.2
Reduced	Q.3	C.3

Step 3: take a random sample of size  $n$  from the lot and measure the desired characteristic of its products.

Step 4: calculate sample mean  $(\bar{X})$  and  $Q_U = \frac{U - \bar{X}}{\sigma}$  &  $Q_L = \frac{\bar{X} - L}{\sigma}$

Step 5: Accept the lot if  $Q_U$  or  $Q_L \geq k$

Reject if  $Q_U$  or  $Q_L$  is negative or  $Q_U$  or  $Q_L < k$ .

#### Example 13.4

If  $N=25, L=LSL=2000, \sigma = 1000$ , and  $AQL=4.5\%$ , specify a single sampling plan from ISO3951 at level II and normal inspection

**Answer** From Table L :  $AQL=6.5\%$  ; the steps are

1: Table M gives code H,

2: Table Q.1 gives  $n=19$  and  $k=1.009$

3: the random sample of size 19 is:

[23308 21843 20568 21313 19848 22762 21108 21671 20008 21204  
20326 20573 20314 19858 19975 20674 19185 21492 20766]

4:  $\bar{X} = 20884$ ,  $Q_L = \frac{\bar{X} - L}{\sigma} = \frac{20884 - 20000}{1000} = 0.884$ ,

6: The lot is rejected because  $Q_L < k = 1.009$ .

#### Example 13.5

A  $\sigma$  – method plan for upper specification limit uses  $n=9$  and  $k=1.466$  to inspect coming lots, if the measures quality characteristic is normal, what is the probability of accepting the lots having 3.75% non conforming products with this plan?

$$X \sim N(\mu, \sigma)$$

$$0.0375 = \Pr(X > U) = \Pr(Z > \frac{U - \mu}{\sigma}) \Rightarrow \frac{U - \mu}{\sigma} = 1.78$$

$$P_a = \Pr\left(\frac{U - \bar{X}}{\sigma} \geq 1.466\right) = \Pr(\bar{X} \leq -1.466\sigma + U) = \Pr\left(Z \leq \frac{-1.466\sigma + U - \mu}{\frac{\sigma}{\sqrt{9}}}\right) \Rightarrow$$

$$P_a = \Pr(Z \leq -1.466 \times 3 + 1.78 \times 3) \cong 82.64\%$$

### 13.4.2.2 $\sigma$ -method, Double Specification

After deciding on i) the AQL, ii) the level and iii) the severity, the steps are (from ISO3951-1-2013 page 23):

Step1: Read  $f_\sigma$  from Table L.1,

Step2: Calculate the maximum allowable value of the process standard deviation from  $\sigma_{max} = (U - L)f_\sigma$ . If  $\sigma$  exceeds this amount (i.e. if  $\sigma > \sigma_{max}$ ) declare the lot unacceptable, and sampling is pointless until it is demonstrated that  $\sigma$  has been adequately reduced.

If  $\sigma \leq \sigma_{max}$  go to the next step.

Step 3: If  $\sigma \leq \sigma_{max}$ , read the code based on the inspection level and the lot size from Table M.

Step4:

Read sample size  $n$ , and acceptability constant  $k$  from :

$\sigma$ -method Double limits Type of inspection	Table in this book	Table in ISO3951-2:2013 Manual
Normal	Q.1	C.1
Tightened	Q.2	C.2
Reduced	Q.3	C.3

Step 5: Calculate  $\bar{X}_U = U - k\sigma$  &  $\bar{X}_L = L + k\sigma$

Step 6: Take a random sample of size  $n$  from the lot and measure the desired characteristic of its products, and calculate  $\bar{X}$

Step 7:

Accept the lot if  $\bar{X}_L \leq \bar{X} \leq \bar{X}_U$ , reject if  $\bar{X}_U$  is negative or  $\bar{X}$  is outside  $[\bar{X}_L, \bar{X}_U]$ .

Note : If  $\bar{X}_L \leq \bar{X} \leq \bar{X}_U$  but  $\sigma > 0.75\sigma_{\max}$  and  $\bar{X}$  is close to  $\bar{X}_L$  or  $\bar{X}_U$ , the exact method given in Section 17.3 ISO 3951-2 shall be preferred (ISO3951-1-2013-E page 23) instead of the approximate procedure given above.

### Example 13.6

The specification for a certain electrical resistance is  $520 \pm 50\Omega$ . In the production process, the characteristic is normally distributed with the standard deviation of 18.5 ohms. To inspect the lots of size 1000 with AQL 1.5% a  $\sigma$  – method with the following steps would be appropriate:

From Table I.1  $f_\sigma = 0.194$ .

$$\sigma_{\max} = (U - L)f_\sigma = 19.4; \text{ since } \sigma_{\max} > \sigma \text{ we continue,}$$

-Given  $N=1000$  and level II from Table M letter code J is read,

-With AQL =1.5% and normal inspection , Table Q.1 gives  $n=19, k=1.677$ ,

$$\bar{X}_U = U - k\sigma = 538.9, \bar{X}_L = L + k\sigma = 501.1,$$

A random sample of size 19 was:

515; 491; 479; 513; 521; 536; 483; 509; 514; 507; 484; 526; 532; 499;  
530; 512; 492; 522; 488  
 $\Rightarrow \bar{X} = 508.0$

Since  $\bar{X}_L \leq \bar{X} \leq \bar{X}_U$  the lot is accepted.

**Notice** that in the above example (from ISO3951-1-2013 page 23):

Although  $\sigma > (0.75)(\sigma_{\max})$ ,  $\bar{X}$  is close to neither  $\bar{X}_L$  nor to  $\bar{X}_U$ , so the above approximate method is acceptable.

If for example,  $\sigma$  were say 20 then  $\sigma > \sigma_{\max}$  and , therefore sampling inspection should not even take place.

**Table 13-1** Some plans of **ISO 3951** for inspecting a normally distributed characteristic

Variability	Method	Specification	form	P R O C E D U R E   S T E P S
				Before performing steps decide on the severity ,level of inspection and AQL (if necessary use Table L)
unknown	s-method	Single	k	1)Read code from Table M 2)Read n &K from Table Tor T.1 or T.2 3)Take a sample of size n 4)Calculate $S$ & $\bar{X}$ 5) Calculate $Q_L = \frac{\bar{X}-L}{s}$ or $Q_U = \frac{U-\bar{X}}{s}$ . 6) Accept the lot if $Q_L$ or $Q_U \geq k$ . When $Q_L$ or $Q_U$ is negative OR( $Q_L$ or $Q_U < k$ ) conclude the lot does not satisfy your desired quality.
			$p^*$	1)Read code from Table M 2) Read n, $p^*$ from Table N or N.1 or N.2 3)Take sample of size n 4) Calculate $S$ , $\bar{X}$ 5)calculate $Q_U = \frac{U-\bar{X}}{s}$ or $Q_L = \frac{\bar{X}-L}{s}$ if negative reject lot, if positive go step 6. 6)Calculate $\hat{p}_L = \text{betacdf}(.5*(1-Q_L*(n^{.5})/(n-1)), (n-2)/2, (n-2)/2)$ or $\hat{p}_U = \text{betacdf}(.5*(1-Q_U*(n^{.5})/(n-1)), (n-2)/2, (n-2)/2)$ 7)Accept the lot if $\hat{p}_L$ or $\hat{p}_U \leq p^*$ , reject it if $\hat{p}_L$ or $\hat{p}_U > p^*$ .
		Double		1)Read code from Table M 2) Read n, $p^*$ from Table N or N.1 or N.2 3) Take a random sample of size n 4)Calculate $s$ & $\bar{X}$ . Read parameter $f_s$ from: Table P.1 or P.2 or P.3 If $\bar{X}$ is outside $[L, U]$ or $s > (U - L)f_s$ , reject the lot, otherwise go to step5. 5)Calculate $Q_L = \frac{\bar{X}-L}{s}$ & $Q_U = \frac{U-\bar{X}}{s}$ if $Q_L$ or $Q_U$ negative, reject the lot, otherwise go to step 6. 6)If $s \leq S_{max} = (U - L)f_s$ calculate $\hat{p}_L$ & $\hat{p}_U$ using MATLAB: $\hat{p}_L = \text{betacdf}(.5*(1-Q_L*(n^{.5})/(n-1)), (n-2)/2, (n-2)/2)$ $\hat{p}_U = \text{betacdf}(.5*(1-Q_U*(n^{.5})/(n-1)), (n-2)/2, (n-2)/2)$ 7) Accept the lot if $\hat{p}_L + \hat{p}_U = \hat{p} \leq p^*$ , reject it $\hat{p} > p^*$ .
known	$\sigma$ -method	Single		1)Read code from Table M, 2) Read n, and acceptability constant k from Table Q.1 or Q.2 or Q.3 . 3)Take a random sample of size n 4) Calculate $\bar{X}$ , $Q_U = \frac{U-\bar{X}}{\sigma}$ & $Q_L = \frac{\bar{X}-L}{\sigma}$ . 5) Accept if $Q_U$ or $Q_L \geq k$ Reject if $Q_U$ or $Q_L$ is negative or $Q_U$ or $Q_L < k$ .
		Double		1) Read $f_\sigma$ from Table L.1. 2)If $\sigma > \sigma_{max} = (U - L)f_\sigma$ reject the lot , otherwise got to step 3. 3) if $\sigma \leq \sigma_{max}$ , read Code from Table M, 4) Read n, k from Table Q.1 or Q.1 or Q.2. 5) Calculate $\bar{X}_U = U - k\sigma$ & $\bar{X}_L = L + k\sigma$ 5) Calculate $\bar{X}_U = U - k\sigma$ & $\bar{X}_L = L + k\sigma$ , 6) Take a random sample and calculate $\bar{X}$ , 7) Accept the lot if $\bar{X}_L \leq \bar{X} \leq \bar{X}_U$ , Reject if $\bar{X}_U$ is negative or $\bar{X}$ is outside $[\bar{X}_L, \bar{X}_U]$ .

## Exercises

### 13.1. (Grant Leavenworth, 1988 page 575)

The lower specification limit for the tensile strength of a certain kind of wire is  $L=215\text{lb}$ ; the variability i.e.  $\sigma$  is unknown. In order to inspect the lots, an s-method of form  $p^*$  in ISO3951 was used with  $AQL=0.15\%$ , normal inspection and sample size code H. Based on the following 5 random subgroups (i.e. the required sample of 25), do the required calculations to decide on the acceptance/rejection of the lots.

Sample No.	Strength(lb)
1	231, 238, 228, 231, 235
2	224, 245, 263, 231, 245
3	224, 228, 235, 238, 235
4	221, 242, 242, 235, 224
5	224, 224, 242, 252, 252

### 13.2. (Grant Leavenworth, 1988 page 575)

A single specification – form  $k$ ,  $\sigma$  – method, is used to inspect a kind of lot given the upper specification limit. Assuming the frequency distribution in the lot is normal and the  $\sigma$  is estimated correctly, compute the probability of accepting lots of 3% non conforming products.

**Answer :0.326,**

Hint:

$$\Pr\left(Z > \frac{U - \mu}{\sigma}\right) = 0.03 \quad \text{or} \quad \frac{U - \mu}{\sigma} = Z_{0.03}$$

### 14.3. (Based on 17-7 Grant Leavenworth, 1988 page 577)

An ISO 2859 single sampling plan calls for  $n=50$  and  $c=2$  for code H, an AQL of 1.5% and normal inspection.

What  $\sigma$  – method of ISO 3951 variable-sampling plan corresponds to this ISO 2859 attributes plan? Assume  $\sigma$  is known and there is a single specification limit for this characteristic.

**You learn more from failure than from success;  
Don't let it stop you;  
Failure builds character**

# Chapter 14

## Introduction to Reliability & Related Subjects



## Chapter 14: Introduction to Reliability & Related Subjects

### Aims

The final chapter deals with reliability theory and related concepts such as, failure rate, bathtub curve for failure, RBD(reliability block diagram), availability, various system configurations related to reliability, OC curves of inspection plans; it also introduces some tables in order to design a plan for accepting /rejecting a lot, based on the mean life of the product.

### 14.1 Definitions and Basic Concepts

To aid understanding of reliability engineering, some of the terms and basic concepts used in the field are presented here.

#### 14.1.1 Reliability

Many definitions have been proposed for reliability by researchers and textbook authors. Some of these are as follows:

-Reliability of a device, an equipment or a system is "the capability of it not to break down in operation. When a equipment works well, and works whenever called upon to do the job for which it was designed, such product or system is said to be reliable"(Bazovsky & Igor,2004).

This capability is often stated in terms of probability or a success ratio, both quantified as a number between 0 and 1.

-“ The reliability of a system, a device,... is the probability that it will give satisfactory performance for a specified period of time under specified operating conditions” (Grant and Leavenworth,1988 page 582).

-“Product reliability is the ability of a unit to perform a required function under stated conditions for a stated period of time”(Feigenbaum,1991, page 574)

For example, the reliability of 90% for a system indicates the system will not breakdown with a probability of 0.9 in the specified time or the specified number of operations, or number of cycles, .... When the product reliability is just given as a number in the interval 0 to 1 (without specifying time , etc), it is meant the probability of successful implementation of the product equals this number. This number could be

the mean of reliability function. It is worth knowing that some components where strength and stress interfere, reliability is defined as “the probability that the unit is strong enough to overcome the stress”(Jonson,1988: Three Stress-Strength models for reliability); or reliability = Probability(Strength or capacity > Stress or load).

Reliability is important in quality control and has been discussed in various books including Kapur & Pecht (2014), O'Connor & Kleyner (2012), Kapur & Lamberson(1977), Lewis(1994), Billinton and Allan (1992), Ross(1985), Chapter 9, Grant & Leavenworth (1988) chapter 18 and Crow & Feinberg(2001).

### 14.1.2 Failure

In reliability, the term "failure" is not necessarily breaking down; it means that the system is not capable of performing the required function with the desired properties under the stated conditions.

In industry, 2 cases could be distinguished for "conditions" which include the application and operating circumstances under which the product is put to use:

- a) In some industries especially military industry, the conditions are set by the customer based on his own requirements.
- b) Reliability under unknown conditions: In many commercial industries, the manufacturer does not know what reliability the customer wants; however, he tries to decide instead of the customer, based on market, psychology, human factors engineering, etc.

### 14.1.3

#### **MTTF, MTBF, MTTR, MTFF & MTBR- Explanation of Terms:**

MTTF, MTBF, MTTR, MTFF, MTB which are terms in reliability engineering stand for the following terms:

-Mean Time To Failure (**MTTF**) is time required for a device or system to fail; it is a basic measure of reliability. For repairable systems, MTTF can be thought of as the average time between a renewal point and the next following failure point. For non repairable systems, the mean time to failure is the same as the mean lifetime (I. Eusgeld et al: Hardware Reliability<sup>1</sup>).

---

<sup>1</sup> <https://citemaster.net/get/c54deb1c-61ac-11e4-a639-00163e009cc7/salfner08hardware.pdf>

-Mean Time Between Failures (**MTBF**) is a reliability term equivalent to the expected number of operating hours before a product fails, or the “service life”. Technically, MTBF should be used only in reference to a repairable item, while MTTF should be used for non-repairable items. However, MTBF might be used for both repairable and non-repairable items.

-**MTTR** stands for Mean Time To Repair or Mean Time To Replacement, it is the mean time needed to repair or replace a failed hardware module.

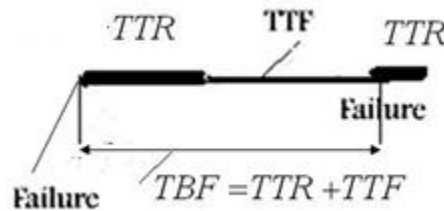
- **MTTF** stands for Mean Time to First Failure.

- **MTBR** stands for Mean Time between Repair;

In most cases, MTBR could be replaced with

**MTBF** which stands for Mean Time Between Failures.

Time Between Failures is the sum of time to repair and time to failure (Fig. 14-1).



**Fig. 14.1 Differentiating Between failure metrics**

Then :  $MTBF = MTTR + MTTF$

A way of predicting the MTBF for a component, device, or system is by the help of the following formula (Tersine,1985, page202)

$$\widehat{MTBF} = \frac{\text{working time}}{\text{no. of failures}}$$

#### 14.1.4 Reliability Function

The reliability function for a product or device etc,  $R(t)$ , is the probability that the lifetime exceeds some time  $t$ , given by

$$R(t) = 1 - F(t) = \int_t^{\infty} f(x)dx$$

where

$f(t)$  is the probability density function(pdf) of  $X$ , the life time of a certain device or system,

$F(t)$  is the cumulative distribution function(cdf) of  $X$ ,

$X$  is the lifetime of the device or the system.

### 14.1.5 Hazard Function

What is called hazard function for a device or system in the subject of reliability and also known as the instantaneous failure rate or instantaneous hazard function is denoted by such symbols as  $\lambda(t)$  or  $h(t)$  and defined as follows:

$$h(t) = \frac{f(t)}{1-F(t)} = \frac{f(t)}{R(t)}$$

where

$f(t)$  is the pdf of  $X$ , the life time

$F(t)$  is the cdf of  $X$ ,

$R(t)$  is the device reliability function.

Figure 14-6 shows the shape of several hazard functions.

It is worth knowing, whichever of the functions  $f(t)$ ,  $F(t)$ ,  $R(t)$  &  $h(t)$  is known, the 3 others could be obtained. The following formula might be helpful in this regard:

$$R(t) = \frac{f(t)}{h(t)} = e^{-\int_0^t h(\tau) d\tau}$$

### 14.1.6 Bathtub-shaped Model for Devices' Failures

A well-known model for the behavior of the device failure rate as a function of time is the so-called bathtub model (Fig 14.2). This model assumes 3 periods for failure:

1) Early infant mortality failure,

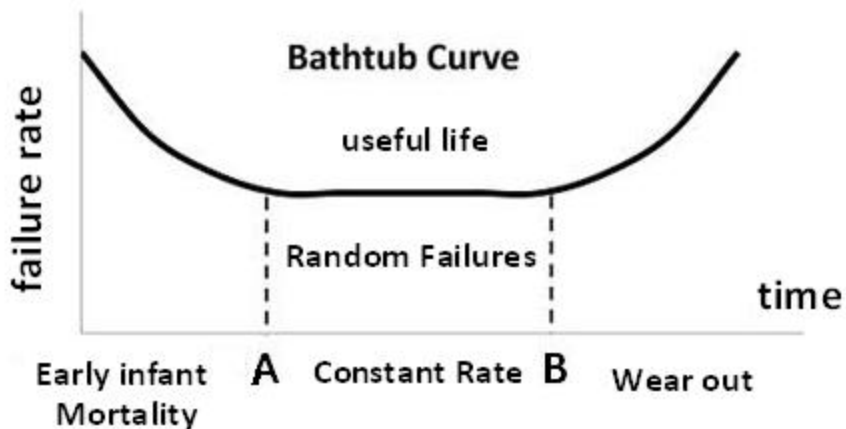


Fig. 14.2 Bathtub Model for devices' Failure Rate

The first period is the period of decreasing rate of failures, where poor components break down,

2)Constant Failures period,

The middle phase of the bath-tub curve is the useful stage of lifetime where the failure rate is constant.

3)Wear- out period

In the last phase of the bath-tub curve, the failure rate increases with time.

It should be added that it is assumed that for all the 3 periods the working conditions are the same.

The middle period of the model is discussed more here.

Consider the following formula and the histogram in Fig. 14-3 related to a sample of 200 electrical switches; we have :

$$\hat{P}_t = \widehat{Pr}(X < t) = \frac{n_t}{N}$$

where

$X$  is a switch life time

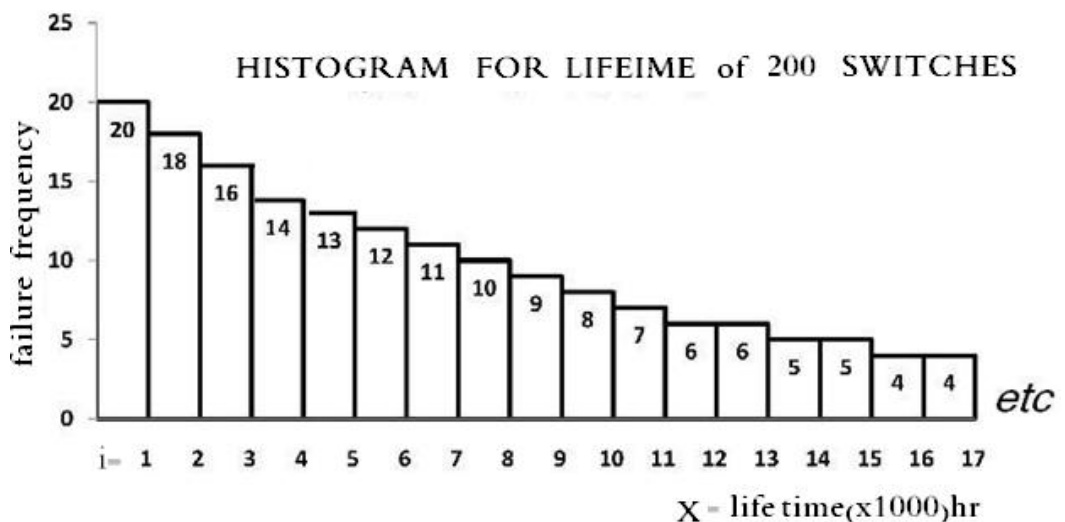
$P_t$  the probability of failure during period  $t$

$\hat{P}_t$  estimate for  $P_t$ , relative frequency of failure during time  $t$

$n_t$  the number of switches which broke down during period  $t$ ,

failure frequency

$N$  total number of switches in the sample



**Fig.14.3 Histogram of lifetime of a sample of 200 electrical switches.**

**Example 14.1**

Consider the data given in Fig 14.3, then the estimate for the lifetime of a switch to be 1000 hours or less would be:

$$\hat{P}_{1000} = \hat{\Pr}(0 < X < 1000) = \frac{20}{200} = 0.1 = 10\%$$

And also

$$\hat{\Pr}(X < 4000) = \frac{20 + 18 + 16 + 14}{200} = 0.34 = 34\%$$

$$\hat{P}_r(5000 < X < 6000) = \frac{12}{200} = 0.06 = 6\%$$

$$\hat{P}_r(16000 < X < 17000) = \frac{4}{200} = 2\%$$

**Reliability and its relationship to failure frequency**

Defining the reliability as the probability of the device survival after say T hours or T cycles or T kilometer etc; denoted by  $R_T$ , then

$$R_T = \Pr(X > T)$$

and given a sample size N of the devices.

$$\hat{R}_T = 1 - P_T = 1 - \frac{n_T}{N}$$

where

X the product life time

$\hat{R}_T$  estimate for  $R_T$

$n_T$  the number of devices in the sample which break down (failure frequency) during period T,

$P_T$  the estimate for the proportion of devices with life time than T

N sample size.

For example for T=6000 hr in Fig. 14.3

$$T = 6000 \quad P_T = \frac{n_T}{N} = \frac{20 + 18 + 16 + 14 + 13 + 12}{200} = 0.465;$$

Then the relative frequency for 6000 hr is 0.465 and the estimate of 6000-hr reliability is

$$R_{6000} = 1 - 0.465 = 0.535 = 53.5\%$$

when the probability density function or the cumulative density function of product lifetime denoted by  $f(x)$  and  $F(x)$  is available then,

$$R_T = \Pr(X > T) = 1 - F(T) = \int_T^{\infty} f(t) dt$$

## 14.2 Introducing some statistical distributions used in reliability theory

The random variables TTF (lifetime) and TTR (repair time) for a device or a system could have various statistical distributions including the ones introduced below:

### 14.2.1 Exponential distribution

One common distribution used for lifetime of products is exponential distribution. In an exponential distribution with parameter  $\lambda > 0$  or mean  $\theta = \frac{1}{\lambda}$ , the cumulative distribution function and the probability density function are:

$$F_X(t) = 1 - e^{-\lambda t} = 1 - e^{-\frac{t}{\theta}}, \quad f_X(t) = \lambda e^{-\lambda t} = \frac{1}{\theta} e^{-\frac{t}{\theta}} \quad t > 0$$

the hazard function is:

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda \text{ or}$$

$$h(t) = \lambda = \frac{1}{\theta}.$$

the hazard function of exponential distribution does not depend on time. Therefore the distribution is appropriate for section A-B in Fig. 14.2 whose failure rate is constant over time.

### Illustrating the fact that hazard function of exponential distribution is constant

The following table shows the estimate for the hazard function, pdf, cdf, reliability function of the lifetime of the switches related to Fig. 14.3. Third row illustrates that the hazard function is approximately the same over time.

To test whether the lifetime data of Fig 14.3 is exponentially distributed or not, one can use a goodness of fit test such as Bartlett goodness of fit test for exponential distribution or Pearson chi-squared test. To estimate  $\lambda$ , the distribution parameter, if the number of failures in  $k$  subinterval are given, either the following formulae could be used:

$$\hat{\lambda} = \frac{\frac{m}{N}}{T} = \frac{m}{NT} \quad \text{or} \quad \hat{\lambda} = \frac{\sum_{i=1}^K (\text{relative frequency of failures})}{K},$$

Table 14-0 Estimate of hazard function, pdf, cdf , reliability function of the switch lifetime related to Fig.14.3.																	
Sub-interval	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Non-defectives	200	180	162	146	132	119	107	96	86	77	69	62	56	50	45	40	36
defectives	20	18	16	14	13	12	11	10	9	8	7	6	6	5	5	4	4
Relative freq. of failure $h(t) \cong 0.1$ t in 1000 hr	$\frac{20}{200}$ =0.1	$\frac{18}{180}$ =0.1	$\frac{16}{162}$ = .098	$\frac{14}{146}$	0.1	$\frac{12}{119}$	$\frac{11}{107}$	$\frac{10}{96}$	$\frac{9}{86}$	$\frac{8}{77}$	$\frac{7}{69}$	$\frac{6}{62}$	$\frac{6}{56}$	0.1	0.1	0.1	$\frac{4}{36}$
$\hat{f}(t)$ time in hr D=200000 hr	$\frac{20}{D}$	$\frac{18}{D}$	$\frac{16}{D}$	$\frac{14}{D}$	$\frac{13}{D}$	$\frac{12}{D}$	$\frac{11}{D}$	$\frac{10}{D}$	$\frac{9}{D}$	$\frac{8}{D}$	$\frac{7}{D}$	$\frac{6}{D}$	$\frac{6}{D}$	$\frac{5}{D}$	$\frac{5}{D}$	$\frac{4}{D}$	$\frac{4}{D}$
$\hat{f}(t)$ time in 1000hr	$\frac{20}{200}$	$\frac{18}{200}$	$\frac{16}{200}$	$\frac{14}{200}$	$\frac{13}{200}$	$\frac{12}{200}$	$\frac{11}{200}$	$\frac{10}{200}$	$\frac{9}{200}$	$\frac{8}{200}$	$\frac{7}{200}$	$\frac{6}{200}$	$\frac{6}{200}$	$\frac{5}{200}$	$\frac{5}{200}$	$\frac{4}{200}$	$\frac{4}{200}$
$\hat{R}(t)$ Time in 1000hr	$\frac{20}{200}$	$\frac{38}{200}$	$\frac{54}{200}$	$\frac{68}{200}$	$\frac{81}{200}$	$\frac{93}{200}$	$\frac{104}{200}$	$\frac{114}{200}$	$\frac{123}{200}$	$\frac{131}{200}$	$\frac{138}{200}$	$\frac{144}{200}$	$\frac{150}{200}$	$\frac{155}{200}$	$\frac{160}{200}$	$\frac{164}{200}$	$\frac{168}{200}$
$\hat{R}(t)=1-\hat{F}(t)$	0.9	0.81	0.73	0.66	0.6	0.54	0.48	0.43	.39	0.36	0.48	0.28	0.25	0.23	0.20	0.18	0.16

where m is the number of the switches broke down in time T, out of sample of size N.

The following reasoning also proves the distribution of the life time is exponential:

Having given the above table, the 4<sup>th</sup> row shows the failure rate is approximately constant and equal 0.1. On the other hand since

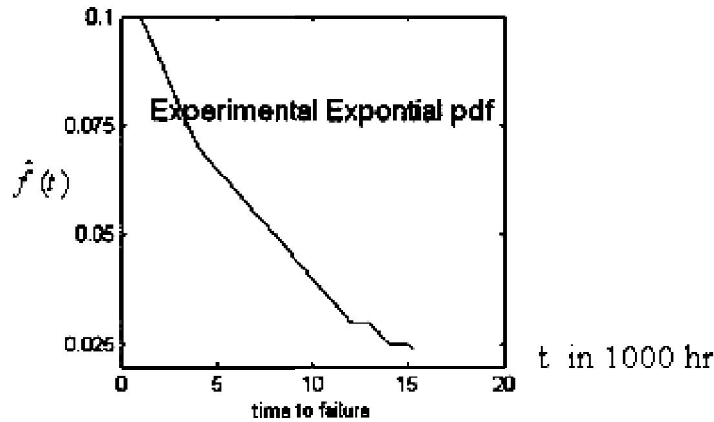
$$R(t) = \frac{f(t)}{h(t)} = e^{-\int_0^t h(\tau) d\tau}$$

then  $1 - F(t) = R(t) = e^{-\int_0^t 0.1 \tau} = e^{-0.1t} \Rightarrow F(t) = 1 - e^{-0.1t}$ .

Therefore the lifetime of the switch follows an exponential distribution with parameter  $\lambda = 0.1$  or mean  $\theta = 10$ .

Figure 14-4 shows the plot of row 5 of the table in terms of row 1; i.e. the proportion of the switches in the sample which broke down over time that is actually  $\hat{f}(t)$ , the estimate of the probability density function(pdf) of the switches lifetime (TTF) at time t.





**Fig. 14.4 Empirical pdf of the switches related to Fig.14.3**  
(extracted from Feigenbaum,1991 page 580)

### Reliability Function of Exponential Distribution

The Reliability function of an exponentially distributed random variable is given by:

$$R_T = 1 - F(T) \rightarrow R_T = e^{-\lambda T} = e^{-\frac{T}{\theta}}$$

For example if we would like to calculate the 6000-hr reliability of a kind of product which has constant failure rate of  $\lambda = 0.1$  failures per 1000 hr or  $\lambda = 10^{-4}$  failures per hour; since the failure rate is constant then the distribution is exponential and

$$R_{6000} = e^{-\lambda T} = e^{-0.0001 \times 6000} = 0.55$$

i.e.

the probability that a unit product does not fail is 0.55; in other words 55% of the product survive are 6000 hours.

If the reliability is estimated from Fig. 14.3, the 6000-hr reliability would be calculated as:

$$\frac{200 - (20 + 18 + 16 + 14 + 13 + 12)}{200} = 0.535$$

Needless to say that the sum (20+18+16+14+13+12) is the number of switches remaining after 6000 hours.

Note that

-It is the exponential distribution that possess a constant hazard function over time; the hazard function of other distributions vary with time;

-Exponential distribution is a special form of gamma:  $\text{gamma}(1, \beta)$  and two-parameter Weibull distribution:  $C=1$ .

-Given a sample of a statistical distribution, its parameters could be estimated with estimation methods such as method of moments or maximum likelihood method(MLE); MATLAB is able to estimate many distributions parameters; e.g. the command expofit estimates the parameter of exponential distributions. It is worth mentioning that if sample  $X_1, \dots, X_n$  is taken from a distribution the sample mean is an unbiased estimate for the distribution mean:  $\hat{\lambda} = \bar{X}$ .

### Example 14.2

The lifetime of a component is exponentially distributed with parameter  $\lambda$ . The component will be replaced either it breaks down or works for a period of T time. Find the average time to replace this component.

#### Solution

Let

X= component lifetime

Y= Time elapsed to replace the component

$$Y = \text{Min}(X, T) \Rightarrow E(Y) = E[\text{Min}(X, T)]$$

$$\text{Min}(X, T) + \text{Max}(X, T) = X + T \Rightarrow$$

$$E[\text{Min}(X, T)] = E(X + T) - E[\text{Max}(X, T)]$$

$$E(X + T) = E(X) + T$$

$$E\{\text{Max}(X, T)\} =$$

$$E(\text{Max}(X, T)|X > T)Pr(X > T) + E(\text{Max}(X, T)|X \leq T)Pr(X \leq T)$$

$$E(\text{Max}(X, T)|X > T) = E(X|X > T)$$

Random variable X is memory-less because it is exponentially distributed then  $E(X|X > T) = T + \frac{1}{\lambda}$ ; on the other hand

$$E(\text{Max}(X, T)|X \leq T) = E(T) = T$$

Then:

$$E\{\text{Max}(X, T)\} = (T + \frac{1}{\lambda}) \times e^{-\lambda T} + T(1 - e^{-\lambda T}) = T + \frac{1}{\lambda}e^{-\lambda T}$$

$$E(Y) = E\{\text{Min}(X, T)\} = E(X + T) - E[\text{Max}(X, T)]$$

$$= \frac{1}{\lambda} + T - (T + \frac{1}{\lambda}e^{-\lambda T}) \Rightarrow$$

$$E(Y) = \frac{1}{\lambda}(1 - e^{-\lambda T}).$$

### 14.2.2 Three- parameter Weibull distribution

3-parameter Weibull distribution is a distribution with:

$$f(t) = \frac{C}{B} \left( \frac{t-A}{B} \right)^{C-1} e^{-\left( \frac{t-A}{B} \right)^C} \quad t \geq A$$

$$R(t) = e^{-\left( \frac{t-A}{B} \right)^C}, \quad F(t) = 1 - e^{-\left( \frac{t-A}{B} \right)^C} \quad t \geq A$$

$$E(X) = A + B \Gamma \left( 1 + \frac{1}{C} \right) \quad \Gamma(\text{integer } n) = (n-1)!$$

$$\text{Var}(X) = B^2 \Gamma \left( 1 + \frac{2}{C} \right) - B^2 \left[ \Gamma \left( 1 + \frac{1}{C} \right) \right]^2$$

where A, B>0 and C>0 are called respectively the location parameter, the scale parameter and the shape parameters.

By the way of reminding, it is added that

- when A=0 & C=1 the distribution is exponential;
- when A=0 & C=2 the distribution is called Rayleigh which is used in modeling sea waves and communication waves.

### 14.2.3 Two- parameter(2-p) Weibull Distribution

A two-parameter weibull is a 3-parameter weibull with A=0; then

$$R(t) = 1 - F_x(t) = e^{-\left( \frac{t}{B} \right)^C}, \quad f(t) = \frac{C}{B} \left( \frac{t}{B} \right)^{C-1} e^{-\left( \frac{t}{B} \right)^C} \quad t > 0$$

Figure 14.5 shows the pdf of several 2- parameter Weibull distributions.

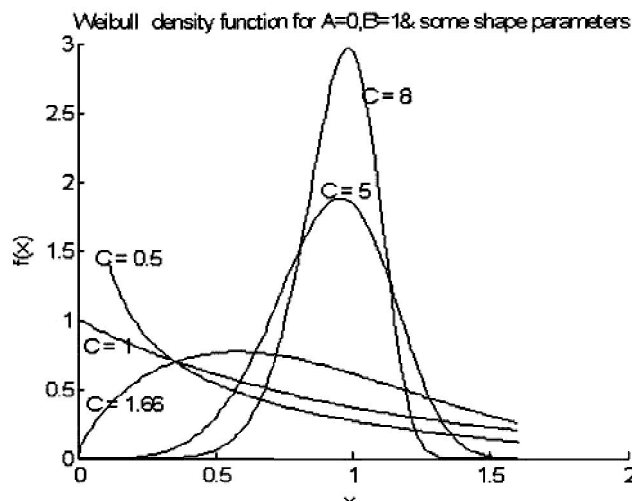
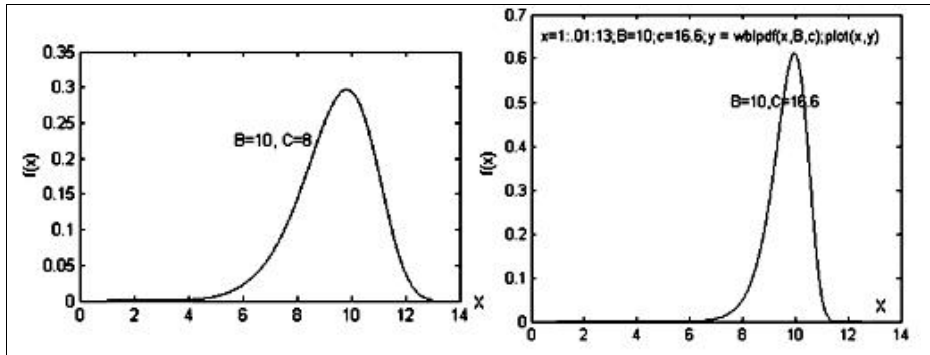


Figure.14.5-a Density functions of 2-p Weibull with B=1 and several C's

(extracted from Grant&amp;Leavenworth,1988 page 605)



**Fig.14.5-b Density functions of 2-p Weibull with B=10 and 2 C's**

It is worth reminding that exponential distribution is the same as a two-parameter Weibull with shape parameter =1; and the same as a gamma distribution with shape parameter =1.

#### Parameter Estimate of Weibull Distribution

Given a random sample of size  $n$ , the parameters of weibull distribution could be estimated graphically<sup>1</sup> or using such methods as MLE. The equations in MLE methods are (Muir & Elsherawi(1986):

$$\begin{cases} B^C = - \sum_{i=1}^n \left( \frac{A - x_i}{B} \right)^C \\ \frac{n}{C} + \sum_{i=1}^n \ln(x_i - A) + \sum_{i=1}^n \left( \frac{A - x_i}{B} \right)^C \ln(A - x_i) = 0 \\ \sum_{i=1}^n \frac{1-C}{x_i - A} + \frac{C}{B} \sum_{i=1}^n \left( \frac{A - x_i}{B} \right)^{C-1} = 0 \end{cases}$$

It is worth knowing that given a sample of size  $n$  an appropriate estimate for the location parameter in 3- p Weibull distribution is  $\hat{A} = \min_{i=1}^n \{x_i\}$ .

The equation for 2-p Weibull are (Montgomeri & Rungers, 1994 page 299):

$$C = \left[ \frac{\sum_{i=1}^n (x_i^C \ln x_i)}{\sum_{i=1}^n x_i^C} - \frac{\sum_{i=1}^n (\ln x_i)}{n} \right]^{-1} \quad B = \left[ \frac{\sum_{i=1}^n x_i^C}{n} \right]^{\frac{1}{C}}$$

<sup>1</sup> Graphical estimation method for Weibull distribution is described in references such as Ireson (1996) p 25-31 or internet.

If the shape parameter C is known, B is estimated using

$$MLE(B) = \hat{B} = \sqrt[C]{\frac{\sum_{i=1}^n x_i^C}{n}}.$$

A suitable estimate for the location parameter (A) of 3-parameter Weibull is  $\hat{A} = \{x_1, \dots, x_n\}$ .

The MATLAB command wblfit estimates the parameters of a 2-p Weibull distribution i.e. estimates B&C.

### Example 14.3

The random sample

X=[113.0634 49.5432 53.4872 93.7147 74.0594 114.3216  
97.1033 61.5069 74.7216 52.8807];

is given from a 2-p weibull distribution. Write a MATLAB code which calculates B and C from the above equations. Estimate B & C with wblfit command in MATLAB as well.

### Solution:

```
%Sample X=[X(1).....X(n)]
X=[113.0634 49.5432 53.4872 93.7147 74.0594 114.3216 97.1033
61.5069 74.7216 52.8807];
for C=.01:0.001:40
for l=1:length(X)
    LNX(l)=log(X(l)); XIC(l)=X(l)^C;XICLNX(l)=XIC(l)* LNX(l);
end
    A= C-(sum(XICLNX)/sum(XIC)-sum(LNX)/length(X))^-1;
    if ( abs(A)<= 0.001 ) C1=C; disp(sprintf('C= %6.4f ', C1))
end
end
B=(sum(X.^C1)/(length(X)))^(1/C1);
disp(sprintf('B= %6.4f ', B))
```

The above code gives

C= 3.7150

B= 87.1546

```
>> wblfit(X)
```

```
ans = 87.1543 3.7149
```

**Example 14.4**

Given the random sample  $x_1 = 2.1$ ,  $x_2 = 2.2$ ,  $x_3 = 0.9$  from a Rayleigh distribution, estimate the parameter of the distribution.

**Solution**

Rayleigh distribution is a 2-p Weibull with  $C=2$ ;

Since when the shape parameter  $C$  is known,  $B$  is estimated using

$$MLE(B) = \hat{B} = \left[ \frac{\sum X^c}{n} \right]^{\frac{1}{c}} \text{ then}$$

$$\hat{B} = \left[ \frac{(2.1)^2 + (2.2)^2 + (.9)^2}{3} \right]^{\frac{1}{2}} = \sqrt{(2.1^2 + 2.2^2 + .9^2)/3} = 1.8312.$$

**Hazard function of Weibull distribution**

The Instantaneous hazard function of Weibull is given by:

$$h(t) = \frac{f(t)}{R(t)} = \frac{C}{B} \left( \frac{t-A}{B} \right)^{C-1} \quad t > A$$

for 2-parameter weibull:

$$h(t) = \frac{C}{B} \left( \frac{t}{B} \right)^{C-1} \quad t > 0.$$

when  $C > 1$   $h(t)$  increases as  $t$  increases

when  $0 < C < 1$   $h(t)$  decreases as  $t$  increases

when  $C = 1$   $h(t)$  is constant with respect to  $t$ .

(Note that for  $C=1$  the distribution reduces to exponential).

**Reliability function of Weibull distribution**

The survival probability after time  $T$  according to Weibull distribution is

$$R_T = e^{-\left(\frac{T-A}{B}\right)^C};$$

when there is no minimum for lifetime i.e.  $A=0$  then  $R_T = e^{-\left(\frac{T}{B}\right)^C}$ .

The interpretation of  $A=0$  is the fact that the device might break down at  $t=0$  i.e. there is no minimum lifetime guarantee for all products.

When  $C=1$  the reliability function would be the same as that of exponential distribution.

**Example 14.5**

The time to failure of a kind of ball bearing follows a Weibull distribution with  $B=5000$  &  $C=0.5$ .

a) Find mean time to failure:

$$MTTF = E(X) = 5000\Gamma\left(1 + \frac{1}{0.5}\right) = 5000\Gamma(3) = 5000 \times 2! = 10000 \text{ hours.}$$

Find the probability of survival after 6000 hr

$$R_{6000} = \Pr(X > 6000) = e^{-\left(\frac{6000}{5000}\right)^2} = 0.334$$

Then the 6000-hr reliability of the product is 0.334; in other words 33.4% of the ball bearings have a minimum life of 6000 hours.

#### 14.2.4 Truncated Normal Distribution

The ordinary normal distribution is not that useful for lifetime of products; however if its negative section is truncated the distribution of nonnegative part is more appropriate. The normal distribution truncated to be positive with parameters  $\mu$  &  $\sigma$  has the following density function:

$$f(t) = \begin{cases} \frac{1}{a\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} & t > 0 \\ 0 & \text{otherwise} \end{cases}, a = \Pr(Z > -\frac{\mu}{\sigma})$$

The mean and variance of this distribution are no longer  $\mu$  &  $\sigma^2$ . The mean is calculated as follows:

$$\begin{aligned} E(T) &= \int_{-\infty}^{\infty} t f(t) dt = \int_0^{\infty} \frac{t}{a\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt \quad t = t - \mu + \mu \Rightarrow \\ E(T) &= \int_0^{\infty} \frac{t - \mu}{a\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt + \int_0^{\infty} \frac{\mu}{a\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt \Rightarrow \\ E(T) &= \left(-\frac{\sigma}{a} \int_0^{\infty} \frac{2(t-\mu)}{-2\sigma^2} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt\right) + \mu = \left(-\frac{\sigma}{a} - e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} \right) \Bigg|_{t=0}^{t=\infty} + \mu \Rightarrow \\ E(T) &= \mu + \frac{\sigma}{a} e^{-\frac{\mu^2}{2\sigma^2}}. \end{aligned}$$

To calculate the probabilities of a random variable  $X$  having this distribution with parameters  $\mu$  and  $\sigma$ , one could write:

$$\Pr(X > x^*) = \frac{1}{a} \int_{x^*}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{a} \Pr(Z > \frac{x^* - \mu}{\sigma})$$

Where  $a = \Pr\left(Z > -\frac{\mu}{\sigma}\right)$ .

Logarithmic normal distribution was mentioned in chapter 4.

Figure 14.6 shows some functions of five distributions used in reliability theory.

### 14.3 Reliability of a system of several components

This section deals with the reliability of a system composed of several components. The configuration or arrangement of components in a system could be series, parallel, k-out-of-n, star-delta or a simple or complex combination of them. Some of the configurations are described below. It is worth knowing the reliability of components may vary with time, or may does not vary with time. The former case is called dynamic and the latter case is called a static model. The arrangement of components in a system is sometimes referred to as RBD<sup>1</sup>.

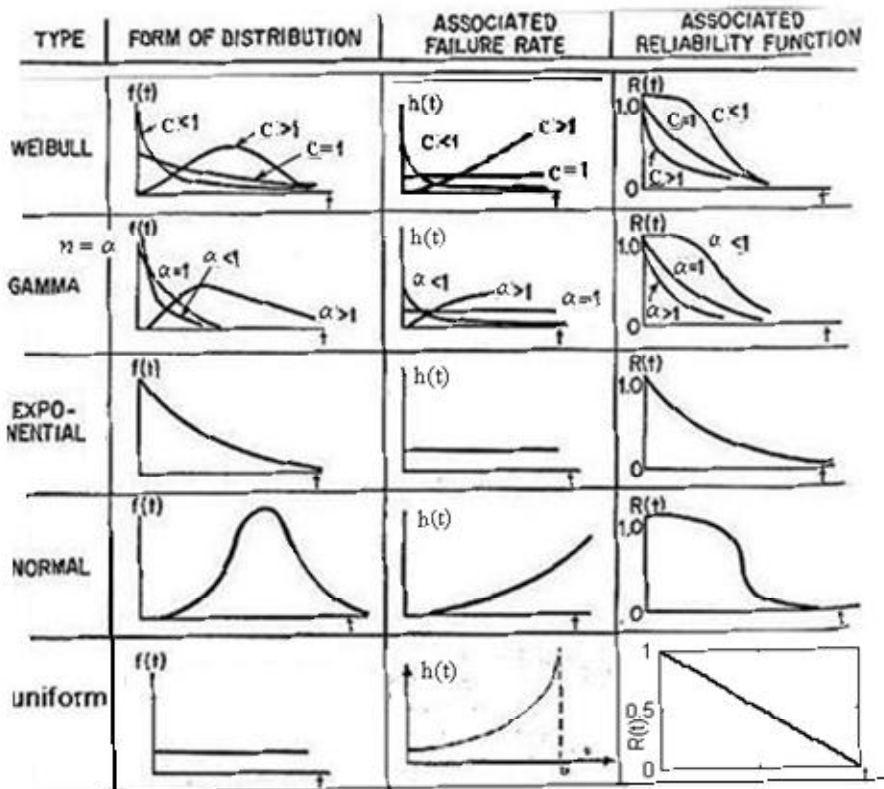


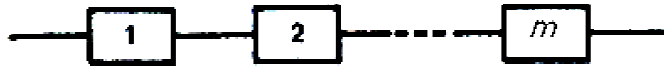
Fig. 14.6 The shape of pdf, hazard function and reliability function some distributions (after Feigenbaum, 1990, Fig.17-5)

<sup>1</sup> Reliability Block Diagram



### 14.3.1 Series configuration

A system of series configuration is such that it works as far as all components work. In other words in this configuration if a component fails the system fails.



**Fig. 14.7 m-component series configuration**

Suppose the system in Fig. 14.7 is composed of  $m$  independent components each having T-time reliability  $(R_T)_i = \Pr(X_i > T)$  where  $X_1, \dots, X_m$  denote the lifetimes of the components. The system fails if at least one component fails.

The reliability of the system, denoted by  $(R_T)_{sys} = \Pr(X > T)$  is computed from:

$$(R_T)_{sys} = (R_T)_1 (R_T)_2 \dots (R_T)_m$$

Where  $X$  denotes the lifetime of the system.

#### Proof

Let  $X$  denotes the lifetime of the system and  $X_m \dots X_1$  denote the independent lifetimes of the components; then:

$$X = \min(X_1 \dots X_m) \quad P_r(X > T) = P_r[\min(X_1 \dots X_m) > T] \Rightarrow$$

$$P_r(X > T) = P_r(X_1 > T, X_2 > T, \dots, X_m > T)$$

$$\Rightarrow (R_T)_{sys} = (R_T)_1 \dots (R_T)_m$$

#### End of proof.

In static case i.e. dependent of time where the reliabilities of independent components are given as  $R_1, \dots, R_n$ , again the system probability would be the product of  $R_i$ 's:

$$R_{sys} = \prod_{i=1}^n R_i.$$

It is worth noting in a series configuration

the less the number of components of the system the more reliable the system, and the more reliable the components, the more reliable the system.

#### Example 14.6

A system of  $m$  components fails when each of the components fails. The lifetimes of the components are independent and exponentially distributed with parameters  $\lambda_1, \lambda_2, \dots, \lambda_m$ . Find the reliability of the system.

**Solution**

$$(R_T)_{sys} = e^{-\lambda_1 T} \dots e^{-\lambda_m T} = e^{-T \sum_{i=1}^m \lambda_i}$$

That is when  $m$  components with constant failure rates  $\lambda_i, i = 1, 2, \dots, m$  constitute a series system, the system itself also would have a constant failure rate equal to  $\sum_{i=1}^m \lambda_i$ . In other words the lifetime of the system is exponentially distributed with parameter  $\sum_{i=1}^m \lambda_i$ .

**Example 14.7**

A system is assembled as a series configuration of 8 types of electrical components, shown in the following table. Each type constitutes a subassembly which has a series configuration in turn. All life times are independent and follow exponential distribution. Given the data in the following table, predict the 10-hour reliability of the system.

Order(i)	Component Name	Number used in series in the system ( $m_i$ )	$\lambda_i$ Failure rate (per hour)	Failure rate of the subassembly per hr ( $m_i \lambda_i$ )
1	Diode	52	$120 \times 10^{-6}$	$52 \times 120 \times 10^{-6} = 6.24 \times 10^{-3}$
2	Motor	3	$100 \times 10^{-6}$	$0.300 \times 10^{-3}$
3	Relay	18	$145 \times 10^{-6}$	$2.610 \times 10^{-3}$
4	Resistor	213	$10 \times 10^{-6}$	$2.130 \times 10^{-3}$
5	Potentiometer	26	$70 \times 10^{-6}$	$1.820 \times 10^{-3}$
6	Switch	82	$25 \times 10^{-6}$	$2.050 \times 10^{-3}$
7	Transform motor	21	$20 \times 10^{-6}$	$0.420 \times 10^{-3}$
8	Soldered point	341	$18 \times 10^{-6}$	$6.138 \times 10^{-3}$
Sum				$\lambda = 21.708 \times 10^{-3}$

The life time distribution of the total assembled product is exponential with  $\lambda = 0.021708$  failure/hr. The 10-hr reliability is 0.805 as calculated below:

$$R_{10} = e^{-(0.0217)(10)} = 0.805 = 80.5\%$$

**Example 14.8**

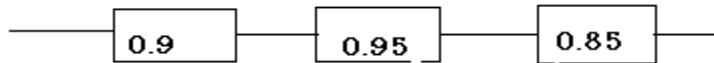
A series system constitutes 3 components having reliabilities  $R_1=0.9$ ,  $R_2=0.95$  and

$R_3=0.85$ . The failure the system imposes a cost of \$5000.

- Find the reliability of the system;
- Find the mean of incurred cost in the long range;
- If a similar systems is added to the system as a standby subsystem. Find the reliability of the new system, assuming perfect switching.
- Calculate the mean of the incurred cost paid for the failure of the new system in the long range.

**Solution**

a)



$$R = R_1 R_2 R_3 = (0.9)(0.95)(0.85) = 0.73$$

b)

probability	Cost
0.73	0
0.27	\$5000

The average cost is  $5000 \cdot 0.27 = \$1350$ .

c)

Let the reliability of the working subsystem be denoted by  $R_{1S}$  and the reliability of the standby subsystem by  $R_{2S}$ ; then the reliability of the whole system would be given by:

$$\begin{aligned}
 R_{sys} &= 1 - (1 - R_{1S})(1 - R_{2S}) = R_{1S} + R_{2S} - R_{1S}R_{2S} \\
 &= R_{1S} + (1 - R_{1S})R_{2S} = \\
 &= 0.73 + (1 - 0.73)(0.73) = 0.93
 \end{aligned}$$

d)

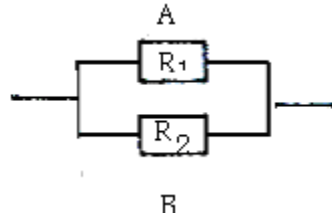
Probability	cost
0.93	0
0.07	\$5000

The average cost is  $5000 \cdot 0.07 = \$350$ .

The difference of the two costs is 1000 dollars; then if the price of the backup is considerably less than 1000 dollars, it could be said the latter system is economic.

### 14.3.2 Parallel configuration

At the outset consider a 2- component system in Fig. 14.8 subassemblies A & B are in parallel; i.e. the system works if at least one of the two works.



**Fig. 14.8 A 2-subassembly parallel system**

From the probability theory we know:

The probability that either A or B or both work =

$$= \Pr(A \overset{\text{or}}{+} B) = \Pr(A) + \Pr(B) - \Pr(AB)$$

Let the reliability of system (R) be the probability that either A or B or both work and denote the probability that A works by  $R_1$  and the probability that B works by  $R_2$ ; if A and B are independent then

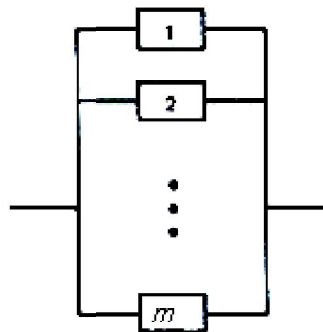
$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(A) \Pr(B)$$

$$\Pr(A \overset{\text{or}}{+} B) = R_1 + R_2 - R_1 R_2 = 1 - 1 + R_1 + R_2 - R_1 R_2 = 1 - (1 - R_1) + R_2(1 - R_2) = 1 - (1 - R_1)(1 - R_2)$$

Therefore the the reliability of a 2 component parallel system is given by

$$R = 1 - (1 - R_1)(1 - R_2).$$

In the general case where there are  $m$  independent components and the system works as far as at least one component works, we have an  $m$ -component parallel system(Fig 14.9). An example of this is the circuit of several parallel resistances, several similar teller in banks, airports.



**Fig. 14.9 A Parallel System**

Denoting the lifetimes of the components by  $X_1, \dots, X_m$ , in this case we have:

System Lifetime =  $X = \max(X_1, \dots, X_m)$  then

$$\Pr(X \leq T) = \Pr[\max(X_1, \dots, X_m) \leq T] = \Pr(X_1 \leq T, X_2 \leq T, \dots, X_m \leq T)$$

Assume  $X_1, \dots, X_m$  are independent and each component has T-time reliability  $(R_T)_i$  i.e.  $(R_T)_i = \Pr(X_i > T)$ . Denote system reliability by

$(R_T)_{sys} = \Pr(X > T)$ , then

$$1 - (R_T)_{sys} = [1 - (R_T)_1] [1 - (R_T)_2] \dots [1 - (R_T)_m] \Rightarrow \\ (R_T)_{sys} = 1 - [1 - (R_T)_1] \dots [1 - (R_T)_m].$$

Further more if the reliability of the components of a parallel system, denoted by  $R_1, \dots, R_m$ , are dependent of time then

$$R_{sys} = 1 - (1 - R_1) \dots (1 - R_m)$$

### 14.3.3 Calculation of reliability when back-up is available

Consider a system in which subsystem 1 works and one or more redundant subsystems are waiting to replace subsystem 1 in case of failure by a "switch" as shown in Fig. 14-10. The "switch" could be a man or a device; it could be always working or be out of order when needed to replace the standby component. The former case is called perfect switching and the latter is called imperfect switching. In addition the standby components themselves might be subject to failure before replacing. That is why these cases (reliability of the switch and the failure of standby redundant components on replacement) come into effect. References such as Kapur & Lamberson (1977) and Kapur & Pecht (2014) deal with these cases in detail. Below some cases regarding to standby redundancy are briefly dealt.

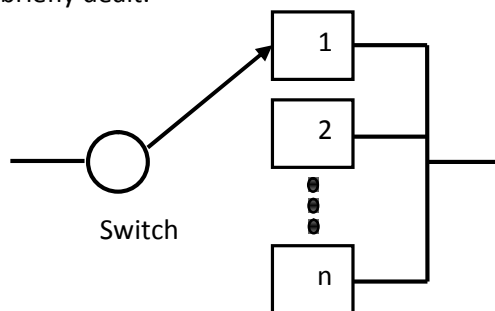


Fig.14-10 A system with standby redundant subsystems

### 14.3.3.1 Two-component standby (imperfect & perfect switching system)

Suppose there are 2 subsystems in a system, one with reliability of  $R_1$  is active(working), the other having a reliability of  $R_2$  is standby or backup;  $R_1$  &  $R_2$  are constants rather than functions of time. In addition suppose, when called upon to be used, the switch is subject to failure but the standby component is not subject to failure. The probability that the switch works when it is needed to replace the standby subsystem is a constant number  $P_s$  (e.g. the mean of the switch reliability function rather than the reliability function itself). Then the reliability of this 2-component system denoted here by

$R_{2 \text{ sys}}$  would be (Billinton & Allen, 1992):

$$R_{2 \text{ sys}} = R_1 + P_s R_2 (1 - R_1) = 1 - (1 - R_1)(1 - P_s R_2)$$

The simplest case is when we have perfect switching i.e.  $P_s=1$  and it is not probable that the standby component become out of order during its period of being standby. When the active component fails, the standby component replaces it by the help of the perfect switch; in this case since  $P_s=1$  then:

$$R_{2 \text{ sys}} = 1 - (1 - R_1)(1 - R_2)$$

This relationship is similar to that of a 2-component parallel system when both are active.

#### Example 14.9

A device with reliability of 90% has a back-up with reliability of 80%. When the device fails it is replaced automatically by a perfect switch. Suppose the backup might not be out of order when replaces. Find the reliability of the system composed of the active and stand by.

#### Solution 1

$\Pr(\text{system works}) = \Pr(\text{the active device works}) + \Pr(\text{the device fails \& the backup works}) =$

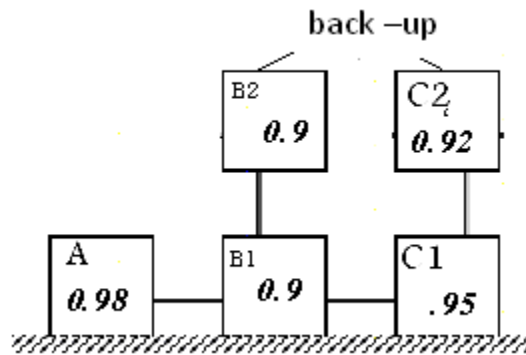
$\Pr(\text{the device works}) + \Pr(\text{the active device fails}) \times \Pr(\text{the backup works}) = 0.9 + 0.1 \times 0.8 = 0.98$

#### Solution 2

$$R_{2 \text{ sys}} = 1 - (1 - R_1)(1 - R_2) = 1 - (1 - 0.9)(1 - 0.8) = 0.98$$

#### Example 14.10

Calculate the reliability of a system whose RBD is as follows. The number in each box represent the reliability of the component.  $B_2$  and  $C_2$  are backups which replace respectively  $B_1$  and  $C_1$  upon failure by a perfect switch. The backups are supposed to be working when replacing. Also calculate the reliability of the system if backups were not available.



### Solution

The A, B and C constitute a series configuration, at the outset note the reliability of " $B_1$  + its backup  $B_2$ " is  $1 - (1 - 0.9)(1 - 0.9)$  and that of " $C_1$  + its backup  $C_2$ " is  $1 - (1 - 0.95)(1 - 0.92)$ . Then the reliability of the series system is given by

$$R_{sys} =$$

$$0.98[1 - (1 - 0.9)(1 - 0.9)][1 - (1 - 0.95)(1 - 0.92)] = 0.966$$

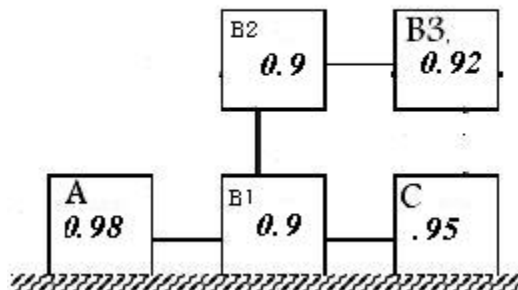
Ignoring the backups would yield the reliability equal to

$$0.98 \times 0.90 \times 0.95 \cong 0.84.$$

### Example14.11

A system has the following RBD. The  $B_2$  and  $C_2$  are backups for  $B_1$  with assumption of no failure while they are in standby mode when  $B_1$  fails,  $B_2$  replaces it by a perfect switch ; when  $B_2$  fails,  $B_3$  replaces it by a perfect switch. Compute the system reliability. The number in each box represent the reliability of the component.

2 backups for E1



**Answer**

It could be shown that the reliability of the middle subsystem is calculated as if it is a 3- component parallel system:  $1-(1-0.9)(1-0.9)(1-0.92)=0.9992$   
 $R_{\text{sys}} = (0.98)0.9992*(0.95)=0.93$  .

**Example 14.12**

Test of 100 units of a kind of valve, each for 20000 hours, resulted in 6 failures;

- a) find the proportion failed,
- b) the failure rate of each unit valve per year( i.e.no. of failure in a year),
- c) find expected number of failures per year for 1000 units,
- d) find MTBF.

**Solution**

a) 6%

b) From the 100 units during the 20000 hours test ,6 units failed . it is supposed that each of these 6 failed items worked  $\left(\frac{0+20000}{2} = 10000\right)$  hours on the average before failure.

The total test time of the units during the test is then:

$$100 \times 20000 - \frac{6 \times 20000}{2} = 1940000 \text{ hr.}$$

The failure rate(FR) of a unit is equal to

$$\begin{aligned} \text{FR} &= \frac{6}{1940000} = 3.09 \times 10^{-6} \text{ failure per hr} \\ &= 3.09 \text{ failures per } 10^6 \text{ hr.} \end{aligned}$$

On the other hand

Failures of a unit per year= Failures of a unit per hour  $\times 365 \times 24$ .

Then failure rate for each unit per year is  $3.3 \times 10^{-6} \times 365 \times 24 = 0.0289$

In other words the amount of failures for a unit per year is 0.029.

If 1000 units are operated; the expected number of failed items per year is  $0.029 \text{ failures per unit per year} \times 1000 \text{ units} = 29 \text{ failure per year}$

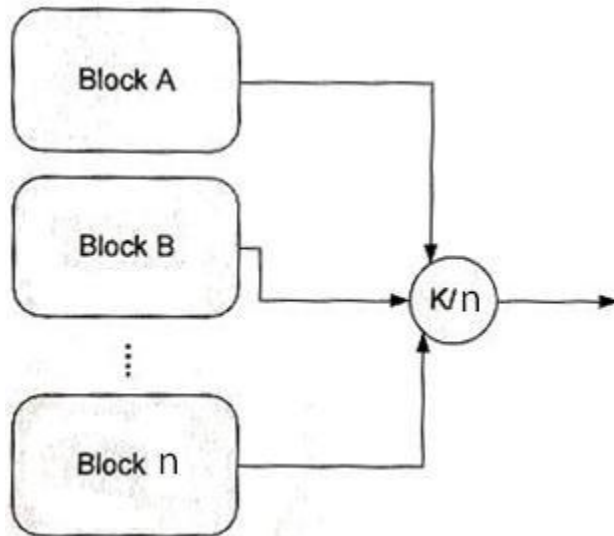
d)

$$\text{MTBF} = \frac{\text{total test time}}{\text{number of failures}} = \frac{1940000}{6} = 323333.3 \text{ hr}$$



### 14.3.4 k-out-of-n configuration

A k-out-of-n reliability configuration is a system with n similar components which fails if and only if at least k consecutive components fail (Fig.14-11)



**Fig.14-11 A "K out n" configuration.**

Needless to say that when  $k=n$ , we have a series configuration; when  $k=1$  we have a parallel system. The reliability of a k-out-of-n configuration consisting n similar components, each with reliability R, is given by the following relationship (Barlow&Proshan,1996, page 218)

$$R_{sys} = \sum_{j=k}^n \binom{n}{j} R^j (1-R)^{n-j} = k \binom{n}{k} \int_0^R x^{k-1} (1-x)^{n-k} dx.$$

In this system the reliability of the system increases,  
when k reduces and R is constant,  
and

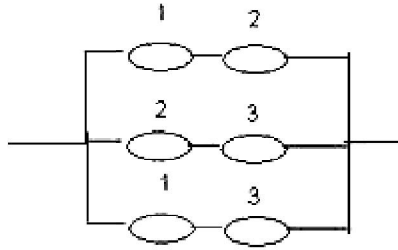
when R increases and k is constant.

As an exercise show that

for  $k=n$ , the above formula gives the reliability of series configuration , and  
for  $k=1$ , the formula gives the reliability of parallel system.

#### **Example 14.13**

The lifetime of components 1,2 &3 follow a Weibull distribution with parameters  $A=0$ ,  $B=200$  hr and  $C=.5$ . Verify the following diagram which contain only 3 components ( not 6 components) could be regarded as a 2-out-of-3 system. Calculate the 100-hour reliability of the system.

**Solution**

Note that according to the diagram, the system works in the following cases:

If 1&2 work;

If 1&3 work;

If 2&3 work;

If 1,2&3 work

Then it is a 2-out-of-3 system.

$$R_1 = R_2 = R_3 = R = e^{-\left(\frac{x-A}{B}\right)^C} = e^{-\left(\frac{100}{200}\right)^{0.5}} = 0.4931$$

$$R_{\text{sys}} = 2 \binom{3}{2} \int_0^{0.4931} x^{2-1} (1-x)^{3-2} dx$$

$$= 2 * 3 \int_0^{0.4931} x(1-x) dx = 0.4897$$

or with MATLAB

$$\text{syms } x; R_{\text{sys}} = \text{int}(6*x*(1-x), 0, 0.4931)$$

$$R_{\text{sys}} =$$

$$244825328509/500000000000 = 0.4897$$

or

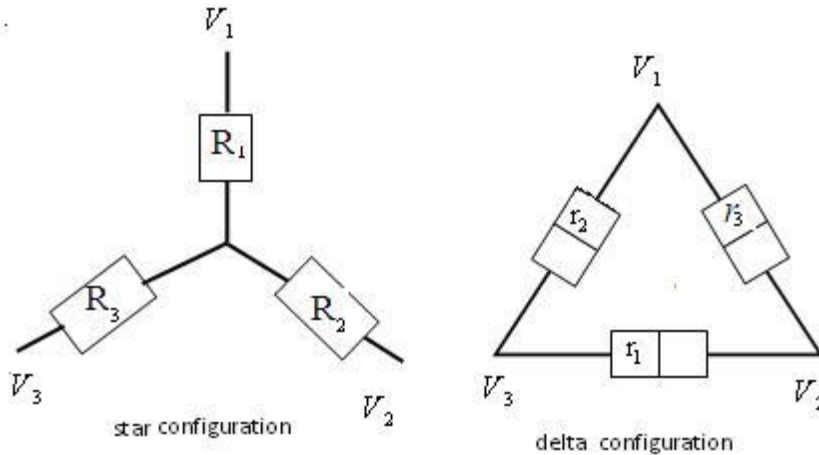
$$R_{\text{sys}} = \sum_{j=2}^3 \binom{3}{j} 0.4931^j (1 - 0.4931)^{3-j}$$

with MATLAB

$$R_{\text{sys}} = 1 - \text{binocdf}(1, 3, 0.4931) = 0.4897.$$

### 14.3.5 Star and Delta configurations

This section deals briefly with star and delta configurations. The following figure shows a star configuration and a delta configuration. These are easily converted into each other, a property which could be used to simplify the RBD of a complex System whose original reliability is hard to compute.



**Fig.14-12 Star and Delta configurations.**

We now deal with a special case where the reliability of the 3 components are the same.

Suppose in a delta configuration the reliability of each component is  $r_\Delta$ , then  $R_3, R_2, R_1$  in the equivalent star configuration is derived from:

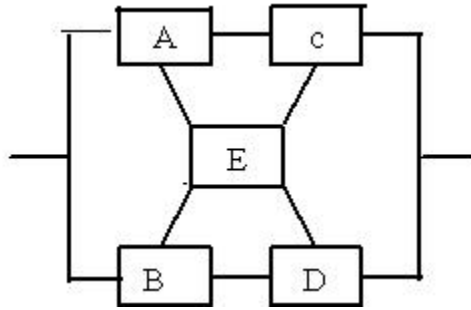
$$R_1 = R_2 = R_3 = \sqrt{r_\Delta + r_\Delta^2 - r_\Delta^3} = R_Y,$$

Conversely, if in a star configuration  $R_1 = R_2 = R_3 = R_Y$ , then  $r_\Delta$  as the reliability of the 3 components of the equivalent delta configuration ( $r_1 = r_2 = r_3 = r_\Delta$ ) is a root of the following equation (based on Grosh, 1989):

$$-r_\Delta^3 + r_\Delta^2 + r_\Delta - R_Y = 0.$$

### 14.3.6 Complex configurations

As well as series, parallel and k-out-of-n configurations, there are other configurations, some of which are complex whose reliability cannot be calculated easily. The following RBD is an example of complex systems.



**Fig. 14.13 An example of a complex RBD**

There are some methods, which might be useful in calculating the reliability of these kinds of systems, including:

- 1-Enumeration Method,
- 2-Path Tracing Algorithm<sup>1</sup>
- 3-Appication of Bayesian Decomposition Theorem
- 4- Star-Delta Conversion
- 5-The method based on Markov Process
- 6-Cut-set and Tie-set Method

The third method is described here.

#### **14.3.6.1 Application of Bayesian Decomposition Theorem**

A way to calculate the reliability of some complex systems is to apply Bayesian decomposition, described below:

a)

Search for an element in the RBD of the system, which is a key one such that if the it works permanently the RBD reduces to an RBD whose reliability can be calculated easily; and if it does not work permanently, the system reduces to a system whose reliability can be calculated easily again.

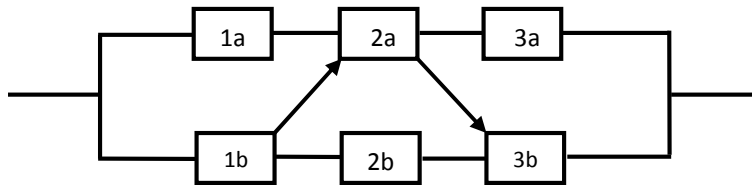
b)

Use Bayes' rule of conditioning to compute the original system reliability.

#### **Example 14.14**

Find the reliability of a system consisting of subsystem types 1, 2, and 3, with reliabilities  $R_1$ ,  $R_2$  and  $R_3$  with the following RBD (Lewis,1992 page 282).

<sup>1</sup> From: Primer of Reliability Theory by Doris Lloyd Grosh; John Wiley



**Fig. 14-14 A complex system**

**Solution**

Component 2a is selected as the key element by trial and error. The survival of the system is conditioned on two events i.e. either subsystem 2a fails or operating.

Let

Y The event denoting successful system operation,

X The event denoting failure of component 2a

$\bar{X}$  The event denoting successful operation of 2a,

$R_1$  denote the reliability of components 1a , 1b,

$R_2$  denote the reliability of components 2a , 2b

$R_3$  denote the reliability of components 3a , 3b.

Consider the sample space(SS) consisting 2 events X ,  $\bar{X}$  ;

i.e.  $SS = \{\bar{X}, X\}$ .

According to Bayesian theorem we have:

$$P\{Y\} = Pr(Y|X)Pr(X) + Pr(Y|\bar{X})Pr(\bar{X})$$

where

$Pr(Y|X) = R^-$  the probability of successful system operation if 2a has failed,

$Pr(Y|\bar{X}) = R^+$  the probability of successful system operation if 2a is operating,

$Pr(Y) = R$  the total probability of successful system operation,

$Pr\{\bar{X}\}$  The probability of successful operation of component 2a,

$Pr\{X\}$  The probability that component 2a fails.

Since  $Pr\{X\} = 1 - R_2$  and  $Pr\{\bar{X}\} = R_2$  then

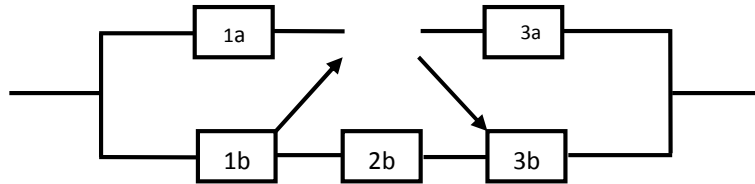
$$R = R^-(1 - R_2) + R^+R_2.$$

**Calculation the conditional reliabilities  $R^-$  and  $R^+$**

a) To compute the probability of successful system operation if component 2a has failed ( $R^-$ ), all the paths through 2a in Fig.14.12 are disconnected (as if 2a is removed); the result appears in Fig. 14.13.a; which is actually the RBD of a series of components 1b,2b

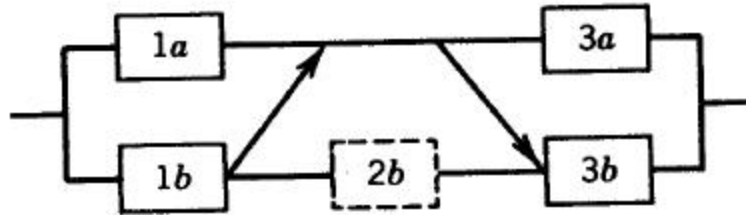
and 3b; subsystems 1a and 3a no longer make any contribution to the value of  $R^-$ . Therefore:

$$\Pr(Y|X) = R^- = R_1 R_2 R_3$$



**Fig.14.14.a Component 2a has failed**

b) For computing the probability of successful system operation if component 2a is operating i.e.  $R^+$ , we say: when 2a is always operating a line could be plotted instead of it as shown



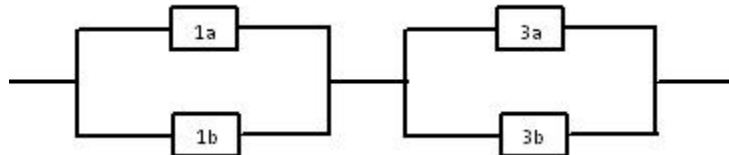
**Fig. 4.14.b Component 2a always functioning in Fig 14.14**

in Fig. 14.14.b or Fig14.14.c. Now we have a series combination of two parallel configurations, 1a and 1b in the first and 3a and 3b in the second; since component 2b is always bypassed, it has no effect on the value of  $R^+$ . Therefore we have an RBD shown in Fig 14.14, whose reliability is given by

$$\Pr(Y|\bar{X}) = R^+ =$$

$$(R_1 + R_1 - R_1 R_1)(R_3 + R_3 - R_3 R_3) = (2R_1 - R_1^2)(2R_3 - R_3^2) \Rightarrow$$

$$R^+ = (1 - (1 - R_1)(1 - R_1))(1 - (1 - R_3)(1 - R_3))$$



**Fig14.14.c RBD of the system of Fig 14.14 when component 2a always functioning**

Substituting  $R^+$  and  $R^-$  in the following formula (Bayes' rule)

$$R = P\{Y\} = \Pr(Y|X)\Pr(X) + \Pr(Y|\bar{X})\Pr(\bar{X}) = R^-(1 - R_2) + R^+R_2,$$

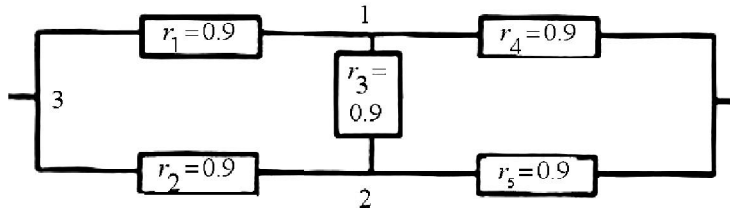
We have:

$$R = \Pr(Y) = R_1 R_2 R_3 (1 - R_2) + (2R_1 - R_1^2)(2R_3 - R_3^2)R_2.$$

As an exercise compute the reliability of the system in Fig14.11 if the reliability of each component is 0.95.

**Example 14.15** (based on Dhillon,2006 page47)

Find the reliability of a system with the following RBD.

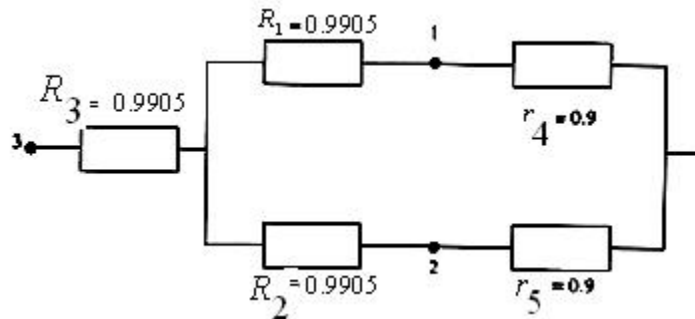


### Solution

The delta configuration of  $r_1=r_2=r_3=0.9=r_\Delta$  has an equivalent star configuration whose 3 components has the reliability of  $R_Y$  derived from:

$$R_1 = R_2 = R_3 = \sqrt{r_\Delta + r_\Delta^2 - r_\Delta^3} = 0.9905.$$

Replacing the delta configuration with its equivalent star results in the following RBD whose reliability could be easily calculated.



### Universal Generating Function( UGF)

In recent years an approach called the universal generating function (UGF) technique has been applied to reliability analysis, especially in systems with multi-state components (MSS<sup>1</sup>). UGF is actually an extension of probability generation function or z-transform. To become acquainted with this function and the related technique, there are a lot of references in the internet.

<sup>1</sup> multi state system

## 14.4 Availability

For repairable systems a fundamental quantity of interest is the availability, defined as follows:

Availability is the probability that a system is available for use at a given time (Lewis,1994 page 300).

### 14.4.1 Availability function

The probability that a system is performing satisfactorily at time  $t$  is referred to as instantaneous or point availability or availability function denoted by  $A(t)$ :

$A(t)$  = probability that a system is performing satisfactorily at time  $t$ .

#### 14.4.1.1

#### Availability function of a device with exponential life

If the lifetime(time to failure) follows an exponential distribution with parameter  $\lambda$  and the time to repair follows another exponential distribution with parameter  $\mu$ , then  $A(t)$

Is given by (Garosh,1989; Kapur&Lamberson,1976 page 227):

$$A(t) = \frac{\mu}{\mu + \lambda} + \frac{\lambda}{\mu + \lambda} e^{-(\lambda + \mu)t} \quad t \geq 0$$

For an irreparable device or equipment whose lifetime is exponentially distributed, the function  $A(t)$  equals the reliability function i.e.  $R(t)$ ; this is easily proved by substituting  $\mu = 0$  in the above relationship.

### 14.4.1 Interval availability

Interval availability is just the value of the availability function averaged over some interval of time  $T$ . This interval may be the design life of the system or the time to accomplish some particular mission. The interval availability,  $A^*(T)$  is defined by (Lewis,1994,page 300)

$$A^*(T) = \frac{1}{T} \int_0^T A(t) dt.$$

### 14.4.2 Steady- state(Long-term) Availability

Long-term availability of the system is the mean proportion of time when the system is functioning (Rausand and Hoyland, 2004) or the probability that the system is functioning in the long time. Limiting or Long-term availability, denoted by  $A$ , is defined as (Ross,1985, pag1 401, Kapur and Laberson,1977 page 228):

$$A = \lim_{t \rightarrow \infty} A(t).$$



If the life time of the product and also the repair time are exponentially distributed with parameters  $\lambda$  and  $\mu$  respectively the Long-term availability would be calculated as follows:

$$A(t) = \frac{\mu}{\mu + \lambda} + \frac{\lambda}{\mu + \lambda} e^{-(\lambda + \mu)t} \quad t \geq 0$$

$$A = \lim_{t \rightarrow \infty} A(t) = \frac{\mu}{\mu + \lambda} = \frac{\frac{\mu}{\lambda\mu}}{\frac{\mu + \lambda}{\lambda\mu}} = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \frac{1}{\mu}}$$

or

$$A = \frac{MTBF}{MTBF + MTTR}$$

Fortunately, if the **repair time follows** another **continuous distribution** other than exponential (such a logarithmic normal) with mean MTTR, **the same above** result would be obtained for the limiting availability A (Kapur and Lamberson, 1977 page 228).

Then the less the time to repair a system or equipment the more its availability i.e. the more the fraction of time that the item performs the required function.

#### Example 14.16

Suppose the lifetime of a device exponentially distributed with mean 100 hours. What should be mean of the repair time in order to have the steady state availability of 98.5%?

$$A = \frac{MTBF}{MTBF + MTTR} \quad 0.985 = \frac{100}{100 + MTTR} \Rightarrow MTTR = 1.52 \text{ hr.}$$

#### 14.4.3 Calculation of system availability from mean on & off times

If in a system, the  $i^{\text{th}}$  component's on & off distributions are arbitrary distributions with means  $\frac{1}{\lambda_i}, \frac{1}{\mu_i}$ , respectively  $i=1,2,\dots,n$ ; it follows from the theory of alternating renewal processes, for the limiting availability of the  $i^{\text{th}}$  component we can write (Ross, 1985 page 402):

$$A_i(t) \rightarrow A_i = \frac{\frac{1}{\lambda_i}}{\frac{1}{\lambda_i} + \frac{1}{\mu_i}}$$

where  $\frac{1}{\lambda_i}$  &  $\frac{1}{\mu_i}$  are the mean time that the  $i^{\text{th}}$  component is on and off times.

The steady state availability of the system is calculated from:

$$A = f(A_1, \dots, A_n) = f\left(\frac{\frac{1}{\lambda_1}}{\frac{1}{\lambda_1} + \frac{1}{\mu_1}}, \dots, \frac{\frac{1}{\lambda_n}}{\frac{1}{\lambda_n} + \frac{1}{\mu_n}}\right)$$

where

$A_i$  is the limiting (steady state) availability of the  $i^{\text{th}}$  component,

$f(R_1, \dots, R_n)$  is the reliability function of the n-component system.

For example if the configuration of n components is series, we have:

$$R_{\text{sys}} = f(R_1, \dots, R_n) = R_1 \times \dots \times R_n$$

$$A_i = \frac{\frac{1}{\lambda_i}}{\frac{1}{\lambda_i} + \frac{1}{\mu_i}}$$

$$A = f(A_1, \dots, A_n) = f\left(\frac{\frac{1}{\lambda_1}}{\frac{1}{\lambda_1} + \frac{1}{\mu_1}}, \dots, \frac{\frac{1}{\lambda_n}}{\frac{1}{\lambda_n} + \frac{1}{\mu_n}}\right) = \frac{\frac{1}{\lambda_1}}{\frac{1}{\lambda_1} + \frac{1}{\mu_1}} \times \dots \times \frac{\frac{1}{\lambda_n}}{\frac{1}{\lambda_n} + \frac{1}{\mu_n}}$$

if the configuration of n-component system is parallel, then:

$$R_{\text{sys}} = f(R_1, \dots, R_n) = 1 - (1 - R_1) \times \dots \times (1 - R_n)$$

$$A = f(A_1, \dots, A_n) = f\left(\frac{\frac{1}{\lambda_1}}{\frac{1}{\lambda_1} + \frac{1}{\mu_1}}, \dots, \frac{\frac{1}{\lambda_n}}{\frac{1}{\lambda_n} + \frac{1}{\mu_n}}\right) = 1 - \left(1 - \frac{\frac{1}{\lambda_1}}{\frac{1}{\lambda_1} + \frac{1}{\mu_1}}\right) \times \dots \times \left(1 - \frac{\frac{1}{\lambda_n}}{\frac{1}{\lambda_n} + \frac{1}{\mu_n}}\right)$$

The steady-state availability of a non-repairable system is zero (Lewis, 1994 page 301)

It is worth mentioning that the reliability and availability of a unit, device equipment, system, etc... have a value between 0 and one. Some times for a special product one of them is very low and the other is high e.g. the reliability of a system might be 0.02 or 2% and its availability be 98%.

#### Example 14.17

A system is compromised of 2 devices :a compressor with constant failure rate 80.37 failures per 1 million working hours and MTTR 899.3 hours. The other device has a constant failure rate 4.78 failures per 1 million hours and MTTR 890.3 hours. Find

- The reliability of the components of the system for a mission time of 26280 hours and the corresponding steady –state availability;
- The reliability of the series system for a mission time of 26280 hours and its steady –state availability.

#### Solution

Constant failure rate imply exponentially distributed lifetime.

a)

Compressor:

$$MTBF=1/(80.37 \times 10^{-6})=12442.4$$

$$R(t)=\exp(-t/MTBF)$$

$$R(26280)=\exp (-26280 \times 80.37 \times 10^{-6})=0.1210$$

$$A_1=\frac{MTBF}{MTBF+MTTR}=\frac{12442.4}{12442.4+89.3}=0.9929$$

The device:

$$MTBF=1/(4.78 \times 10^{-6})=209205 \text{ hr}$$

$$R(26280)=\exp(-26280 \times 4.78 \times 10^{-6})=0.8819$$

$$A_2=\frac{MTBF}{MTBF+MTTR}=\frac{209205}{209205+89.3}=0.99957$$

b)

The system has a series configuration then

$$R_{sys} = f(R_1, R_2) = R_1 \times R_2$$

$$R_{sys} = 0.1210 \times 0.8819 = 0.1067$$

$$A = f(R_1 = A_1, R_2 = A_2) = A_1 \times A_2$$

$$A = 0.9929 \times 0.99957 = 0.9925.$$

#### 14.4.2.1 Steady- state Availability and Preventive Maintenance

Maintainability affects availability. The time required to repair the system or run the preventive maintenance makes the system unavailable. There is a close relationship between reliability and maintainability ; both of the affect availability and costs. In the long-term where the repair and preventive maintenance (PM) take place with constant rate after the transient period, the steady-state availability is given by (O'Connor, 2004, page 401):

$$A = \frac{MTBF}{MTBF + MTTR} \times \frac{PM_{cycle}}{PM_{cycle} \times PM_{time}(total)}$$

where

$PM_{cycle}$

is the time period required to run the preventive maintenance say every  $10^3$  hours,

$PM_{time}(total)$

Is the total time required for accomplishing all repair and PM actions.

### 14.5 Acceptance sampling based on product lifetime

This section introduces some acceptance sampling procedures which accept or reject a lot based on lifetime; also a handbook containing such procedures will be introduced. But before dealing with the procedures, you will get acquainted with plotting OC curves in the field of reliability.

Operating characteristic (OC) curves are strong tools in quality control; they display the discriminatory power of sampling procedures and plans. One application of them is helping to compare acceptance sampling plans with each other. In Chapter 10, a kind of OC curve for a single sampling plan  $n$  &  $c$  was discussed; the curve showed the probability of acceptance as a function of proportion of defective items. The following section deals with those OC curves in which the acceptance probability of a lot of a special product is plotted versus a parameter such as the product mean lifetime or the product reliability.

### 14.5.1 OC Curve for Acceptance Sampling Plans based on Mean Lifetime

To plot the curve a random sample from the lot is taken and tested; if the sample meets a special criterion, the lot will be accepted.

**Example 14.18** (based on Grant&Leevenworth,1988 page 585)

Consider the following acceptance sampling plan to be taken from a largish lot and plot its OC curves using various MTTF.

The plan is as follows:

Take a random sample of 22 units from the lot of product, and apply life test. Whenever an item fails, replace it with another item selected at random from the lot. If the test continues for 500 hr with no more than 2 failures, accept the lot. If 3 failures [ or more] occur before 500 hr of testing. Terminate the test and reject the lot.

#### Solution

We suppose simultaneous testing of 22 units for 500 hours or 110 units for 100 hours or 11000 units for 1 hour give the same results. To plot the OC curve, note that totally we have  $22 \times 500 = 11000$  item-hours with acceptance number  $c = 2$ . Assume the failure probability is the same for all 11000 unit-hours. Define the problem as a single acceptance sampling plan with  $n=11000$ , and  $c=2$ , let:

$X$  = number failures in 11000 unit-hours during 1 hour test, Then

$$P_a = \Pr(X \leq 2 | \text{given})$$

$\lambda =$  The probability of failure of one unit in an hour,  
or

$\lambda =$  the failure rate of one unit per hour or the failure probability of one unit-hour

or

$\lambda =$  the proportion of binomial distribution  $p'$ )

$$= \Pr(X \leq 2 | \text{given})$$

mean of number of failures in a sample of 11000  $= 11000\lambda = np'$ )

Here  $\lambda$  has a value less than 1 and is interpreted as the probability of the failure of one unit in an hour.

The failure rate  $\lambda$  takes the place of fraction nonconformities (Grant & Leavenworth, 1988, page 586). Then the probability that a unit fails in an hour is  $\lambda$ . To plot the OC curve, the exact value of lot acceptance probability corresponding to a particular  $p'$  could be calculated from:

$$P_a = \text{binocdf}(2, n=11000, p'=\lambda).$$

The approximate probability for various  $np'$  may be calculated from

$$P_a \approx \text{poisscdf}(2, np').$$

As stated earlier, in chapter 10,  $P_a$  were plotted versus  $p'$ . But in this chapter the horizontal axis is either the product mean life ( $\theta = \frac{1}{\lambda}$ ) or the product reliability. Table 14-1 shows the probabilities for some values of  $\theta$ . Figure 14-15 shows the corresponding OC curve.

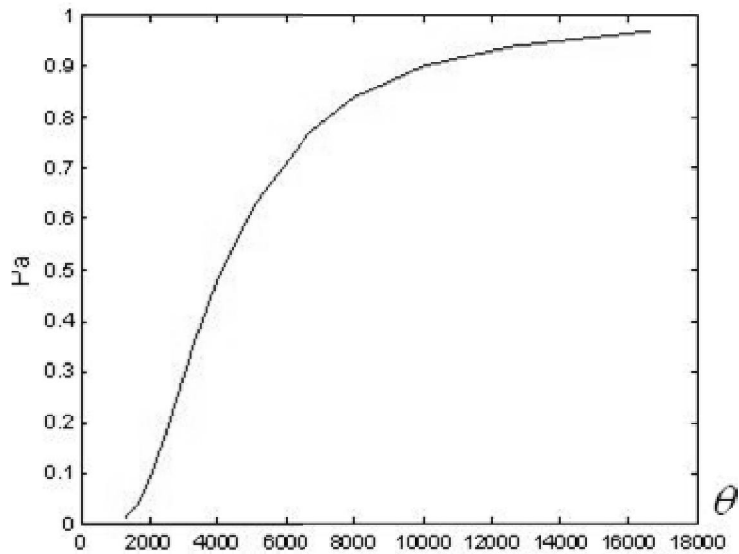
To know how  $P_a$  is calculated, suppose  $\lambda = 0.0003$ ; since  $n=11000$  then  $np' = 11000\lambda = 3.3$ . The approximate value for  $P_a$  from Poisson CDF table :  $P_a = \Pr(X \leq 2) = 0.359 \approx 0.360$ . The exact value of  $P_a$  is calculable from MATLAB :

$$P_a = \Pr(\leq 2) = \text{binocdf}(2, 11000, 0.0003) = 0.3594.$$

<b>Table 14-1 Acceptance probability of in Example 14.15 for various mean lifetimes</b> (Grant Leavenworth, 1988, Page 586)			
Calculation of OC curve for acceptance sampling plan requiring 11,000 item hours of life testing with an acceptance number of 2. Calculation assumes that the failure rate $\lambda$ is independent of the <u>age of the item tested</u>			
Failure rate per hour, $\lambda = p'$	Mean life $\theta = \frac{1}{\lambda}$ hours	Expected average number of failures in 11,000 test hours ( $np' = 11000\lambda$ )	Probability of acceptance (probability of 2 or less failures) from Poisson distribution $P_a = \Pr(X \leq 2)$
0.00002	50000	0.22	0.999
0.00005	20000	0.55	0.982
0.00006	16667	0.66	0.971
0.00008	12500	0.88	0.939
0.00010	10000	1.1	0.900
0.000125	8000	1.375	0.839
0.00015	6667	1.65	0.770
0.00020	5000	2.2	0.623
0.00025	4000	2.75	0.480
0.00030	3333	3.3	0.360
0.00040	2500	4.4	0.185
0.00050	2000	5.5	0.088
0.00060	1667	6.6	0.040
0.00080	1250	8.8	0.007

The following MATLAB commands plots the OC curve.

```
p=1/17000:.00001:1/1000;Pa=binocdf(2,11000,p);plot(1./p,Pa)
```



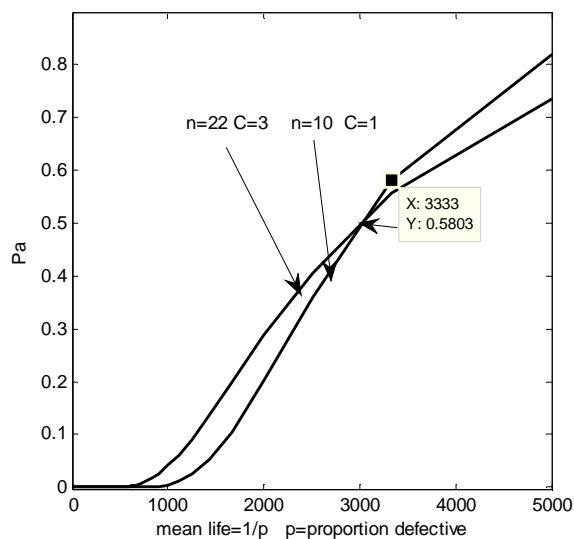
**Fig. 14-15 OC curve for Example 14.15**  
(Probability of acceptance versus mean life).

**Example 14.19**

Figure 14-16 shows the OC curves for the following 2 acceptance sampling plans:  $n=10, c=1, T=500$ ,  $n=22, c=3, T=500$ .

The commands used for plotting the curves in MATLAB were:

$p=1:-.0001:.00020$ ;  $Pa=\text{binocdf}(1,5000,p)$ ;  $\text{plot}(1./p,Pa)$ ; hold on;  
 $Pa=\text{binocdf}(3,11000,p)$ ;  $\text{plot}(1./p,Pa)$ ;



**Fig 14.16 The OC curves of Example 14.16.**

The figures could be interpreted according to what you have learned in the subject of OC curve in Chapter 10.

### 14.5.2 OC Curve for Acceptance Sampling Plans based on Reliability

To plot  $P_a$  versus  $R$  for the OC curve of a single sampling plan with sample size  $n$  & acceptance number  $c$ , used for inspecting large lots, take a random sample of size  $n$  and test all the  $n$  products for a time period  $T$ . Each product has a succeeding probability of  $R$ , which is the product reliability for time  $T$ . The probability that the number of failures observed during the test time,  $X$ , is less than or equal to  $c$ , which is the number of allowable failures in  $n$  trials is given by:

$$P_a = \Pr(X \leq c) = \sum_{x=0}^c \binom{n}{x} (1-R)^x R^{n-x}.$$

This is a relationship for plotting  $P_a$  versus  $R$ .

#### Example 14.20

Plot the OC curve based on reliability for the single sampling plan of  $n=10$ ,  $c=2$ . Take the following reliability points on the X-axis:  
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9

#### Solution

The following MATLAB commands are used to plot the curve shown in Fig.14.17:

```
R=0:0.1:1; Pa= binocdf(2,10,1-R); plot(R,Pa)
```

The figure shows the probability of accepting the lots versus various product reliability.

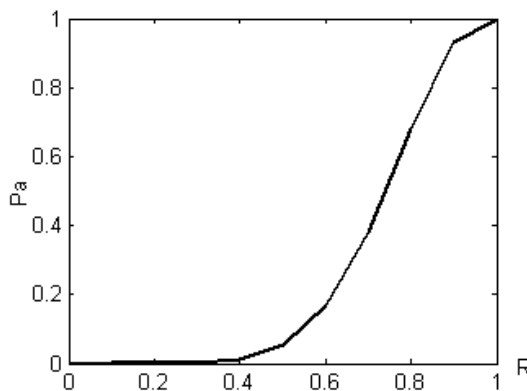


Fig 14.17 OC curve for Plan  $n=10$ ,  $c=2$ .



## 14.6 Acceptance Sampling based on life test

This section deals with introducing some tables for designing acceptance sampling plans based on product lifetime. Several types of tables have been developed for acceptance sampling plans based on lifetime, with distributions such as Weibull and exponential.

### H 108 Sampling Procedures and Tables for Life and Reliability Testing

is a work in this area based on exponential distribution and provides procedures and tables for life and reliability testing. The exponential distribution<sup>1</sup> is suitable for the portion A-B of bathtub-shaped failure rate model shown in Fig. 14-2 whose failure rate is some constant  $\lambda$ . The handbook includes

- definitions required for the use of the life test sampling plans and procedures;
- general description of life test sampling plans;
- life tests terminated upon occurrence of pre-assigned number of failures;
- life tests terminated at pre-assigned time; and
- sequential life test sampling plans.

For each test there are 2 options:

replacing the failed items during the test time by new items or not replacing.

Below the type "life tests terminated at pre-assigned time" is used for testing the products of the sample.

### 14.6.1 Acceptance Sampling with life tests terminated at pre-assigned time together with replacement

In this procedure a number of products is taken at random from the lot and are tested simultaneously. Whenever a product fails it is replaced by a new one. The test continues for a pre-assigned time (T). If the recorded number of failures at the end of the time point is less than a rejection number (r), the lot will be accepted.

It has to be emphasized that If before the end of T- time test, a failure occurs it is replaced by a new one, unless the number of failures before time T is equal to rejection number r.

The rest of this section deal with specifying a procedure for accepting or rejecting the lot based on lifetime, from the handbook, given the producer and consumer risks( $\alpha$  &  $\beta$ ). The procedures are described for the following

---

<sup>1</sup> A goodness of fit test or Q-Q plot could be used to examine whether a distribution is exponential or not. Bartlett's test described at the end of this chapter deals with this.

3 cases, each having an exponential distribution for product lifetime and any of the following cases:

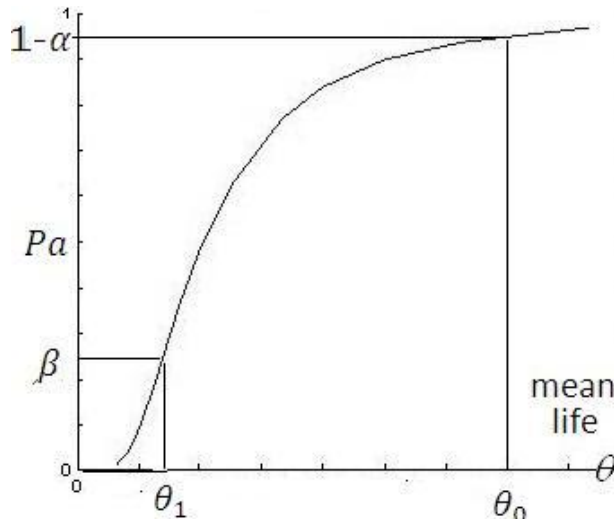
- known  $\alpha$  &  $\beta$  together with a restriction is on the sample size;
- known  $\alpha$ ,  $\beta$ ,  $r$  and  $n$
- known  $\alpha$ ,  $\beta$  and  $T$

#### 14.6.1.1

##### *Case1: Designing a plan subject to a restriction on sample size( $n$ ) given producer risk( $\alpha$ ) and consumer risk ( $\beta$ )*

In this case, as well as a constraint on sample size,  $\alpha$  i.e. the probability of type I error for a mean life  $\theta_0$ , and  $\beta$  i.e. the probability of type II error for mean life  $\theta_1$ , are given (see Fig. 14.18). To specify the plan do the followings

- a) Calculate  $\frac{\theta_1}{\theta_0}$ ,
- b) Extract the code of the plan from Table 14-2. The codes begins with letters A,B,C,....
- c) Extract the rejection number( $r$ ) from the table corresponding to the letter code. For code use A Table 14-3 and for code B use Table 14-4. For other codes refer to the handbook.



**Fig 14.18 the producer and consumer risks( $\alpha$  &  $\beta$ )**

- d) To specify the test time, having calculated the given the ratio  $\frac{\theta_1}{\theta_0}$ , read  $\frac{T}{\theta_0}$  from Table 14-3 if the code is A &  $\alpha = 0.01$  and from Table 14-4 if the code is B &  $\alpha = 0.05$ . If the exact ratio is not found in the table; find the next greater ratio available in the table. Table 14-3, and Table 14-4 give also the sample size (as a multipliers of  $r$ ). The steps are illustrated below.

<b>Table 14-2 Codes of acceptance sampling plans</b> (Table 2A-1of H-108)									
$\alpha$	<b>0.01</b>	<b>0.05</b>		<b>0.10</b>		<b>0.25</b>		<b>0.50</b>	
$\beta$	<b>0.01</b>	<b>0.10</b>		<b>0.10</b>		<b>0.10</b>		<b>0.10</b>	
code	$\frac{\theta_1}{\theta_0}$	code	$\frac{\theta_1}{\theta_0}$	code	$\frac{\theta_1}{\theta_0}$	code	$\frac{\theta_1}{\theta_0}$	code	$\frac{\theta_1}{\theta_0}$
A- 1	0.004	B- 1	0.022	C- 1	0.046	D- 1	0.125	E- 1	0,301
A- 2	0.038	B- 2	0.09	C- 2	0.137	D- 2	0.247	E- 2	0,432
A- 3	0.082	B- 3	0.154	C- 3	0.207	D- 3	0.325	E- 3	0,502
A- 4	0.123	B- 4	0.205	C- 4	0.261	D- 4	0.379	E- 4	0.550
A- 5	0.160	B- 5	0.246	C- 5	0.304	D- 5	0.421	E- 5	0.584
A- 6	0.193	B- 6	0.282	C- 6	0.340	D- 6	0.455	E- 6	0.611
A- 7	0.22	B- 7	0.31	C- 7	0.370	D- 7	0.483	E- 7	0.633
A- 8	0.247	B- 8	0.338	C- 8	0.396	D- 8	0.506	E- 8	0.652
A- 9	0.270	B- 9	0.36	C- 9	0.418	D- 9	0.526	E- 9	0.667
A-10	0.29	B-10	0.38	C-10	0.438	D-10	0.544	E-10	0.681
A-11	0.37	B-11	0.459	C-11	0.512	D-11	0.608	E-11	0.729
A-12	0.428	B-12	0.512	C-12	0.561	D-12	0.650	E-12	0.759
A-13	0.470	B-13	0.55	C-13	0.597	D-13	0.680	E-13	0.781
A-14	0.504	B-14	0.58	C-14	0.624	D-14	0.703	E-14	0.798
A-15	0.554	B-15	0.62	C-15	0.666	D-15	0.737	E-15	0.821
A-16	0.591	B-16	0.65	C-16	0.695	D-16	0.761	E-16	0.838
A-17	0.653	B-17	0.71	C-17	0.743	D-17	0.800	E-17	0,865
A-18	0.698	B-18	0.745	C-18	0.774	D-18	0.824	E-18	0,882
$\alpha$ =producer risk, the probability of rejecting the lot if the mean life is a desired lifetime $\theta_0$ $\beta$ =consumer risk, the probability of accepting the lot if the mean life is a undesired life $\theta_1$									

### Example 14.21

To inspect lots of a kind of electrical resistance, design an accepting sampling plan with the following specifications. The life of the resistance is exponentially distributed.

If the mean life  $\theta_0$  is 900 hours, 95% of the lots be accepted;

If the mean life  $\theta_1$  is 300 hours, 10% of the lots be accepted;

**Answer**

a)Calculation of  $\frac{\theta_1}{\theta_0}$  :

$$\frac{\theta_1}{\theta_0} = \frac{\text{mean life realted to } \beta}{\text{mean life realted to } 1 - \alpha} = \frac{300}{900} = 0.33,$$

b)Reading the code *B-8* from Table 14-2 under the column  $\alpha=5\%$  &  $\beta = 10\%$ .

c)From Table 14-4 rejection  $no=r= 8$  in front of code *B-8*.

d)To specify sample size( $n$ ) and a time for the test i.e.  $T$  from Table 14-4, various  $n$  can be selected. Suppose for some reasons  $n=4r$ .i.e  $n=32$  is selected.

From Table 14-4 at the intersection of column  $4r$  and row *B-8*  $\Rightarrow$

$$\frac{T}{\theta_0} = 0.124. \Rightarrow T = 0.124 \times \theta_0 = 0.124 \times 900 = 111.6 \text{ hr.}$$

Therefore a random sample of size  $n=32$  is selected from the lot; all items of the sample are put on test; when an item fails, it is replaced by a new item. If before the end of the test time  $T=111.6$  hr the recorded number failures reached 8 or exceeded it, reject the lot; otherwise if the time  $T=111.6$  passed and the failures did not reach 8, accept the lot.

Note that if  $r=8$ , and  $n=64, \Rightarrow \frac{T}{\theta_0} = 0.062 \Rightarrow T = 55.8 < 111.6$ ; then a factor affecting the sample size is a possible restriction on the test time .

### Example 14.22

Given  $\frac{\theta_1}{\theta_0}=0.022$ , specify a sampling plan for inspecting lots containing products with exponential distributed lifetime ,  $\alpha = 5\%$ ,  $\beta = 1\%$  and  $\theta_0 = 100 \text{ hr}$ . Design the test for about 15 hours.

**Solution** We would examine some sample sizes by trial and error

$$\begin{aligned} \frac{\theta_1}{\theta_0}=0.022 \text{ Table 14-2} & \rightarrow \text{code= B-1} \Rightarrow r = 1 \\ n = 6 \times r = 6 \text{ Table 14-4} & \rightarrow \frac{T}{\theta_0} = 0.009 \Rightarrow T = 9 \text{ hr} \\ n = 3 \times r = 3 \text{ Table 14-4} & \rightarrow \frac{T}{\theta_0} = 0.017 \Rightarrow T = 17 \text{ hr} \\ n = 2 \times r = 2 \text{ Table 14-4} & \rightarrow \frac{T}{\theta_0} = 0.026 \Rightarrow T = 26 \text{ hr} \end{aligned}$$

The procedure  $n=3$ ,  $r=1$  and  $T=17$  is selected. That is a random sample of size 3 has to be selected and the 3 products be tested simultaneously. If before the end of 17-hr test no failure happens, the lot has to be accepted; otherwise, rejected. If before the end of 17- hr test, a failure occurs it is replaced by a new one, unless the number of failures before 176 hr is equal to rejection number  $r=1$ .

**Table 14-3 Values of  $\frac{T}{t_0}$  for code A and  $\alpha = 0.01$** 

life tests terminated at pre-assigned time with replacement  
 Master Table for Life Tests Terminated at Pre-assigned Time-Testing with Replacement  
 (Table 2C-2(a) Handbook H108)

Code	r	Sample Size(n)									
		2r	3r	4r	5r	6r	7r	8r	9r	10r	20r
A-1	1	0.005	0.003	0.003	0.002	0.002	0.001	0.001	0.001	0.001	0.0005
A-2	2	.037	.025	.019	.015	.012	.011	.009	.008	.007	.004
A-3	3	.073	.048	.036	.029	.024	.021	.018	.016	.015	.007
A-4	4	.103	.069	.051	.041	.034	.029	.026	.023	.021	.010
A-5	5	.128	.085	.064	.051	.043	.037	.032	.028	.026	.013
A-6	6	.149	.099	.074	.060	.050	.043	.037	.033	.030	.015
A-7	7	.166	.111	.083	.067	.055	.048	.042	.037	.033	.017
A-8	8	.182	.121	.091	.073	.061	.052	.045	.040	.036	.018
A-9	9	.195	.130	.097	.078	.065	.056	.049	.043	.039	.019
A-10	10	.207	.138	.103	.083	.069	.059	.052	.046	.041	.021
A-11	15	.249	.166	.125	.100	.083	.071	.062	.055	.050	.025
A-12	20	.277	.185	.139	.111	.092	.079	.069	.062	.055	.028
A-13	25	.297	.198	.149	.119	.099	.085	.074	.066	.059	.030
A-14	30	.312	.208	.156	.125	.104	.089	.078	.069	.062	.031
A-15	40	.335	.223	.167	.134	.112	.096	.084	.074	.067	.033
A-16	50	.350	.234	.175	.140	.117	.100	.088	.078	.070	.035
A-17	75	.376	.250	.188	.150	.125	.107	.094	.083	.075	.038
A-18	100	.391	.261	.196	.156	.130	.112	.098	.087	.078	.039

**Table 14-4 Values of  $\frac{T}{\theta_0}$  for code B and  $\alpha = 0.05$** 

life test terminated at pre-assigned time with replacement

(Table 2C-2(b) of Handbook H-108 From Naghanderian ,1372 page 543)

Code	r	Sample size(n)									
		2r	3r	4r	5r	6r	7r	8r	9r	10r	11r
B-1	1	0,026	0.017	0.013	0.010	0.009	0.007	0.006	0.006	0.005	0.003
B-2	2	0,089	0.059	0.044	0.036	0.030	0.025	0.022	0.020	0.018	0.09
B-3	3	0,136	0.091	0.068	0.055	0.045	0.039	0.034	0.030	0.027	0.014
B-4	4	0,171	0.114	0.096	0.068	0.057	0.049	0.043	0.038	0.034	0.017
B-5	5	0,197	0.131	0.099	0.079	0.066	0.056	0.049	0.044	0.039	0.020
B-6	6	0.218	0.145	0.109	0.087	0.073	0.062	0.054	0.048	0.044	0.022
B-7	7	0.235	0.156	0.117	0.094	0.078	0.67	0.059	0.052	0.047	0.023
B-8	8	0.249	0.165	0.124	0.100	0.083	0.71	0.062	0.055	0.050	0.025
B-9	9	0.261	0.174	0.130	0.104	0.087	0.75	0.065	0.058	0.052	0.026
B-10	10	0.271	0.181	0,136	0.109	0.090	0.78	0.068	0.060	0.054	0.027
B-11	15	0.308	0.205	0,154	0.123	0.103	0.88	0.077	0.068	0.062	0.031
B-12	20	0.331	0.221	0.166	0,133	0.110	0.95	0.083	0.074	0.066	0.033
B-13	25	0.348	0.232	0.174	0,139	0.116	0.99	0.087	0.077	0.070	00.03
B-14	30	0.360	0.240	0. 180	0.144	0.120	0.103	0.090	0.080	0.072	0.036
B-15	40	0,377	0252	0.189	0.161	0.126	0.108	0.094	0.084	0.076	0.038
B-16	50	0.390	0.260	0,195	0.156	0.130	0.111	0.097	0.087	0.078	0.039
B-17	75	0.400	0.273	0,204	0.164	0.136	0.117	0.102	0.091	0.082	0.041
B-18	100	0.421	0.280	0210	0.168	0.140	0.120	0.105	0.093	0.084	0.042

**14.6.1.2****Case2:Designing a plan, given sample size(n),producer risk( $\alpha$ ) and rejection number(r)**

To designing a plan, given n,  $\alpha$  and rejection number r , at the outset read the code from Table 14-2 given  $\alpha$ . Since n and r are known, compute

$b = \frac{n}{r}$ . under the column of  $b \times r$  read  $\frac{T}{\theta_0}$  and then calculate T.

**Example 14.23**

For lots of products having a lifetime exponentially distributed, design a plan n=10 and r=5 including a test terminated at pre-assigned time with replacement in such a way that for lots of mean life  $\theta_0 = 1200 \text{ hr}$ ,  $\alpha = 0.05$ .

**Solution**

From Table 14-2 with  $\alpha = 0.05$  and  $r = 5$ , B-5 is read as the code. This indicate we have to go to Table 14-4, related to letter B; the sample size is  $2 \times r = 10$ . In Table 14-4 under the column  $2r$ , in front of B-5  $\Rightarrow \frac{T}{\theta_0} = 0.197$ . The test time is:  $T = 0.197(\theta_0) = 0.197(1200) \cong 236$ . Then a sample of size 10 is taken from the lot and the items are simultaneously tested. The failed items are replaced by new items. If before the end of the test time i.e.  $T = 236 \text{ hr}$  5 failures occurred, the lot is rejected; if at the end of 236 hours less than 5 failures happened the lot is accepted. It is worth mentioning that if the test time has not ended and a failure happens, the failed product has to be replaced by a new one unless the number of failures has reached the allowed number i.e.  $r-1$ .

**14.6.1.3****Case3: Designing a plan, given producer risk( $\alpha$ ), consumer risk( $\beta$ ) and test time( $T$ )**

In this case given  $\alpha, \beta$  and  $T$ , after calculating  $\frac{T}{\theta_0}$  &  $\frac{\theta_1}{\theta_0}$ , read  $n$  &  $r$  from Table 14-5.

**Example 14.23**

Design a plan whose test is of type "life tests terminated at pre-assigned time" in such a way the test time does not exceed  $T = 500 \text{ hr}$ . The plan is wanted to accept 90% the lots having mean life  $\theta_0 = 10000 \text{ hr}$  ( $\alpha = 0.10$ ), and to reject 95% of the lots with mean life  $\theta_1 = 2000 \text{ hr}$  ( $\beta = 0.05$ ). The life is assumed to be exponentially distributed.

**Solution**

$$\frac{\theta_1}{\theta_0} = \frac{2000}{10000} = \frac{1}{5} \quad \frac{T}{\theta_0} = \frac{500}{10000} = \frac{1}{20}$$

We read  $n=34$  &  $r=4$  from Table 15-4 with  $\frac{T}{\theta_0} = \frac{1}{20}, \frac{\theta_1}{\theta_0} = \frac{1}{5}$ ,

$\alpha = 0.10$ , and  $\beta = 0.05$ . That is a random sample of size 34 is taken from lot and its items are tested simultaneously; if before the end of 500- hr test, a failure occurs it is replaced by a new one, unless the number of failures before 500 hr is equal to  $r=4$ .

**Table 14-5 Acceptance Sampling Plans for some  $\alpha, \beta, \frac{T}{\theta_0}$  &  $\frac{\theta_1}{\theta_0}$  with a test terminated at pre-assigned time with replacement (Table 2 – C4 H108)**

$\theta_1 / \theta_0$	$r$	$T / \theta_0$				$r$	$T / \theta_0$			
		$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{20}$		$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{20}$
		n	N	n	n		n	n	n	n
		$\alpha = 0.01, \beta = 0.01$					$\alpha = 0.05, \beta = 0.001$			
2 Thirds	136	331	551	1103	2207	95	238	397	795	1591
1 half	46	95	158	317	634	33	72	120	241	483
1 Third	19	31	51	103	206	13	25	38	76	153
1 Fifth	9	10	17	35	70	7	9	16	32	65
1 Tenth	5	4	6	12	25	4	4	6	13	27
		$\alpha = 0.01, \beta = 0.05$				$\alpha = 0.05, \beta = 0.05$				
2 Thirds	101	237	395	790	1581	67	162	270	541	1082
1 half	35	68	113	227	454	23	47	78	157	314
1 Third	15	22	37	74	149	10	16	27	54	108
1 Fifth	8	8	14	29	58	5	6	10	19	39
1 Tenth	4	3	5	8	16	3	3	4	8	16
		$\alpha = 0.01, \beta = 0.1$				$\alpha = 0.05, \beta = 0.1$				
2 Thirds	83	189	316	632	1265	55	130	216	433	867
1 half	30	56	93	187	374	19	37	62	124	248
1 Third	13	18	30	60	121	8	11	19	39	79
1 Fifth	7	7	11	23	46	4	4	7	13	27
1 Tenth	4	2	4	8	16	3	3	4	8	16
		$\alpha = 0.01, \beta = 0.25$				$\alpha = 0.05, \beta = 0.25$				
2 Thirds	60	130	217	434	869	35	77	129	258	517
1 half	22	37	62	125	251	13	23	38	76	153
1 Third	10	12	20	41	82	6	7	13	26	52
1 Fifth	5	4	7	13	25	3	3	4	8	16
1 Tenth	3	2	2	4	8	2	1	2	3	7
		$\alpha = 0.1, \beta = 0.01$				$\alpha = 0.25, \beta = 0.01$				
2 Thirds	77	197	329	659	1319	52	140	234	469	939
1 half	26	59	98	197	394	17	42	70	140	281
1 Third	11	21	35	70	140	7	15	25	50	101
1 Fifth	5	7	12	24	48	3	5	8	17	34
1 Tenth	3	3	5	11	22	2	2	4	9	19
		$\alpha = 0.1, \beta = 0.05$				$\alpha = 0.25, \beta = 0.05$				
2 Thirds	52	128	214	429	859	32	84	140	280	560
1 half	18	38	64	128	256	11	25	43	86	172
1 Third	8	13	23	46	93	5	10	16	33	67
1 Fifth	4	5	8	17	34	2	3	5	10	19
1 Tenth	2	2	3	5	10	2	2	4	9	19
		$\alpha = 0.1, \beta = 0.1$				$\alpha = 0.25, \beta = 0.1$				
2 Thirds	41	99	165	330	660	23	58	98	196	392
1 half	15	30	51	102	205	8	17	29	59	119
1 Third	6	9	15	31	63	4	7	12	25	50
1 Fifth	3	4	6	11	22	2	3	4	9	19
1 Tenth	2	2	2	5	10	1	1	2	3	5
		$\alpha = 0.1, \beta = 0.25$				$\alpha = 0.25, \beta = 0.25$				
2 Thirds	25	56	94	188	376	12	28	47	95	190
1 half	9	16	27	54	108	5	10	16	33	67
1 Third	4	5	8	17	34	2	2	4	9	19
1 Fifth	3	3	5	11	22	1	1	2	3	6
1 Tenth	2	1	2	5	10	1	1	1	2	5



### Example 14.24

(For designing a plan, having a constraint on the test time)

We would like to inspect lots on the basis of the lifetime of their products. If the mean life is 22 hr the lot is supposed to be rejected with the probability of 0.95; if the mean life is 1000 hr the lot is wanted to be accepted with the probability of 0.95. What plan do you suggest to apply whose test time is about 20 hours.

### Solution

With mean life=1000  $P_a = 0.95 \rightarrow \alpha = 0.05$ ,

With mean life=20  $P_a = 0.05 \rightarrow \beta = 0.05$ ,

$\theta_0 = 1000, \alpha = 0.05, \beta = 0.05, \theta_1 = 20$

$$\frac{\theta_1}{\theta_0} = 0.022, \frac{T}{\theta_0} = \frac{20}{1000} = 0.02$$

This problem should be solved using Table 14-5 of case 3 in Sec 14.6.1.3, however, the value of  $\frac{\theta_1}{\theta_0} = 0.022$  is not in the table;

therefore we apply case 1 (Sec. 14.6.1.1) as follows:

From Table 14-2 read code B-1 for  $\frac{\theta_1}{\theta_0} = 0.022$ . In Table 14-4, with  $\frac{T}{\theta_0} = 0.02$ , in front of code B-1 the next immediate available number greater than 0.02 is 0.026 for  $r=1$ . Then the sample size is  $n=2r=2$ .

The procedure is as follows:

Take a sample of size 2 randomly from the lot and place the 2 devices on test; If the during  $T = 0.026 \times 1000 = 26$  hr no device fails the lot is accepted; if at least one device fails the lot is rejected. Although generally if during test time  $T$  a device fails it has to be replaced by a new one unless the number of failed items exceeds  $r-1$ , but in this case since  $r=1$ , if one failure happens the test is terminated.

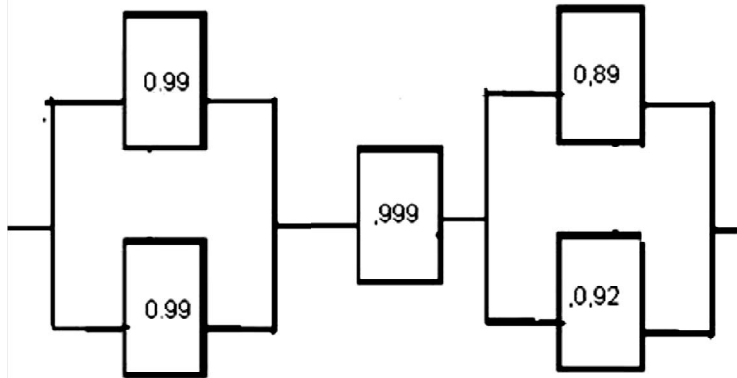
## Exercises

### 14.1

An electrical device has a constant rate of failure 0.00001 per hour (1 failure per  $10^5$  hr). Calculate the 10,000-hour and 20,000-hr reliability of the device.

### 14.2

Calculate the reliability of the following RBD. The number in each box indicates the component reliability.



### 14.3<sup>1</sup>

The lifetime of a direct current battery is normally distributed with mean  $\mu=30$  hr,  $\sigma = 4$  hr.

- Find 25-hr reliability of the battery.
- When should it be replaced to ensure that the probability of failure at that time prior to replacement does not exceed 10%.
- Two batteries are connected in parallel to power a light. Assuming that the light does not fail, what is the 35-hour reliability for the power source.
- A particular battery has been in continuous use for 30 hours. What is the probability that this battery will last another 4 hours?

### 14.4

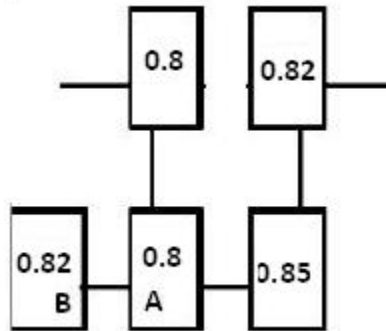
An electronic power system includes power source, receiver and amplifier, connected in series. The mission time for this system is  $10^4$  hours. The constant failure rates are as follows:

subsystem	Failure rate per hour
Receiver	$30 \times 10^{-7}$
Amplifier	
Power Source	

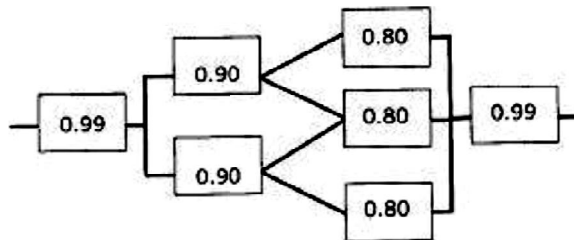
Calculate the system reliability for the mission time.

<sup>1</sup> Kapur & Lamberson(1977) page 72

**14.5** Calculate the reliability of a system with the following RBD; B is a backup for A.



**14.6** The lifetime of a kind of electrical resistance is exponentially distributed. Design an acceptance sampling plan with the test time of around 100 hours for inspecting lots containing this product in such a way that if the mean lifetime of the product is 220 hours, the lots acceptance probability should be 1% and if this mean is 1000 hours the probability would be 99%. **14.7** Calculate the reliability of the a system with the following RBD:



**14.7** In a 5-component system, if each of the components, having the reliability of 0.85 fails, the system will fail. Calculate the reliability of the system.

**14.8** Design a life-based plan for inspecting lots containing products with exponentially distributed lifetime, in such a way that if the mean life is 500 hours, 95% of the lot would be accepted; and if the mean is 250 hours, 90% of the lots would be rejected.

**14.9** Generate 10 random numbers from an exponential distribution with  $\lambda = 0.001$ , using a software such as MATLAB. Apply the hypothesis test stated in Appendix B of Chapter 4 (with  $r=n$ ) with  $\alpha = 1\%$ , to verify the mean of the exponential distribution is 1000.

**14.10** In a "2 out of 3" system, the constant failure rate of the individual components is 0.0002 per hour (2 failures per 10,000 hours). The mean of the down time for each is 10 hours. Calculate the long-term availability of the system.

**14.11** In three consecutive 2000-hour interval, a system has had 250, 200, and 250 failures respectively. With 90% level of confidence, could we consider the distribution of the system lifetime exponential? Apply a goodness of fit test.

**14.12**

The lifetime of a device is exponentially distributed with mean of 5000 hours. Find 10000-hr reliability of the device. Repeat the solution for a roughly normal distribution with mean 5000 hr and standard deviation 1500 hr. What would be the reliability if the lifetime had a Weibull distribution with  $A=200$ ,  $B=5000$  and  $C=1.5$ .

**14.13**

Write a MATLAB code for calculating the parameters  $A$ ,  $B$ ,  $C$ , of a Weibull from maximum likelihood estimation equations:

$$\begin{cases} B^C = - \sum_{i=1}^n \left( \frac{A - x_i}{B} \right)^C \\ \frac{n}{C} + \sum_{i=1}^n \ln(x_i - A) + \sum_{i=1}^n \left( \frac{A - x_i}{B} \right)^C \ln(A - x_i) = 0 \\ \sum_{i=1}^n \frac{1-C}{x_i - A} + \frac{C}{B} \sum_{i=1}^n \left( \frac{A - x_i}{B} \right)^{C-1} = 0 \end{cases}$$

**Three sentences for getting success(WILLIAM SHEAKSPER)**

- 1: Know more than other**
- 2: Work more than other**
- 3: Expect less than other**

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available 14<sup>th</sup> Oct 2016

**Understanding the past  
Challenging the present  
Shaping the future**



## Tables

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**Table 1 Some Continuous Distributions (From Ross, 1983)**

Continuous Distribution	Density Function	Mean	Variance	$\varphi(t)$ MGF
Uniform in the interval (a, b)	$\frac{1}{b-a}, a < x < b$	$\frac{(a+b)}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$
Exponential with parameters $\lambda > 0$ or $\theta > 0$	$\lambda e^{-\lambda x}, x \geq 0$ or $\frac{1}{\theta} e^{-\frac{x}{\theta}}$	$\theta = \frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda - t}$
Gamma with parameters Integer n and $\lambda > 0$	$\frac{\lambda e^{-\lambda x} (\lambda x)^{n-1}}{(n-1)!}, x \geq 0$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$	$\left(\frac{\lambda}{\lambda - t}\right)^n$
Normal with parameters $(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} -\infty < x < \infty$	$\mu$	$\sigma^2$	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$
Beta with parameters a>0 & b>0	$c x^{a-1} (1-x)^{b-1}, 0 < x < 1 \quad c = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$	$\frac{a}{a+b}$	$\frac{ab}{(a+b)^2(a+b+1)}$	

**Table 1 Continuous Distributions ( continued)**

Continuous Distribution	Density Function	Cumulative Function	Mean	Variance
Weibull	$\frac{C}{B} \left(\frac{x-A}{B}\right)^{C-1} e^{-\left(\frac{x-A}{B}\right)^C} \quad x \geq A$	$1 - e^{-\left(\frac{x-A}{B}\right)^C} \quad x \geq A$	$A + B\Gamma\left(1 + \frac{1}{C}\right)$	$B^2\Gamma\left(1 + \frac{2}{C}\right) + B^2[\Gamma\left(1 + \frac{1}{C}\right)]$
GPD	$\frac{1}{B} (1 + C \frac{x-A}{B})^{-\frac{1}{C}} \text{ valid on } A < x < \infty \text{ when } C \geq 0 \text{ or } A < x < A - \frac{B}{C} \text{ when } C < 0$	$1 - (1 + C \frac{x-A}{B})^{-\frac{1}{C}} \text{ valid on } A < x < \infty \text{ when } C \geq 0 \text{ or } A < x < A - \frac{B}{C} \text{ when } C < 0$	$A + \frac{B}{1-C}$	
GEV	$\begin{cases} 0 & x < A + \frac{B}{C} \text{ \& } C < 0 \\ (1 - C \frac{x-A}{B})^{\frac{1}{C}-1} \times e^{-(1 - C \frac{x-A}{B})^{\frac{1}{C}}} & \text{for } \begin{cases} x \geq A + \frac{B}{C} \text{ \& } C < 0 \\ x \leq A + \frac{B}{C} \text{ \& } C > 0 \end{cases} \\ 1 & x > A + \frac{B}{C} \text{ \& } C > 0 \end{cases}$	$\begin{cases} 0 & x < A + \frac{B}{C} \text{ \& } C < 0 \\ e^{-(1 - C \frac{x-A}{B})^{\frac{1}{C}}} & \text{for } \begin{cases} x \geq A + \frac{B}{C} \text{ \& } C < 0 \\ x \leq A + \frac{B}{C} \text{ \& } C > 0 \end{cases} \\ 1 & x > A + \frac{B}{C} \text{ \& } C > 0 \end{cases}$		
FT1	$\frac{1}{B} \exp\left(-\frac{x-A}{B} - e^{-\frac{x-A}{B}}\right)$	$e^{-e^{-\frac{x-A}{B}}}$	$A + \gamma B$ $\gamma = 0.05772$	$\frac{\pi^2 B^2}{6}$
In Weibul Distribution $A=0 \& C=1 \rightarrow \text{Exponential}$ $A=0 \rightarrow 2\text{-parameter Weibull}$				
In GPD $A=0 \& C=0 \rightarrow \text{Exponential Distribution}$				

**Table 2 Some Discrete Distributions<sup>1</sup>**

Distribution	p(x) Probability function	MGF $\phi(t)$	z-transform	Mean	Var
Binomial with parameters $n, 0 < p < 1$	$\binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, \dots, n$	$[pe^t + (1-p)]^n$	$[pz + (1-p)]^n$	$np$	$np(1-p)$
Poisson with parameters $\lambda > 0$	$e^{-\lambda} \frac{\lambda^x}{x!}, x = 0, 1, 2, \dots$	$\exp[\lambda(e^t - 1)]$	$e^{\lambda(z-1)}$	$\lambda$	$\lambda$
Geometric for success with parameter $p, 0 < p < 1$	$p(1-p)^{x-1}, x = 1, 2, \dots$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{pz}{1-(1-p)z}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Geometric for failures with parameter <sup>2</sup> $p, 0 < p < 1$	$p(1-p)^x, x = 0, 1, 2, \dots$	$\frac{p}{1-(1-p)e^t}, t < -\ln(1-p)$	$\frac{p}{1-(1-p)z}$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$
Negative Binomial with parameters $r, p$	$\binom{x-1}{r-1} p^r (1-p)^{x-r}, x = r, r+1, \dots$	$\left[ \frac{pe^t}{1-(1-p)e^t} \right]^r$	$\left[ \frac{pz}{1-(1-p)z} \right]^r$	$\frac{r(1-p)}{p^2}$	$\frac{r}{p}$

**Table 3 MATLAB Commands for Distributions<sup>3</sup>**

Distribution	Density Function	Cumulative Function	Inverse Function	Random Number Generator	Parameter estimator
Beta	betapdf(x,A,B)	betacdf(x,A,B)	betainv(P,A,B)	betarnd(A,B,m,n,o,...)	betafit(X)
Bionomial	binopdf(x,N,P)	binocdf(x,N,P)	binoinv(Y,N,P)	binornd(N,P,m,n)	binofit(X,n)
Chi square	chi2pdf(x,V)	chi2cdf(x,V)	chi2inv(P,V)	chi2rnd(V,m,n)	
Exponential	expdpdf(x, mu)	expcdf(x, mu)	expinv(P,mu)	exprnd(mu,m,n)	expfit(X)
F	fpdf(x, V1,V2)	fcdf(x, V1,V2)	finv(P, V1,V2)	frnd(V1,V2,m,n)	
Gamma	gampdf(x,A,B)	gamcdf(x,A,B)	gaminv(P,A,B)	gamrnd(A,B,m,n)	gamfit(X)
Geometric	geopdf(x,P)	geocdf(x,P)	geoinv(Y,P)	geornd(P,m,n)	
Hyper Geometric	hygepdf(x,M,K,N)	hygecdf(x,M,K,N)	hygeinv(P,M,K,N)	hygernd(M,K,N,m,n)	
Normal	normpdf(x, , $\sigma$ )	normcdf(x, , $\sigma$ )	norminv(P, , $\sigma$ )	normrnd( , $\sigma$ ,m,n)	normfit(X)
Log Normal	lognpdf(x, , $\sigma$ )	logncdf(x, , $\sigma$ )	logninv(P, , $\sigma$ )	lognrnd( , $\sigma$ ,m,n)	lognfit(X)
Negative Bionomial	nbinpdf(x,R,P)	nbincdf(x,R,P)	nbininv(Y,R,P)	nbinrnd(R,P,m,n)	nbinfit(X)
Poisson	poisspdf(x, $\lambda$ )	poisscdf(x, $\lambda$ )	poissinv(P, $\lambda$ )	poissrnd( $\lambda$ ,m,n)	poissfit(X)
Rayleigh	raylpdf(x,B)	raylcdf(x,B)	raylinv(P,B)	raylrnd(B,m,n)	raylfit(X)
t	tpdf(x,V)	tcdf(x,V)	tinvs(P,V)	trnd(V,m,n)	
Uniform	unifpdf(x,A,B)	unifcdf(x,A,B)	unifinv(P,A,B)	unifrnd(A,B,m,n)	unifit(X)
Weibull	wblpdf(x, B,C)	wblcdf(x, B,C)	wblinv(P, B,C)	wblrnd(B,C,m,n)	wblfit(X)
GEV	gevpdf(C,B,A)	gevcdf(x,C,B,A)	gevinv (P,C,B,A)	gevrnd(C,B,A)	gevfit(X)
GPD	gppdf	gpcdf	gpinv	gprnd	gpfit

<sup>1</sup> Extracted from Ross (1983).<sup>2</sup> Geometric for success: x= number of trials until the 1st success      Geometric for failures: x= number of failures before the 1st success<sup>3</sup> With thanks to Mr. Mohsen Abbar, former student of college of engineering of Shahid Bahonar University of Kerman, Iran.

# Table 4 Poisson Distribution-Cumulative Probability<sup>1</sup>

Example :  $\lambda=0.2$     $\Pr(X \leq 1)=0.988$

$\lambda$ $\frac{\lambda}{np}$	$\Pr(X \leq x)$ <b>Poisson CDF</b>														
	$k$														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0.01	0.990	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.02	0.980	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.03	0.970	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.04	0.961	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.05	0.951	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.06	0.942	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.07	0.932	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.08	0.923	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.09	0.914	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.10	0.905	0.995	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.11	0.896	0.994	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.12	0.887	0.993	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.13	0.878	0.992	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.14	0.869	0.991	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.15	0.861	0.990	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.16	0.852	0.988	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.17	0.844	0.987	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.18	0.835	0.986	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.19	0.827	0.984	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.20	0.819	0.982	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

1- Reproduced by MATLAB from HHousyar<1985



**Table 4 Poisson Distribution-Cumulative Probability (Continued)**

$\lambda$ (np)	Pr( $X \leq x$ ) Poisson CDF (continued)														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0.25	0.779	0.974	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.30	0.741	0.963	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.35	0.705	0.951	0.994	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.40	0.670	0.938	0.992	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.45	0.638	0.925	0.989	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.50	0.607	0.910	0.986	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.55	0.577	0.894	0.982	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.60	0.549	0.878	0.977	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.65	0.522	0.861	0.972	0.996	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.70	0.497	0.844	0.966	0.994	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.75	0.472	0.827	0.959	0.993	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.80	0.449	0.809	0.953	0.991	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.85	0.427	0.791	0.945	0.989	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	0.407	0.772	0.937	0.987	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.95	0.387	0.754	0.929	0.984	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.00	0.368	0.736	0.920	0.981	0.996	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.10	0.333	0.699	0.900	0.974	0.995	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.20	0.301	0.663	0.879	0.966	0.992	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.30	0.273	0.627	0.857	0.957	0.989	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.40	0.247	0.592	0.833	0.946	0.986	0.997	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.50	0.223	0.558	0.809	0.934	0.981	0.996	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.60	0.202	0.525	0.783	0.921	0.976	0.994	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.70	0.183	0.493	0.757	0.907	0.970	0.992	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

**Table 4 Poisson Distribution-Cumulative Probability (Continued)**

$\lambda$	$\Pr(X \leq x)$ Poisson CDF (continued)														
(np)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1.80	0.165	0.463	0.731	0.891	0.964	0.990	0.997	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.90	0.150	0.434	0.704	0.875	0.956	0.987	0.997	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2.00	0.135	0.406	0.677	0.857	0.947	0.983	0.995	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2.10	0.122	0.380	0.650	0.839	0.938	0.980	0.994	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2.20	0.111	0.355	0.623	0.819	0.928	0.975	0.993	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2.30	0.100	0.331	0.596	0.799	0.916	0.970	0.991	0.997	0.999	1.000	1.000	1.000	1.000	1.000	1.000
2.40	0.091	0.308	0.570	0.779	0.904	0.964	0.988	0.997	0.999	1.000	1.000	1.000	1.000	1.000	1.000
2.50	0.082	0.287	0.544	0.758	0.891	0.958	0.986	0.996	0.999	1.000	1.000	1.000	1.000	1.000	1.000
2.60	0.074	0.267	0.518	0.736	0.877	0.951	0.983	0.995	0.999	1.000	1.000	1.000	1.000	1.000	1.000
2.70	0.067	0.249	0.494	0.714	0.863	0.943	0.979	0.993	0.998	0.999	1.000	1.000	1.000	1.000	1.000
2.80	0.061	0.231	0.469	0.692	0.848	0.935	0.976	0.992	0.998	0.999	1.000	1.000	1.000	1.000	1.000
2.90	0.055	0.215	0.446	0.670	0.832	0.926	0.971	0.990	0.997	0.999	1.000	1.000	1.000	1.000	1.000
3.00	0.050	0.199	0.423	0.647	0.815	0.916	0.966	0.988	0.996	0.999	1.000	1.000	1.000	1.000	1.000
3.10	0.045	0.185	0.401	0.625	0.798	0.906	0.961	0.986	0.995	0.999	1.000	1.000	1.000	1.000	1.000
3.20	0.041	0.171	0.380	0.603	0.781	0.895	0.955	0.983	0.994	0.998	1.000	1.000	1.000	1.000	1.000
3.30	0.037	0.159	0.359	0.580	0.763	0.883	0.949	0.980	0.993	0.998	1.000	1.000	1.000	1.000	1.000
3.40	0.033	0.147	0.340	0.558	0.744	0.871	0.942	0.977	0.992	0.997	1.000	1.000	1.000	1.000	1.000
3.50	0.030	0.136	0.321	0.537	0.725	0.858	0.935	0.973	0.990	0.997	1.000	1.000	1.000	1.000	1.000
3.60	0.027	0.126	0.303	0.515	0.706	0.844	0.927	0.969	0.988	0.996	1.000	1.000	1.000	1.000	1.000
3.70	0.025	0.116	0.285	0.494	0.687	0.830	0.918	0.965	0.986	0.995	1.000	1.000	1.000	1.000	1.000
3.80	0.022	0.107	0.269	0.473	0.668	0.816	0.909	0.960	0.984	0.994	0.998	0.999	1.000	1.000	1.000
3.90	0.020	0.099	0.253	0.453	0.648	0.801	0.899	0.955	0.981	0.993	0.998	0.999	1.000	1.000	1.000
4.00	0.018	0.092	0.238	0.433	0.629	0.785	0.889	0.949	0.979	0.992	0.997	0.999	1.000	1.000	1.000

**Table 4 Poisson Distribution-Cumulative Probability (Continued)**

$\lambda$	$\Pr(X \leq x)$ Poisson CDF (continued)														
(np)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
4.10	0.017	0.085	0.224	0.414	0.609	0.769	0.879	0.943	0.976	0.990	1.000	0.999	1.000	1.000	1.000
4.20	0.015	0.078	0.210	0.395	0.590	0.753	0.867	0.936	0.972	0.989	1.000	0.999	1.000	1.000	1.000
4.30	0.014	0.072	0.197	0.377	0.570	0.737	0.856	0.929	0.968	0.987	1.000	0.998	0.999	1.000	1.000
4.40	0.012	0.066	0.185	0.359	0.551	0.720	0.844	0.921	0.964	0.985	0.990	0.998	0.999	1.000	1.000
4.50	0.011	0.061	0.174	0.342	0.532	0.703	0.831	0.913	0.960	0.983	0.990	0.998	0.999	1.000	1.000
4.60	0.010	0.056	0.163	0.326	0.513	0.686	0.818	0.905	0.955	0.980	0.990	0.997	0.999	1.000	1.000
4.70	0.009	0.052	0.152	0.310	0.495	0.668	0.805	0.896	0.950	0.978	0.990	0.997	0.999	1.000	1.000
4.80	0.008	0.048	0.143	0.294	0.476	0.651	0.791	0.887	0.944	0.975	0.990	0.996	0.999	1.000	1.000
4.90	0.007	0.044	0.133	0.279	0.458	0.634	0.777	0.877	0.938	0.972	0.990	0.995	0.998	0.999	1.000
5.00	0.007	0.040	0.125	0.265	0.440	0.616	0.762	0.867	0.932	0.968	0.990	0.995	0.998	0.999	1.000
5.20	0.006	0.034	0.109	0.238	0.406	0.581	0.732	0.845	0.918	0.960	0.980	0.993	0.997	0.999	1.000
5.40	0.005	0.029	0.095	0.213	0.373	0.546	0.702	0.822	0.903	0.951	0.980	0.990	0.996	0.999	1.000
5.60	0.004	0.024	0.082	0.191	0.342	0.512	0.670	0.797	0.886	0.941	0.970	0.988	0.995	0.998	0.999
5.80	0.003	0.021	0.072	0.170	0.313	0.478	0.638	0.771	0.867	0.929	0.970	0.984	0.993	0.997	0.999
6.00	0.002	0.017	0.062	0.151	0.285	0.446	0.606	0.744	0.847	0.916	0.960	0.980	0.991	0.996	0.999
6.20	0.002	0.015	0.054	0.134	0.259	0.414	0.574	0.716	0.826	0.902	0.950	0.975	0.989	0.995	0.998
6.40	0.002	0.012	0.046	0.119	0.235	0.384	0.542	0.687	0.803	0.886	0.940	0.969	0.986	0.994	0.997
6.60	0.001	0.010	0.040	0.105	0.213	0.355	0.511	0.658	0.780	0.869	0.930	0.963	0.982	0.992	0.997
6.80	0.001	0.009	0.034	0.093	0.192	0.327	0.480	0.628	0.755	0.850	0.920	0.955	0.978	0.990	0.996
7.00	0.001	0.007	0.030	0.082	0.173	0.301	0.450	0.599	0.729	0.830	0.900	0.947	0.973	0.987	0.994
7.20	0.001	0.006	0.025	0.072	0.156	0.276	0.420	0.569	0.703	0.810	0.890	0.937	0.967	0.984	0.993
7.40	0.001	0.005	0.022	0.063	0.140	0.253	0.392	0.539	0.676	0.788	0.870	0.926	0.961	0.980	0.991
7.60	0.001	0.004	0.019	0.055	0.125	0.231	0.365	0.510	0.648	0.765	0.850	0.915	0.954	0.976	0.989

**Table 4 Poisson Distribution-Cumulative Probability (Continued)**

$\lambda$ (np)	Pr( $X \leq x$ ) Poisson CDF (continued)														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
7.80	0.000	0.004	0.016	0.048	0.112	0.210	0.338	0.481	0.620	0.741	0.840	0.902	0.945	0.971	0.986
8.00	0.000	0.003	0.014	0.042	0.100	0.191	0.313	0.453	0.593	0.717	0.820	0.888	0.936	0.966	0.983
8.20	0.000	0.003	0.012	0.037	0.089	0.174	0.290	0.425	0.565	0.692	0.800	0.873	0.926	0.960	0.979
8.40	0.000	0.002	0.010	0.032	0.079	0.157	0.267	0.399	0.537	0.666	0.770	0.857	0.915	0.952	0.975
8.60	0.000	0.002	0.009	0.028	0.070	0.142	0.246	0.373	0.509	0.640	0.750	0.840	0.903	0.945	0.970
8.80	0.000	0.001	0.007	0.024	0.062	0.128	0.226	0.348	0.482	0.614	0.730	0.822	0.890	0.936	0.965
9.00	0.000	0.001	0.006	0.021	0.055	0.116	0.207	0.324	0.456	0.587	0.710	0.803	0.876	0.926	0.959
9.20	0.000	0.001	0.005	0.018	0.049	0.104	0.189	0.301	0.430	0.561	0.680	0.783	0.861	0.916	0.952
9.40	0.000	0.001	0.005	0.016	0.043	0.093	0.173	0.279	0.404	0.535	0.660	0.763	0.845	0.904	0.944
9.60	0.000	0.001	0.004	0.014	0.038	0.084	0.157	0.258	0.380	0.509	0.630	0.741	0.828	0.892	0.936
9.80	0.000	0.001	0.003	0.012	0.033	0.075	0.143	0.239	0.356	0.483	0.610	0.719	0.810	0.879	0.927
10.00	0.000	0.000	0.003	0.010	0.029	0.067	0.130	0.220	0.333	0.458	0.580	0.697	0.792	0.864	0.917
10.50	0.000	0.000	0.002	0.007	0.021	0.050	0.102	0.179	0.279	0.397	0.520	0.639	0.742	0.825	0.888
11.00	0.000	0.000	0.001	0.005	0.015	0.038	0.079	0.143	0.232	0.341	0.460	0.579	0.689	0.781	0.854
11.50	0.000	0.000	0.001	0.003	0.011	0.028	0.060	0.114	0.191	0.289	0.400	0.520	0.633	0.733	0.815
12.00	0.000	0.000	0.001	0.002	0.008	0.020	0.046	0.090	0.155	0.242	0.350	0.462	0.576	0.682	0.772
12.50	0.000	0.000	0.000	0.002	0.005	0.015	0.035	0.070	0.125	0.201	0.300	0.406	0.519	0.628	0.725
13.00	0.000	0.000	0.000	0.001	0.004	0.011	0.026	0.054	0.100	0.166	0.250	0.353	0.463	0.573	0.675
13.50	0.000	0.000	0.000	0.001	0.003	0.008	0.019	0.041	0.079	0.135	0.210	0.304	0.409	0.518	0.623
14.00	0.000	0.000	0.000	0.000	0.002	0.006	0.014	0.032	0.062	0.109	0.180	0.260	0.358	0.464	0.570
14.50	0.000	0.000	0.000	0.000	0.001	0.004	0.010	0.024	0.048	0.088	0.140	0.220	0.311	0.413	0.518
15.00	0.000	0.000	0.000	0.000	0.001	0.003	0.008	0.018	0.037	0.070	0.120	0.185	0.268	0.363	0.466

**Table 4 Poisson Distribution-Cumulative Probability (Continued)**

Pr( $X \leq x$ ) Poisson CDF (continued)																
$\lambda$	(np)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
15.50	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.006	0.013	0.029	0.055	0.100	0.154	0.228	0.317	0.415
16.00	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.004	0.010	0.022	0.043	0.080	0.127	0.193	0.275	0.368
16.50	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.003	0.007	0.017	0.034	0.060	0.104	0.162	0.236	0.323
17.00	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.005	0.013	0.026	0.050	0.085	0.135	0.201	0.281
17.50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.004	0.009	0.020	0.040	0.068	0.112	0.170	0.243
18.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.003	0.007	0.015	0.030	0.055	0.092	0.143	0.208
18.50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.005	0.012	0.020	0.044	0.075	0.119	0.177
19.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.004	0.009	0.020	0.035	0.061	0.098	0.150
19.50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.003	0.007	0.010	0.027	0.049	0.081	0.126
20.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.005	0.010	0.021	0.039	0.066	0.105
20.50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.004	0.010	0.017	0.031	0.054	0.087
21.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.003	0.010	0.013	0.025	0.043	0.072
21.50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.005	0.010	0.019	0.035	0.059
22.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.004	0.008	0.015	0.028	0.048
22.50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.003	0.006	0.012	0.022	0.039	0.063
23.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.004	0.009	0.017	0.031
23.50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.003	0.007	0.014	0.025
24.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.003	0.005	0.011	0.020
24.50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.004	0.008	0.016

**Table 5** Standard Normal CDFExample:  $\Pr(Z \leq -3.09) = 0.001$ 

$z = (x - \mu) / \sigma$	CDF values of Normal distribution , $\Pr(Z \leq z)$									
	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00
-3.5	0.00017	0.00017	0.00018	0.00019	0.00019	0.0002	0.00021	0.00022	0.00022	0.00023
-3.4	0.00024	0.00025	0.00026	0.00027	0.00028	0.00029	0.0003	0.00031	0.00032	0.00034
-3.3	0.00035	0.00036	0.00038	0.00039	0.0004	0.00042	0.00043	0.00045	0.00047	0.00048
-3.2	0.0005	0.00052	0.00054	0.00056	0.00058	0.0006	0.00062	0.00064	0.00066	0.00069
-3.1	0.00071	0.00074	0.00076	0.00079	0.00082	0.00084	0.00087	0.0009	0.00094	0.00097
-3	0.001	0.00104	0.00107	0.00111	0.00114	0.00118	0.00122	0.00126	0.00131	0.00135
-2.9	0.00139	0.00144	0.00149	0.00154	0.00159	0.00164	0.00169	0.00175	0.00181	0.00187
-2.8	0.00193	0.00199	0.00205	0.00212	0.00219	0.00226	0.00233	0.0024	0.00248	0.00256
-2.7	0.00264	0.00272	0.0028	0.00289	0.00298	0.00307	0.00317	0.00326	0.00336	0.00347
-2.6	0.00357	0.00368	0.00379	0.00391	0.00402	0.00415	0.00427	0.0044	0.00453	0.00466
-2.5	0.0048	0.00494	0.00508	0.00523	0.00539	0.00554	0.0057	0.00587	0.00604	0.00621
-2.4	0.00639	0.00657	0.00676	0.00695	0.00714	0.00734	0.00755	0.00776	0.00798	0.0082
-2.3	0.00842	0.00866	0.00889	0.00914	0.00939	0.00964	0.0099	0.01017	0.01044	0.01072
-2.2	0.01101	0.01130	0.0116	0.01191	0.01222	0.01255	0.01287	0.01321	0.01355	0.01390
-2.1	0.01426	0.01463	0.015	0.01539	0.01578	0.01618	0.01659	0.01700	0.01743	0.01786
-2	0.01831	0.01876	0.01923	0.0197	0.02018	0.02068	0.02118	0.02169	0.02222	0.02275
-1.9	0.0233	0.02385	0.02442	0.025	0.02559	0.02619	0.0268	0.02743	0.02807	0.02872
-1.8	0.02938	0.03005	0.03074	0.03144	0.03216	0.03288	0.03362	0.03438	0.03515	0.03593
-1.7	0.03673	0.03754	0.03836	0.0392	0.04006	0.04093	0.04182	0.04272	0.04363	0.04457

Extracted from Grant&amp;Leavenworth (1988) Reproduced By MATLAB

Table 5 Normal CDF -continued

 $\Pr(Z \leq z)$ 

$z = (x - \mu)/\sigma$	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00
-1.6	0.04551	0.04648	0.04746	0.04846	0.04947	0.0505	0.05155	0.05262	0.0537	0.0548
-1.5	0.05592	0.05705	0.05821	0.05938	0.06057	0.06178	0.06301	0.06426	0.06552	0.06681
-1.4	0.06811	0.06944	0.07078	0.07215	0.07353	0.07493	0.07636	0.0778	0.07927	0.08076
-1.3	0.08226	0.08379	0.08534	0.08691	0.08851	0.09012	0.09176	0.09342	0.0951	0.0968
-1.2	0.09853	0.10027	0.10204	0.10383	0.10565	0.10749	0.10935	0.11123	0.11314	0.11507
-1.1	0.11702	0.119	0.121	0.12302	0.12507	0.12714	0.12924	0.13136	0.1335	0.13567
-1	0.13786	0.14007	0.14231	0.14457	0.14686	0.14917	0.15151	0.15386	0.15625	0.15866
-0.9	0.16109	0.16354	0.16602	0.16853	0.17106	0.17361	0.17619	0.17879	0.18141	0.18406
-0.8	0.18673	0.18943	0.19215	0.19489	0.19766	0.20045	0.20327	0.20611	0.20897	0.21186
-0.7	0.21476	0.2177	0.22065	0.22363	0.22663	0.22965	0.2327	0.23576	0.23885	0.24196
-0.6	0.2451	0.24825	0.25143	0.25463	0.25785	0.26109	0.26435	0.26763	0.27093	0.27425
-0.5	0.2776	0.28096	0.28434	0.28774	0.29116	0.2946	0.29806	0.30153	0.30503	0.30854
-0.4	0.31207	0.31561	0.31918	0.32276	0.32636	0.32997	0.3336	0.33724	0.3409	0.34458
-0.3	0.34827	0.35197	0.35569	0.35942	0.36317	0.36693	0.3707	0.37448	0.37828	0.38209
-0.2	0.38591	0.38974	0.39358	0.39743	0.40129	0.40517	0.40905	0.41294	0.41683	0.42074
-0.1	0.42465	0.42858	0.43251	0.43644	0.44038	0.44433	0.44828	0.45224	0.4562	0.46017
0	0.46414	0.46812	0.4721	0.47608	0.48006	0.48405	0.48803	0.49202	0.49601	0.5

Example  $\Pr(z < -1.06) = 0.14457$

Table 5 Normal CDF-continued

$z = (x - \mu) / \sigma$	$\Pr(Z \leq z)$									
	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
<b>0</b>	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
<b>0.1</b>	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
<b>0.2</b>	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
<b>0.3</b>	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
<b>0.4</b>	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
<b>0.5</b>	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
<b>0.6</b>	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
<b>0.7</b>	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
<b>0.8</b>	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
<b>0.9</b>	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
<b>1</b>	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
<b>1.1</b>	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
<b>1.2</b>	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
<b>1.3</b>	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
<b>1.4</b>	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
<b>1.5</b>	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
<b>1.6</b>	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
<b>1.7</b>	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
<b>1.8</b>	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062



Table 5 Standard Normal CDF-continued

 $\Pr(Z \leq z)$ 

$z = (x - \mu)/\sigma$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999
3.1	0.99903	0.99906	0.9991	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.9994	0.99942	0.99944	0.99946	0.99948	0.9995
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.9996	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.9997	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.9998	0.99981	0.99981	0.99982	0.99983	0.99983

Example  $\Pr(z < 3.44) = 0.99971$

Table 6 Critical Values of Standard Normal Distribution

Example : .  $\alpha=0.05$   $Z_{\alpha}=1.64$   $Z_{\frac{\alpha}{2}}=1.96$   $\alpha=0.05$

$Z_{\alpha}$	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5	0.49601	0.49202	0.48803	0.48405	0.48006	0.47608	0.4721	0.46812	0.46414
0.1	0.46017	0.4562	0.45224	0.44828	0.44433	0.44038	0.43644	0.43251	0.42858	0.42465
0.2	0.42074	0.41683	0.41294	0.40905	0.40517	0.40129	0.39743	0.39358	0.38974	0.38591
0.3	0.38209	0.37828	0.37448	0.3707	0.36693	0.36317	0.35942	0.35569	0.35197	0.34827
0.4	0.34458	0.3409	0.33724	0.3336	0.32997	0.32636	0.32276	0.31918	0.31561	0.31207
0.5	0.30854	0.30503	0.30153	0.29806	0.2946	0.29116	0.28774	0.28434	0.28096	0.2776
0.6	0.27425	0.27093	0.26763	0.26435	0.26109	0.25785	0.25463	0.25143	0.24825	0.2451
0.7	0.24196	0.23885	0.23576	0.2327	0.22965	0.22663	0.22363	0.22065	0.2177	0.21476
0.8	0.21186	0.20897	0.20611	0.20327	0.20045	0.19766	0.19489	0.19215	0.18943	0.18673
0.9	0.18406	0.18141	0.17879	0.17619	0.17361	0.17106	0.16853	0.16602	0.16354	0.16109
1	0.15866	0.15625	0.15386	0.15151	0.14917	0.14686	0.14457	0.14231	0.14007	0.13786
1.1	0.13567	0.1335	0.13136	0.12924	0.12714	0.12507	0.12302	0.121	0.119	0.11702
1.2	0.11507	0.11314	0.11123	0.10935	0.10749	0.10565	0.10383	0.10204	0.10027	0.09853
1.3	0.0968	0.0951	0.09342	0.09176	0.09012	0.08851	0.08691	0.08534	0.08379	0.08226
1.4	0.08076	0.07927	0.0778	0.07636	0.07493	0.07353	0.07215	0.07078	0.06944	0.06811
1.5	0.06681	0.06552	0.06426	0.06301	0.06178	0.06057	0.05938	0.05821	0.05705	0.05592
1.6	0.0548	0.0537	0.05262	0.05155	0.0505	0.04947	0.04846	0.04746	0.04648	0.04551
1.7	0.04457	0.04363	0.04272	0.04182	0.04093	0.04006	0.0392	0.03836	0.03754	0.03673

Table 6 continued

Example  $\alpha=0.01$   $Z\alpha=1.33$ 

$Z_\alpha$	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.8	0.03593	0.03515	0.03438	0.03362	0.03288	0.03216	0.03144	0.03074	0.03005	0.02938
1.9	0.02872	0.02807	0.02743	0.0268	0.02619	0.02559	0.025	0.02442	0.02385	0.0233
2	0.02275	0.02222	0.02169	0.02118	0.02068	0.02018	0.0197	0.01923	0.01876	0.01831
2.1	0.01786	0.01743	0.017	0.01659	0.01618	0.01578	0.01539	0.015	0.01463	0.01426
2.2	0.0139	0.01355	0.01321	0.01287	0.01255	0.01222	0.01191	0.0116	0.0113	0.01101
2.3	0.01072	0.01044	0.01017	0.0099	0.00964	0.00939	0.00914	0.00889	0.00866	0.00842
2.4	0.0082	0.00798	0.00776	0.00755	0.00734	0.00714	0.00695	0.00676	0.00657	0.00639
2.5	0.00621	0.00604	0.00587	0.0057	0.00554	0.00539	0.00523	0.00508	0.00494	0.0048
2.6	0.00466	0.00453	0.0044	0.00427	0.00415	0.00402	0.00391	0.00379	0.00368	0.00357
2.7	0.00347	0.00336	0.00326	0.00317	0.00307	0.00298	0.00289	0.0028	0.00272	0.00264
2.8	0.00256	0.00248	0.0024	0.00233	0.00226	0.00219	0.00212	0.00205	0.00199	0.00193
2.9	0.00187	0.00181	0.00175	0.00169	0.00164	0.00159	0.00154	0.00149	0.00144	0.00139
3	0.00135	0.00131	0.00126	0.00122	0.00118	0.00114	0.00111	0.00107	0.00104	0.001

Table 7 Critical values of t Distribution

Example :  $n=3$   $\alpha=0.01$  ,  $t(\alpha,n)=1.33$ 

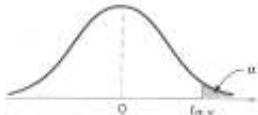
									
$\alpha \backslash v$	0.25	0.10	0.05	0.025	0.01	0.005	0.0025	0.001	0.0005
1	1.000	3.078	6.314	12.706	31.821	63.657	127.32	318.31	636.62
2	.816	1.886	2.920	4.303	6.965	9.925	14.089	22.327	31.598
3	.765	1.638	2.353	3.182	4.541	5.841	7.453	10.213	12.924
4	.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	.727	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	.718	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	.711	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	.700	1.372	1.813	2.228	2.764	3.169	3.581	4.144	4.587
11	.697	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	.694	1.350	1.771	2.160	2.650	3.012	3.373	3.852	4.221
14	.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883

Table 7 Critical values of t distribution- continued


Example

n=30  $\alpha=0.001$ ,  $t_{(\alpha,n)}=3.385$ 

$\alpha \backslash v$	0.25	0.10	0.05	0.025	0.01	0.005	0.0025	0.001	0.0005
20	.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.767
24	.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	.683	1.312	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	.683	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	.681	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
50	.680	1.300	1.676	2.009	2.403	2.678	2.937	3.261	3.496
60	.679	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
80	.678	1.292	1.6264	1.990	2.384	2.639	2.887	3.195	3.416
100	.677	1.290	1.660	1.984	2.364	2.626	2.871	3.174	3.390
120	.677	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
$\infty$	.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

1reproduced from Bowker &amp; Lieberman, (1974)-using MATLAB

Table 8 Critical values of chi square distribution

Example :  $v=3$   $\alpha=0.01$  ,  $\chi^2(\alpha,v)=16.27$ 


$\alpha \backslash v$	$\alpha$											
	.001	.005	.010	.025	.050	.100	.900	.950	.975	.990	.995	.999
1	1.83	7.88	6.63	5.02	3.84	2.71	0.02	0.00	0.00	0.00	0.00	0.00
2	13.82	10.60	9.21	7.38	5.99	4.61	0.21	0.10	0.05	0.02	0.01	0.00
3	16.27	12.84	11.34	9.35	7.81	6.25	0.58	0.35	0.22	0.11	0.07	0.02
4	18.47	14.86	13.28	11.14	9.49	7.78	1.06	0.71	0.48	0.30	0.21	0.09
5	20.52	16.75	15.09	12.83	11.07	9.24	1.61	1.15	0.83	0.55	0.41	0.21
6	22.46	18.55	16.81	14.45	12.59	10.64	2.20	1.64	1.24	0.87	0.68	0.38
7	24.32	20.28	18.48	16.01	14.07	12.02	2.83	2.17	1.69	1.24	0.99	0.60
8	26.13	21.95	20.09	17.53	15.51	13.36	3.49	2.73	2.18	1.65	1.34	0.86
9	27.88	23.59	21.67	19.02	16.92	14.68	4.17	3.33	2.70	2.09	1.73	1.15
10	29.59	25.19	23.21	20.48	18.31	15.99	4.87	3.94	3.25	2.56	2.16	1.48
11	31.26	26.76	24.72	21.92	19.68	17.28	5.58	4.57	3.82	3.05	2.60	1.83
12	32.91	28.30	26.22	23.34	21.03	18.55	6.30	5.23	4.40	3.57	3.07	2.21
13	34.53	29.82	27.69	24.74	22.36	19.81	7.04	5.89	5.01	4.11	3.57	2.62
14	36.12	31.32	29.14	26.12	23.68	21.06	7.79	6.57	5.63	4.66	4.07	3.04
15	37.70	32.80	30.58	27.49	25.00	22.31	8.55	7.26	6.26	5.23	4.60	3.48
16	39.25	34.27	32.00	28.85	26.30	23.54	9.31	7.96	6.91	5.81	5.14	3.94

Table 8 Critical values of  $\chi^2$  distribution- continuedExample :  $v=30$   $\alpha=0.001$  ,  $\chi^2(\alpha,v)=59.70$ 

$\nu \backslash \alpha$	.001	.005	.010	.025	.050	.100	.900	.950	.975	.990	.995	.999
17	4.79	35.72	33.41	30.19	27.59	24.77	10.09	8.67	7.56	6.41	5.70	4.42
18	42.31	37.16	34.81	31.53	28.87	25.99	10.86	9.39	8.23	7.01	6.26	4.91
19	43.82	38.58	36.19	32.85	30.14	27.20	11.65	10.12	8.91	7.63	6.84	5.41
20	45.32	40.00	37.57	34.17	31.41	28.41	12.44	10.85	9.59	8.26	7.43	5.92
21	46.80	41.40	38.93	35.48	32.67	29.62	13.24	11.59	10.28	8.90	8.03	6.45
22	48.27	42.80	40.29	36.78	33.92	30.81	14.04	12.34	10.98	9.54	8.64	6.98
23	49.73	44.18	41.64	38.08	35.17	32.01	14.85	13.09	11.69	10.20	9.26	7.53
24	51.18	45.56	42.98	39.36	36.42	33.20	15.66	13.85	12.40	1.86	9.89	8.09
25	52.62	46.93	44.31	40.65	37.65	34.38	16.47	14.61	13.12	11.52	10.52	8.65
26	54.05	48.29	45.64	41.92	38.89	35.56	17.29	15.38	13.84	12.20	11.16	9.22
27	55.47	49.64	46.96	43.19	40.11	36.74	18.11	16.15	14.57	12.88	11.81	9.80
28	56.89	50.99	48.28	44.46	41.34	37.92	18.94	16.93	15.31	13.56	12.46	10.39
29	58.30	52.34	49.59	45.72	42.56	39.09	19.77	17.71	16.05	14.26	13.12	10.99
30	59.70	53.67	50.89	46.98	43.77	40.26	20.60	18.49	16.79	14.95	13.79	11.59
40	73.40	66.77	63.69	59.34	55.76	51.81	29.05	26.51	24.43	22.16	20.71	17.92
50	86.67	79.49	76.15	71.42	67.50	63.17	37.69	34.76	32.36	29.71	27.99	24.67
60	99.61	91.95	88.38	83.30	79.08	74.40	46.46	43.19	40.48	37.48	35.53	31.74
70	112.32	104.21	100.43	95.02	90.53	85.53	55.33	51.74	48.76	45.44	43.28	39.04
80	124.84	116.32	112.33	106.63	101.88	96.58	64.28	60.39	57.15	53.54	51.17	46.52
90	137.20	128.30	124.12	118.14	113.15	107.57	73.29	69.13	65.65	61.75	59.20	54.16
100	149.45	140.17	135.81	129.56	124.34	118.50	82.36	77.93	74.22	70.06	67.33	61.92

**.Table A**    **Example of Dodge and Romig Single Sampling Lot Inspection Tables**

**Based on Stated Value of LQL (LTPD) = 5.0 % and Consumer’s Risk =β= 0.10.**

LOT SIZE	Process Average (%)																	
	0 - 0.05			0.06 - 0.50			0.51 - 1.00			1.01 - 1.50			1.51 - 2.00			2.01 - 2.50		
	<i>n</i>	<i>c</i>	AOQL (%)	<i>n</i>	<i>c</i>	AOQL (%)	<i>n</i>	<i>c</i>	AOQL (%)	<i>n</i>	<i>c</i>	AOQL (%)	<i>n</i>	<i>c</i>	AOQL (%)	<i>n</i>	<i>c</i>	AOQL (%)
1 - 30	All	0	0.00	All	0	0.00	All	0	0.00	All	0	0.00	All	0	0.00	All	0	0.00
31 - 50	30	0	0.49	30	0	0.49	30	0	0.49	30	0	0.49	30	0	0.49	30	0	0.49
51 - 100	37	0	0.63	37	0	0.63	37	0	0.63	37	0	0.63	37	0	0.63	37	0	0.63
101 - 200	40	0	0.74	40	0	0.74	40	0	0.74	40	0	0.74	40	0	0.74	40	0	0.74
201 - 300	43	0	0.74	43	0	0.74	70	1	0.92	70	1	0.92	95	2	0.99	95	2	0.99
301 - 400	44	0	0.74	44	0	0.74	70	1	0.99	100	2	1.00	120	3	1.00	145	4	1.10
401 - 500	45	0	0.75	75	1	0.95	100	2	1.10	100	2	1.10	125	3	1.20	150	4	1.20
501 - 600	45	0	0.76	75	1	0.98	100	2	1.10	125	3	1.20	150	4	1.30	175	5	1.30
601 - 800	45	0	0.77	75	1	1.00	100	2	1.20	130	3	1.20	175	5	1.40	200	6	1.40
801 - 1000	45	0	0.78	75	1	1.00	105	2	1.20	155	4	1.40	180	5	1.40	225	7	1.50
1001 - 2000	45	0	0.80	75	1	1.00	130	3	1.40	180	5	1.60	230	7	1.70	280	9	1.80
2001 - 3000	75	1	1.10	105	2	1.30	135	3	1.40	210	6	1.70	280	9	1.90	370	13	2.10
3001 - 4000	75	1	1.10	105	2	1.30	160	4	1.50	210	6	1.70	305	10	2.00	420	15	2.20
4001 - 5000	75	1	1.10	105	2	1.30	160	4	1.50	235	7	1.80	330	11	2.00	440	16	2.20
5001 - 7000	75	1	1.10	105	2	1.30	185	5	1.70	260	8	1.90	350	12	2.20	490	18	2.40
7001 - 10000	75	1	1.10	105	2	1.30	185	5	1.70	260	8	1.90	380	13	2.20	535	20	2.50
10001 - 20000	75	1	1.10	135	3	1.40	210	6	1.80	285	9	2.00	425	15	2.30	610	23	2.60
20001 - 50000	75	1	1.10	135	3	1.40	235	7	1.90	305	10	2.10	470	17	2.40	700	27	2.70
50001 - 100000	75	1	1.10	160	4	1.60	235	7	1.90	355	12	2.20	515	19	2.50	770	30	2.80

.n: Size of Sample; entry of “All” indicates that each piece in lot is to be inspected:  
.c: Allowable Defect Number for Sample  
.AOQL : Average Outgoing Quality Limit

from H. F. Dodge and H. .,G. Romig, 1959,"Sampling Inspection Tables—Single and Double Sampling, " John Wiley & Sons, Inc., New York



**Table B**      Example of Dodge and Romig **Single** Sampling Lot Inspection Tables  
Based on Stated Value of LQL = 1.0 % and Consumer's Risk= $\beta$  = 0.10.

LOT SIZE	PROCESS AVERAGE (%)																	
	0 - 0.010			0.011 - 0.10			0.11 - 0.20			0.21 - 0.30			0.31 - 0.40			0.41 - 0.50		
	<i>n</i>	<i>c</i>	AOQL (%)	<i>n</i>	<i>c</i>	AOQL (%)	<i>n</i>	<i>c</i>	AOQL (%)	<i>n</i>	<i>c</i>	AOQL (%)	<i>n</i>	<i>c</i>	AOQL (%)	<i>n</i>	<i>c</i>	AOQL (%)
1 - 120	All	0	0.00	All	0	0.00	All	0	0.00	All	0	0.00	All	0	0.00	All	0	0.00
121 - 150	120	0	0.06	120	0	0.06	120	0	0.06	120	0	0.06	120	0	0.06	120	0	0.06
151 - 200	140	0	0.08	140	0	0.08	140	0	0.08	140	0	0.08	140	0	0.08	140	0	0.08
201 - 300	165	0	0.10	165	0	0.10	165	0	0.10	165	0	0.10	165	0	0.10	165	0	0.10
301 - 400	175	0	0.12	175	0	0.12	175	0	0.12	175	0	0.12	175	0	0.12	175	0	0.12
401 - 500	180	0	0.13	180	0	0.13	180	0	0.13	180	0	0.13	180	0	0.13	180	0	0.13
501 - 600	190	0	0.13	190	0	0.13	190	0	0.13	190	0	0.13	190	0	0.13	305	1	0.14
601 - 800	200	0	0.14	200	0	0.14	200	0	0.14	330	1	0.15	330	1	0.15	330	1	0.15
801 - 1000	205	0	0.14	205	0	0.14	205	0	0.14	335	1	0.17	335	1	0.17	335	1	0.17
1001 - 2000	220	0	0.15	220	0	0.15	360	1	0.19	490	2	0.21	490	2	0.21	610	3	0.22
2001 - 3000	220	0	0.15	375	1	0.20	505	2	0.23	630	3	0.24	745	4	0.26	870	5	0.26
3001 - 4000	225	0	0.15	380	1	0.20	510	2	0.24	645	3	0.25	880	5	0.28	1000	6	0.29
4001 - 5000	225	0	0.16	380	1	0.20	520	2	0.24	770	4	0.28	895	5	0.29	1120	7	0.31
5001 - 7000	230	0	0.16	385	1	0.21	655	3	0.27	780	4	0.29	1020	6	0.32	1260	8	0.34
7001 - 10000	230	0	0.16	520	2	0.25	660	3	0.28	910	5	0.32	1150	7	0.34	1500	10	0.37
10001 - 20000	390	1	0.21	525	2	0.26	785	4	0.31	1040	6	0.35	1400	9	0.39	1980	14	0.43
20001 - 50000	390	1	0.21	530	2	0.26	920	5	0.34	1300	8	0.39	1890	13	0.44	2570	19	0.48
50001 - 100000	390	1	0.21	670	3	0.29	1040	6	0.36	1420	9	0.41	2120	15	0.47	3150	23	0.50

*n*: size of sample; entry of “All” indicates that each piece in lot is to be inspected.

*c* : acceptance number for sample.

AOQL : average outgoing quality limit.

Tables of Quality Control By Hamid Bazargan  
**Table C.** Example of Dodge-Romig Double Sampling Lot Inspection Tables.

**.Based on Stated Value of LQL) = 1.0 % and     Consumer’s Risk=  $\beta$  = 0.10**

LOT SIZE	PROCESS AVERAGE (%)																	
	0 - 0.010						0.011 - 0.10						0.11 - 0.20					
	Trial 1		Trial 2			AOQL (%)	Trial 1		Trial 2			AOQL (%)	Trial 1		Trial 2			AOQL (%)
	$n_1$	$c_1$	$n_2$	$n_1 + n_2$	$c_2$		$n_1$	$c_1$	$n_2$	$n_1 + n_2$	$c_2$		$n_1$	$c_1$	$n_2$	$n_1 + n_2$	$c_2$	
1 - 120	All	0	-	-	-	0.00	All	0	-	-	-	0.00	All	0	-	-	-	0.00
121 - 150	120	0	-	-	-	0.06	120	0	-	-	-	0.06	120	0	-	-	-	0.06
151 - 200	140	0	-	-	-	0.08	140	0	-	-	-	0.08	140	0	-	-	-	0.08
201 - 260	165	0	-	-	-	0.10	165	0	-	-	-	0.10	165	0	-	-	-	0.10
261 - 300	180	0	75	255	1	0.10	180	0	75	255	1	0.10	180	0	75	255	1	0.10
301 - 400	200	0	90	290	1	0.12	200	0	90	290	1	0.12	200	0	90	290	1	0.12
401 - 500	215	0	100	315	1	0.14	215	0	100	315	1	0.14	215	0	100	315	1	0.14
501 - 600	225	0	115	340	1	0.15	225	0	115	340	1	0.15	225	0	115	340	1	0.15
601 - 800	235	0	125	360	1	0.16	235	0	125	360	1	0.16	235	0	125	360	1	0.16
801 - 1000	245	0	135	380	1	0.17	245	0	135	380	1	0.17	245	0	250	495	2	0.19
1001 - 2000	265	0	155	420	1	0.18	265	0	155	420	1	0.18	265	0	285	550	2	0.21
2001 - 3000	270	0	160	430	1	0.19	270	0	300	570	2	0.22	270	0	420	690	3	0.25
3001 - 4000	275	0	160	435	1	0.19	275	0	305	580	2	0.22	275	0	435	710	3	0.25
4001 - 5000	275	0	165	440	1	0.19	275	0	310	585	2	0.23	275	0	565	840	4	0.28
5001 - 7000	275	0	170	445	1	0.20	275	0	315	590	2	0.23	275	0	580	855	4	0.29
7001 - 10000	280	0	320	600	1	0.24	280	0	460	740	3	0.26	280	0	590	870	4	0.30
10001 - 20000	280	0	325	605	1	0.24	280	0	465	745	3	0.27	450	1	700	1150	6	0.33
20001 - 50000	280	0	325	605	1	0.25	280	0	605	885	4	0.30	450	1	830	1280	7	0.36
50001 - 100000	280	0	325	605	1	0.25	280	0	605	885	4	0.30	450	1	960	1410	8	0.38

$n_1$  = size of first sample;  $n_2$  = size of second sample; entry of “All” indicates that each piece in lot is to be inspected.  
 $c_1$  = acceptance number for first sample.  $c_2$  = acceptance number for first and second samples combined.  
 AOQL = average outgoing quality limit.

Table C -continued

LOT SIZE	PROCESS AVERAGE (%)																	
	0.21 - 0.30						0.31 - 0.40						0.41 - 0.50					
	Trial 1		Trial 2			AOQL (%)	Trial 1		Trial 2			AOQL (%)	Trial 1		Trial 2			AOQL (%)
	$n_1$	$c_1$	$n_2$	$n_1 + n_2$	$c_2$		$n_1$	$c_1$	$n_2$	$n_1 + n_2$	$c_2$		$n_1$	$c_1$	$n_2$	$n_1 + n_2$	$c_2$	
1 - 120	All	0	-	-	-	0.00	All	0	-	-	-	0.00	All	0	-	-	-	0.00
121 - 150	120	0	-	-	-	0.06	120	0	-	-	-	0.06	120	0	-	-	-	0.06
151 - 200	140	0	-	-	-	0.08	140	0	-	-	-	0.08	140	0	-	-	-	0.08
201 - 260	165	0	-	-	-	0.10	165	0	-	-	-	0.10	165	0	-	-	-	0.10
261 - 300	180	0	75	255	1	0.10	180	0	75	255	1	0.10	180	0	75	255	1	0.10
301 - 400	200	0	90	290	1	0.12	200	0	90	290	1	0.12	200	0	90	290	1	0.12
401 - 500	215	0	100	315	1	0.14	215	0	100	315	1	0.14	215	0	100	315	1	0.14
501 - 600	225	0	115	340	1	0.15	225	0	115	340	1	0.15	225	0	205	430	2	0.16
601 - 800	235	0	230	465	2	0.18	235	0	230	465	2	0.18	235	0	230	465	2	0.18
801 - 1000	245	0	250	495	2	0.19	245	0	250	495	2	0.19	245	0	250	495	2	0.19
1001 - 2000	265	0	405	670	3	0.23	265	0	515	780	4	0.24	265	0	515	780	4	0.24
2001 - 3000	270	0	545	815	4	0.26	430	1	620	1050	6	0.28	430	1	830	1260	8	0.30
3001 - 4000	435	1	645	1080	6	0.29	435	1	865	1300	8	0.30	580	2	940	1520	10	0.33
4001 - 5000	440	1	660	1100	6	0.30	440	1	1000	1440	9	0.33	585	2	1075	1660	11	0.35
5001 - 7000	445	1	785	1230	7	0.33	590	2	990	1580	10	0.36	730	3	1190	1920	13	0.38
7001 - 10000	450	1	920	1370	8	0.35	600	2	1240	1840	12	0.39	870	4	1540	2410	17	0.41
10001 - 20000	605	2	1035	1640	10	0.39	745	3	1485	2230	15	0.43	1150	6	1990	3140	23	0.44
20001 - 50000	605	2	1295	1900	12	0.42	885	4	1845	2730	19	0.47	1280	7	2600	3880	29	0.52
50001 - 100000	605	2	1545	2150	14	0.44	885	4	2085	2970	21	0.49	1410	8	3280	4690	36	0.55

$n_1$  = size of first sample;  $n_2$  = size of second sample; entry of "All" indicates that each piece in lot is to be inspected.

$c_1$  = acceptance number for first sample.  $c_2$  = acceptance number for first and second samples combined.

AOQL = average outgoing quality limit.

Table C.1 Example of Dodge-Romig LQL(LTPD) Double Sampling Tables

LQL = 5.0%<sub>r</sub> β = 0.10.

Lot Size	Process average, %																																			
	0-0.05						0.06-0.50						0.51-1.00						1.01-1.50						1.51-2.00						2.01-2.50					
	Trial 1		Trial 2			AOQL	Trial 1		Trial 2			AOQL	Trial 1		Ttrial2			AOQL	Trial 1		Trial 2			AOQL	Trial 1		Trial 2			AOQL	Trial 1		Trial 2			AOQL
	n <sub>1</sub>	c <sub>1</sub>	n <sub>2</sub>	n <sub>1</sub> + n <sub>2</sub>	C <sub>2</sub>	%	n <sub>1</sub>	c <sub>1</sub>	n <sub>2</sub>	n <sub>1</sub> + n <sub>2</sub>	C <sub>2</sub>	%	n <sub>1</sub>	c <sub>1</sub>	n <sub>2</sub>	n <sub>1</sub> + n <sub>2</sub>	C <sub>2</sub>	%	n <sub>1</sub>	c <sub>1</sub>	n <sub>2</sub>	n <sub>1</sub> + n <sub>2</sub>	C <sub>2</sub>	%	n <sub>1</sub>	c <sub>1</sub>	n <sub>2</sub>	n <sub>1</sub> + n <sub>2</sub>	C <sub>2</sub>	%	n <sub>1</sub>	c <sub>1</sub>	n <sub>2</sub>	n <sub>1</sub> +	C <sub>2</sub>	%
1-30	all	0	—	—	—	0	all	0	—	—	—	0	all	0	—	—	—	0	all	0	—	—	—	0	all	0	—	—	—	0	all	0	—	—	—	0
31-50	30	0	—	—	—	0.49	30	0	—	—	—	0.49	30	0	—	—	—	0.49	30	0	—	—	—	0.49	30	0	—	—	—	0.49	30	0	—	—	—	0.49
51-75	38	0	—	—	—	0.59	38	0	—	—	—	0.59	38	0	—	—	—	0.59	38	0	—	—	—	0.59	38	0	—	—	—	0.59	38	0	—	—	—	0.59
76-100	44	0	21	65	1	0.64	44	0	21	65	1	0.64	44	0	21	65	1	0.64	44	0	21	65	1	0.64	44	0	21	65	1	0.64	44	0	21	65	1	0.64
101-200	49	0	26	75	1	0.84	49	0	26	75	1	0.84	49	0	26	75	1	0.84	49	0	51	100	2	0.91	49	0	51	100	2	0.91	49	0	51	100	2	0.91
201-300	50	0	30	80	1	0.91	50	0	30	80	1	0.91	50	0	35	105	2	1.0	50	0	55	105	2	1.0	50	0	80	130	3	1.1	50	0	100	150	4	1.1
301-400	55	0	30	85	1	0.92	55	0	55	110	2	1.1	55	0	55	110	2	1.1	55	0	80	135	3	1.1	55	0	100	155	4	1.2	85	1	105	190	6	1.3
401-500	55	0	30	85	1	0.93	55	0	55	110	2	1.1	55	0	80	135	3	1.2	55	0	105	160	4	1.3	85	1	120	205	6	1.4	85	1	140	225	7	1.4
501-600	55	0	30	85	1	0.94	55	0	60	115	2	1.1	55	0	85	140	3	1.2	55	0	110	165	4	1.3	85	1	145	230	7	1.4	85	1	165	250	8	1.5
601-800	55	0	35	90	1	0.95	55	0	65	120	2	1.1	55	0	85	140	3	1.3	90	1	125	215	6	1.5	90	1	170	260	8	1.5	120	2	185	305	10	1.6
801-1000	55	0	35	90	1	0.96	55	0	65	120	2	1.1	55	0	115	170	4	1.4	90	1	150	240	7	1.5	90	1	200	290	9	1.6	120	2	210	330	11	1.7
1001-2000	55	0	35	90	1	0.98	55	0	95	150	3	1.3	55	0	120	175	4	1.4	90	1	185	275	8	1.7	120	2	225	345	11	1.9	175	4	260	435	15	2.0
2001-3000	55	0	65	120	2	1.2	55	0	95	150	3	1.3	55	0	150	205	5	1.5	120	2	180	300	9	1.9	150	3	270	420	14	2.1	205	5	375	580	21	2.3
3001-4000	55	0	65	120	2	1.2	55	0	95	150	3	1.3	90	1	140	230	6	1.6	120	2	210	330	10	2.0	150	3	295	445	15	2.3	230	6	420	650	24	2.4
4001-5000	55	0	65	120	2	1.2	55	0	95	150	3	1.4	90	1	165	255	7	1.8	120	2	255	375	12	2.1	150	3	345	495	17	2.3	255	7	445	700	26	2.5
5001-7000	55	0	65	120	2	1.2	55	0	95	150	3	1.4	90	1	165	255	7	1.8	120	2	260	380	12	2.1	150	3	370	520	18	2.3	255	7	495	750	23	2.6
7001-10,000	55	0	65	120	2	1.2	55	0	120	175	4	1.5	90	1	190	280	8	1.9	120	2	285	405	13	2.1	175	4	370	545	19	2.4	280	8	540	820	31	2.7
10,001-20,000	55	0	65	120	2	1.2	55	0	120	175	4	1.5	90	1	190	280	8	1.9	120	2	310	430	14	2.2	175	4	420	595	21	2.4	280	8	660	940	36	2.8
20,001-50,000	55	0	65	120	2	1.2	55	0	150	205	5	1.7	90	1	215	305	9	2.0	120	2	335	455	15	2.2	205	5	485	690	25	2.5	305	9	745	1050	41	2.9
50,001-100,000	55	0	65	120	2	1.2	55	0	150	205	5	1.7	90	1	240	330	10	2.1	120	2	360	480	16	2.3	205	5	555	760	28	2.6	330	10	810	1140	45	3.0

Reroduced,from Grant and Leavenworth,1988 Table 13.3,

Table D Philips Standard Sampling System

	Point of Control=Indifference Point																				
Lot Size	0.25%		0.5%		1%		2%		3%		5%		7%		10%						
Single Sampling	n	c	n	c	n	c	n	c	n	c	n	c	n	c	n	c					
20-50	all all all 175 225	-	all all 100 100 225	-	all 60 60 135 150	-	30	0	20	0	13	0	10	0	7	0					
51-100		-		-		60	0	30	0	20	0	13	0	10	0	7	0				
101-200		-		100		0	60	0	35	0	55	1	35	1	25	1	17	1			
201-500		0		100		0	135	1	75	1	55	1	35	1	40	1	25	2			
501-1,000		0		225		1	150	1	85	1	85	2	55	2	55	2	35	3			
Double Sampling	n <sub>1</sub>	c <sub>1</sub>	c <sub>2</sub>	n <sub>1</sub>	c <sub>1</sub>	c <sub>2</sub>	n <sub>1</sub>	c <sub>1</sub>	c <sub>2</sub>	n <sub>1</sub>	c <sub>1</sub>	c <sub>2</sub>	n <sub>1</sub>	c <sub>1</sub>	c <sub>2</sub>	n <sub>1</sub>	c <sub>1</sub>	c <sub>2</sub>	n <sub>1</sub>	c <sub>1</sub>	c <sub>2</sub>
1,001-2,000	330	0	1	150	0	1	110	0	2	55	0	2	45	0	3	25	0	3	30	1	5
2,001-5,000	425	0	2	200	0	2	135	0	3	70	0	3	70	1	5	45	1	5	55	2	10
5,001-10,000	525	0	3	260	0	3	220	1	5	110	1	5	125	2	10	75	2	10	75	3	15
10,001-20,000	875	1	5	440	1	5	380	2	10	190	2	10	180	3	15	110	3	15	100	4	20
20,001-50,000	1,500	2	10	750	2	10	540	3	15	270	3	15	240	4	20	140	4	20	120	5	25
50,001 & more	2,200	3	15	1,100	3	15	700	4	20	350	4	20	290	5	25	175	5	25	145	6	30

Reference Grant&amp;Levenworth 1988

all means whole lot has to be inspected  
 in Double case:  $n_2=2 n_1$  Rejection no=  $c_2+1$

Table E Code of sample size for ISO2859

Lot size	S-1	S-2	S-3	S-4	I	II	III
2-8	A	A	A	A	A	A	B
9-15	A	A	A	A	A	B	C
16-25	A	A	B	B	B	C	D
26-50	A	B	B	C	C	D	E
51-90	B	B	C	C	C	E	F
91-150	B	B	C	D	D	F	G
151-280	B	C	D	E	E	G	H
281-500	B	C	D	E	F	H	J
501-1200	C	C	E	F	G	J	K
1201-3200	C	D	E	G	H	K	L
3201-10,000	C	D	F	G	J	L	M
10,001-35,000	C	D	F	H	K	M	N
35,001-150,000	D	E	G	J	L	N	P
150,001-500,000	D	E	G	J	M	P	Q
500,001 & more	D	E	H	K	N	Q	R

reference Bestrefield (1990)

# Table F single sampling Plan Tightened ISO2859-1

[illegible]

↓ = Use the first sampling plan below the arrow. If sample size equals, or exceeds, lot size, carry out 100 % inspection.

↑ = Use the first sampling plan above the arrow.

**Ac = Acceptance number**

**Re = Rejection number**

## Table G single sampling Plan Reduced

ISO2859

Sample size code letter	Sample size	AQL(%)																											
		0.010	0.015	0.025	0.040	0.065	0.10	0.15	0.25	0.40	0.65	1.0	1.5	2.5	4.0	6.5	10	15	25	40	65	100	150	250	400	650	1000		
		Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re
A	2	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↓	↓	1 2	2 3	3 4	5 6	7 8	10 11	14 15	21 22	30 31		
B	2	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↓	0 2	1 3	2 4	3 5	5 6	7 8	10 11	14 15	21 22	30 31	
C	2	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↑	↑	0 2	1 3	1 4	2 5	3 6	5 8	7 10	10 13	14 17	21 24	↑	↑
D	3	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↓	0 2	1 3	1 4	2 5	3 6	5 8	7 10	10 13	14 17	21 24	↑	↑	↑	↑
E	5	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↓	0 2	1 3	1 4	2 5	3 6	5 8	7 10	10 13	14 17	21 24	↑	↑	↑	↑	↑
F	8	↓	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↑	0 2	1 3	1 4	2 5	3 6	5 8	7 10	10 13	↑	↑	↑	↑	↑	↑	↑	↑
G	13	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↓	0 2	1 3	1 4	2 5	3 6	5 8	7 10	10 13	↑	↑	↑	↑	↑	↑	↑	↑	↑
H	20	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↑	0 2	1 3	1 4	2 5	3 6	5 8	7 10	10 13	↑	↑	↑	↑	↑	↑	↑	↑	↑
J	32	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↑	0 2	1 3	1 4	2 5	3 6	5 8	7 10	10 13	↑	↑	↑	↑	↑	↑	↑	↑	↑
K	50	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↑	0 2	1 3	1 4	2 5	3 6	5 8	7 10	10 13	↑	↑	↑	↑	↑	↑	↑	↑	↑
L	80	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↑	0 2	1 3	1 4	2 5	3 6	5 8	7 10	10 13	↑	↑	↑	↑	↑	↑	↑	↑	↑
M	125	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↑	0 2	1 3	1 4	2 5	3 6	5 8	7 10	10 13	↑	↑	↑	↑	↑	↑	↑	↑	↑
N	200	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↑	0 2	1 3	1 4	2 5	3 6	5 8	7 10	10 13	↑	↑	↑	↑	↑	↑	↑	↑	↑
P	315	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↑	0 2	1 3	1 4	2 5	3 6	5 8	7 10	10 13	↑	↑	↑	↑	↑	↑	↑	↑	↑
Q	500	0 1	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑
R	800	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑



# Table H single sampling Plan Normal

ISO2859

 IS 2500 (Part 1) : 2000  
 ISO 2859-1 : 1999

Sample size code letter	Sample size	AQL(%)																											
		0,010	0,015	0,025	0,040	0,065	0,10	0,15	0,25	0,40	0,65	1,0	1,5	2,5	4,0	6,5	10	15	25	40	65	100	150	250	400	650	1 000		
		Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re	Ac Re
A	2	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↓	↓	1 2	2 3	3 4	5 6	7 8	10 11	14 15	21 22	30 31		
B	3	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↓	1 2	2 3	3 4	5 6	7 8	10 11	14 15	21 22	30 31	44 45		
C	5	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↓	1 2	2 3	3 4	5 6	7 8	10 11	14 15	21 22	30 31	44 45	↑		
D	8	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↓	1 2	2 3	3 4	5 6	7 8	10 11	14 15	21 22	30 31	44 45	↑	↑		
E	13	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↓	1 2	2 3	3 4	5 6	7 8	10 11	14 15	21 22	30 31	44 45	↑	↑	↑		
F	20	↓	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↓	1 2	2 3	3 4	5 6	7 8	10 11	14 15	21 22	↑	↑	↑	↑	↑	↑		
G	32	↓	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↓	1 2	2 3	3 4	5 6	7 8	10 11	14 15	21 22	↑	↑	↑	↑	↑	↑	↑		
H	50	↓	↓	↓	↓	↓	↓	↓	0 1	↑	↓	1 2	2 3	3 4	5 6	7 8	10 11	14 15	21 22	↑	↑	↑	↑	↑	↑	↑	↑		
J	80	↓	↓	↓	↓	↓	0 1	↑	↓	1 2	2 3	3 4	5 6	7 8	10 11	14 15	21 22	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑		
K	125	↓	↓	↓	↓	0 1	↑	↓	1 2	2 3	3 4	5 6	7 8	10 11	14 15	21 22	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑		
L	200	↓	↓	↓	0 1	↑	↓	1 2	2 3	3 4	5 6	7 8	10 11	14 15	21 22	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑		
M	315	↓	↓	0 1	↑	↓	1 2	2 3	3 4	5 6	7 8	10 11	14 15	21 22	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑		
N	500	↓	0 1	↑	↓	1 2	2 3	3 4	5 6	7 8	10 11	14 15	21 22	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑		
P	800	↓	0 1	↑	↓	1 2	2 3	3 4	5 6	7 8	10 11	14 15	21 22	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑		
Q	1 250	0 1	↑	↓	1 2	2 3	3 4	5 6	7 8	10 11	14 15	21 22	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑		
R	2 000	↑	↑	1 2	2 3	3 4	5 6	7 8	10 11	14 15	21 22	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑		

↓ = Use the first sampling plan below the arrow. If sample size equals, or exceeds, lot size, carry out 100 % inspection.

↑ = Use the first sampling plan above the arrow.

Ac = Acceptance number

Re = Rejection number

.Based on Average Outgoing Quality Limit (AOQL) = 2.0% **Table I. Dodge-Romig Single Sampling** **AOQL=2%**

LOT SIZE	PROCESS AVERAGE (%)																	
	0.00 - 0.04			0.05 - 0.40			0.41 - 0.80			0.81 - 1.20			1.21 - 1.60			1.61 - 2.00		
	<i>n</i>	<i>c</i>	LQL (%)	<i>n</i>	<i>c</i>	LQL (%)	<i>n</i>	<i>c</i>	LQL (%)	<i>n</i>	<i>c</i>	LQL (%)	<i>n</i>	<i>c</i>	LQL (%)	<i>n</i>	<i>c</i>	LQL (%)
1 - 15	All	0	-	All	0	-	All	0	-	All	0	-	All	0	-	All	0	-
16 - 50	14	0	13.6	14	0	13.6	14	0	13.6	14	0	13.6	14	0	13.6	14	0	13.6
51 - 100	16	0	12.4	16	0	12.4	16	0	12.4	16	0	12.4	16	0	12.4	16	0	12.4
101 - 200	17	0	12.2	17	0	12.2	17	0	12.2	17	0	12.2	35	1	10.5	35	1	10.5
201 - 300	17	0	12.3	17	0	12.3	17	0	12.3	37	1	10.2	37	1	10.2	37	1	10.2
301 - 400	18	0	11.8	18	0	11.8	38	1	10.0	38	1	10.0	38	1	10.0	60	2	8.5
401 - 500	18	0	11.9	18	0	11.9	39	1	9.8	39	1	9.8	60	2	8.6	60	2	8.6
501 - 600	18	0	11.9	18	0	11.9	39	1	9.8	39	1	9.8	60	2	8.6	60	2	8.6
601 - 800	18	0	11.9	40	1	9.6	40	1	9.6	65	2	8.0	65	2	8.0	85	3	7.5
801 - 1000	18	0	12.0	40	1	9.6	40	1	9.6	65	2	8.1	65	2	8.1	90	3	7.4
1001 - 2000	18	0	12.0	41	1	9.4	65	2	8.2	65	2	8.2	95	3	7.0	120	4	6.5
2001 - 3000	18	0	12.0	41	1	9.4	65	2	8.2	95	3	7.0	120	4	6.5	180	6	5.8
3001 - 4000	18	0	12.0	42	1	9.3	65	2	8.2	95	3	7.0	155	5	6.0	210	7	5.5
4001 - 5000	18	0	12.0	42	1	9.3	70	2	7.5	125	4	6.4	155	5	6.0	245	8	5.3
5001 - 7000	18	0	12.0	42	1	9.3	95	3	7.0	125	4	6.4	185	6	5.6	280	9	5.1
7001 - 10000	42	1	9.3	70	2	7.5	95	3	7.0	155	5	6.0	220	7	5.4	350	11	4.8
10001 - 20000	42	1	9.3	70	2	7.6	95	3	7.0	190	6	5.6	290	9	4.9	460	14	4.4
20001 - 50000	42	1	9.3	70	2	7.6	125	4	6.4	220	7	5.4	395	12	4.5	720	21	3.9
50001 - 100000	42	1	9.3	95	3	7.0	160	5	5.9	290	9	4.9	505	15	4.2	955	27	3.7

*n* : size of sample; entry of “All” indicates that each piece in lot is to be inspected.

*c* : acceptance number for sample.

LQL : limiting quality level corresponding to a consumer’s risk ( $\beta$ ) = 0.10.

Refrence [www.galitshmueli.com/system/files/Table\\_DODGE\\_ROMIG.pdf](http://www.galitshmueli.com/system/files/Table_DODGE_ROMIG.pdf)

Table J. Dodge-Romig Single Sampling Lot Inspection Table AOQL=3%.

Based on Average Outgoing Quality Limit (AOQL) = 3.0 %.

LOT SIZE	PROCESS AVERAGE (%)																	
	0.00 - 0.06			0.07 - 0.60			0.61 - 1.20			1.21 - 1.80			1.81 - 2.40			2.41 - 3.00		
	<i>n</i>	<i>c</i>	LQL (%)	<i>n</i>	<i>c</i>	LQL (%)	<i>n</i>	<i>c</i>	LQL (%)	<i>n</i>	<i>c</i>	LQL (%)	<i>n</i>	<i>c</i>	LQL (%)	<i>n</i>	<i>c</i>	LQL (%)
1 - 10	All	0	-	All	0	-	All	0	-	All	0	-	All	0	-	All	0	-
11 - 50	10	0	19.0	10	0	19.0	10	0	19.0	10	0	19.0	10	0	19.0	10	0	19.0
51 - 100	11	0	18.0	11	0	18.0	11	0	18.0	11	0	18.0	11	0	18.0	22	1	16.4
101 - 200	12	0	17.0	12	0	17.0	12	0	17.0	25	1	15.1	25	1	15.1	25	1	15.1
201 - 300	12	0	17.0	12	0	17.0	26	1	14.6	26	1	14.6	26	1	14.6	40	2	12.8
301 - 400	12	0	17.1	12	0	17.1	26	1	14.7	26	1	14.7	41	2	12.7	41	2	12.7
401 - 500	12	0	17.2	27	1	14.1	27	1	14.1	42	2	12.4	42	2	12.4	42	2	12.4
501 - 600	12	0	17.3	27	1	14.2	27	1	14.2	42	2	12.4	42	2	12.4	60	3	10.8
601 - 800	12	0	17.3	27	1	14.2	27	1	14.2	43	2	12.1	60	3	10.9	60	3	10.9
801 - 1000	12	0	17.4	27	1	14.2	44	2	11.8	44	2	11.8	60	3	11.0	80	4	9.8
1001 - 2000	12	0	17.5	28	1	13.8	45	2	11.7	65	3	10.2	80	4	9.8	100	5	9.1
2001 - 3000	12	0	17.5	28	1	13.8	45	2	11.7	65	3	10.2	100	5	9.1	140	7	8.2
3001 - 4000	12	0	17.5	28	1	13.8	65	3	10.3	85	4	9.5	125	6	8.4	165	8	7.8
4001 - 5000	28	1	13.8	28	1	13.8	65	3	10.3	85	4	9.5	125	6	8.4	210	10	7.4
5001 - 7000	28	1	13.8	45	2	11.8	65	3	10.3	105	5	8.8	145	7	8.1	235	11	7.1
7001 - 10000	28	1	13.9	46	2	11.6	65	3	10.3	105	5	8.8	170	8	7.6	280	13	6.8
10001 - 20000	28	1	13.9	46	2	11.7	85	4	9.5	125	6	8.4	215	10	7.2	380	17	6.2
20001 - 50000	28	1	13.9	65	3	10.3	105	5	8.8	170	8	7.6	310	14	6.5	560	24	5.7
50001 - 100000	28	1	13.9	65	3	10.3	125	6	8.4	215	10	7.2	385	17	6.2	690	29	5.4

*n* : size of sample; entry of “All” indicates that each piece in lot is to be inspected.

*c* : acceptance number for sample.

LQL : limiting quality level corresponding to a consumer's risk ( $\beta$ ) = 0.10.

Table K Dodge-Romig Double Sampling Lot Inspection AOQL=1%

LOT SIZE	PROCESS AVERAGE (%)																	
	0.00 - 0.02						0.03 - 0.20						0.21 – 0.40					
	Trial 1		Trial 2			LQL (%)	Trial 1		Trial 2			LQL (%)	Trial 1		Trial 2			LQL (%)
	$n_1$	$c_1$	$n_2$	$n_1 + n_2$	$c_2$		$n_1$	$c_1$	$n_2$	$n_1 + n_2$	$c_2$		$n_1$	$c_1$	$n_2$	$n_1 + n_2$	$c_2$	
1 - 25	All	0	-	-	-	-	All	0	-	-	-	-	All	0	-	-	-	-
26 - 50	22	0	-	-	-	7.7	22	0	-	-	-	7.7	22	0	-	-	-	7.7
51 - 100	33	0	17	50	1	6.9	33	0	17	50	1	6.9	33	0	17	50	1	6.9
101 - 200	43	0	22	65	1	5.8	43	0	22	65	1	5.8	43	0	22	65	1	5.8
201 - 300	47	0	28	75	1	5.5	47	0	28	75	1	5.5	47	0	28	75	1	5.5
301 - 400	49	0	31	80	1	5.4	49	0	31	80	1	5.4	55	0	60	115	2	4.8
401 - 500	50	0	30	80	1	5.4	50	0	30	80	1	5.4	55	0	65	120	2	4.7
501 - 600	50	0	30	80	1	5.4	50	0	30	80	1	5.4	60	0	65	125	2	4.6
601 - 800	50	0	35	85	1	5.3	60	0	70	130	2	4.5	60	0	70	130	2	4.5
801 - 1000	55	0	30	85	1	5.2	60	0	75	135	2	4.4	60	0	75	135	2	4.4
1001 - 2000	55	0	35	90	1	5.1	65	0	75	140	2	4.3	75	0	120	195	3	3.8
2001 - 3000	65	0	80	145	2	4.2	65	0	80	145	2	4.2	75	0	125	200	3	3.7
3001 - 4000	70	0	80	150	2	4.1	70	0	80	150	2	4.1	80	0	175	255	4	3.5
4001 - 5000	70	0	80	150	2	4.1	70	0	80	150	2	4.1	80	0	180	260	4	3.4
5001 - 7000	70	0	80	150	2	4.1	75	0	125	200	3	3.7	80	0	180	260	4	3.4
7001 - 10000	70	0	80	150	2	4.1	80	0	125	205	3	3.6	85	0	180	265	4	3.3
10001 - 20000	70	0	80	150	2	4.1	80	0	130	210	3	3.6	90	0	230	320	5	3.2
20001 - 50000	75	0	80	155	2	4.0	80	0	135	215	3	3.6	95	0	300	395	6	2.9
50001 - 100000	75	0	80	155	2	4.0	85	0	180	265	4	3.3	170	1	380	550	8	2.6

$n_1$  = size of sample;  $n_2$  = size of second sample; entry of “All” indicates that each piece in lot is to be inspected.

.  $c_1$  = acceptance number for sample.  $c_2$  = acceptance number for first and second samples combined

.LQL = limiting quality level corresponding to a consumer’s risk ( $\beta$ ) = 0.10

Reference [www.galitshmueli.com/system/files/Table\\_DODGE\\_ROMIG.pdf](http://www.galitshmueli.com/system/files/Table_DODGE_ROMIG.pdf)

Table K Continued

.% Based on AOQL = 1.0

LOT SIZE	PROCESS AVERAGE (%)																	
	0.41 - 0.60						0.61 - 0.80						0.81 - 1.00					
	Trial 1		Trial 2			LQL (%)	Trial 1		Trial 2			LQL (%)	Trial 1		Trial 2			LQL (%)
	$n_1$	$c_1$	$n_2$	$n_1 + n_2$	$c_2$		$n_1$	$c_1$	$n_2$	$n_1 + n_2$	$c_2$		$n_1$	$c_1$	$n_2$	$n_1 + n_2$	$c_2$	
1 - 25	All	0	-	-	-	-	All	0	-	-	-	-	All	0	-	-	-	-
26 - 50	22	0	-	-	-	7.7	22	0	-	-	-	7.7	22	0	-	-	-	7.7
51 - 100	33	0	17	50	1	6.9	33	0	17	50	1	6.9	33	0	17	50	1	6.9
101 - 200	43	0	22	65	1	5.8	43	0	22	65	1	5.8	47	0	43	90	2	5.4
201 - 300	55	0	50	105	2	4.9	55	0	50	105	2	4.9	55	0	50	105	2	4.9
301 - 400	55	0	60	115	2	4.8	55	0	60	115	2	4.8	60	0	80	140	3	4.5
401 - 500	55	0	65	120	2	4.7	60	0	95	155	3	4.3	60	0	95	155	3	4.3
501 - 600	60	0	65	125	2	4.6	65	0	100	165	3	4.2	65	0	100	165	3	4.2
601 - 800	65	0	105	170	3	4.1	65	0	105	170	3	4.1	70	0	140	210	4	3.9
801 - 1000	65	0	110	175	3	4.0	70	0	150	220	4	3.8	125	1	180	305	6	3.5
1001 - 2000	80	0	165	245	4	3.7	135	1	200	335	6	3.3	140	1	245	385	7	3.2
2001 - 3000	80	0	170	250	4	3.6	150	1	265	415	7	3.0	215	2	355	570	10	2.8
3001 - 4000	85	0	220	305	5	3.3	160	1	330	490	8	2.8	225	2	455	680	12	2.7
4001 - 5000	145	1	225	370	6	3.1	225	2	375	600	10	2.7	240	2	595	835	14	2.5
5001 - 7000	155	1	285	440	7	2.9	235	2	440	675	11	2.6	310	3	665	975	16	2.4
7001 - 10000	165	1	355	520	8	2.7	250	2	585	835	13	2.4	385	4	785	1170	19	2.3
10001 - 20000	175	1	415	590	9	2.6	325	3	655	980	15	2.3	520	6	980	1500	24	2.2
20001 - 50000	250	2	490	740	11	2.4	340	3	910	1250	19	2.2	610	7	1410	2020	32	2.1
50001 - 100000	275	2	700	975	14	2.2	420	4	1050	1470	22	2.1	770	9	1850	2620	41	2.0

$n_1$  = size of sample;  $n_2$  = size of second sample; entry of “All” indicates that each piece in lot is to be inspected.

$c_1$  = acceptance number for sample.  $c_2$  = acceptance number for first and second samples combined

.LQL = limiting quality level corresponding to a consumer’s risk ( $\beta$ ) = 0.10

# Table L

Table AQL conversion Etracted from MilSTD414 Manual	
If your AQL is in the interval	Use This AQL
• -----	0.01
• -----	0.015
• -----	0.025
• 0.050 - 0.069	0.065
• 0.070 - 0.109	0.10
• 0.110 - 0.164	0.15
• 0.165 - 0.279	0.25
• 0.280 - 0.439	0.40
• 0.440 - 0.699	0.65
• 0.700 - 1.09	1.0
• 1.10 - 1.64	1.5
• 1.65 - 2.79	2.5
• 2.80 - 4.39	4.0
• 4.40 - 6.99	6.5
• 7.00 - 10.9	10.0

## Table L.1

(Table E.1 ISO 3951.1)

Values of $f_e$ for calculating $\sigma_{max} = ((U-L))/f_e$ : $\sigma$ -Method																
AQL %	0,0107	0,015	0,025	0,04 0	0,065	0,10	0,15	0,25	0,40	0,65	1,0	1,5	2,5	4	6,5	10
$f_\sigma$	,125	,129	,132	,137	,141	,147	,152	,157	,165	,174	,184	,194	,206	,223	,243	,273

**Table M** Sample Size based on lot size and inspection level  
ISO 3951

Lot or batch size	Special inspection levels				General inspection levels		
	S-I	S-2	S-3	S-4	I	II	III
2 to 8	B	B	B	B	B	B	B
9 to 15	B	B	B	B	B	B	C
16 to 25	B	B	B	B	B	C	D
26 to 50	B	B	B	C	C	D	E
91 to 150	B	B	C	D	D	F	G
151 to 2 80	B	C	D	E	E	G	H
281 to 500	B	C	D	E	F	H	J
501 to 1200	C	C	E	F	G	J	K
1201 to 3200	C	D	E	G	H	K	L
3201 to 10000	C	D	F	G	J	L	M
10001 to 35000	C	D	F	H	K	M	N
35 001 to 150 000	D	E	G	J	L	N	P
150001 to 500000	D	E	G	J	M	P	Q
500 000 and over	D	E	H	K	N	Q	R

Refrence:Table A.1 ISO 3951-2

Table N Single Sampling S-method Form p\*

Normal inspection

Code letter	Acceptance quality limit (in percent nonconforming)															
	0,01	0,015	0,025	0,04	0,065	0,10	0,15	0,25	0,40	0,65	1,0	1,5	2,5	4,0	6,5	10,0
	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$
B	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	3	4	4
C	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	4	6	6	5
D	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	6	9	9	6	7
E	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	9	13	13	9	9	9
F	↓	↓	↓	↓	↓	↓	↓	↓	↓	11	17	18	13	14	14	14
G	↓	↓	↓	↓	↓	↓	↓	↓	15	22	23	18	20	21	21	21
H	↓	↓	↓	↓	↓	↓	↓	18	28	30	24	27	30	32	33	33
J	↓	↓	↓	↓	↓	↓	23	36	38	31	37	41	46	49	52	53
K	↓	↓	↓	↓	↓	28	44	47	40	48	54	63	69	75	79	82
L	↓	↓	↓	↓	34	54	58	50	61	71	84	94	105	115	124	↑
M	↓	↓	↓	40	64	69	60	76	89	108	124	143	159	178	↑	↑
N	↓	↓	47	75	82	73	93	110	137	159	186	213	247	↑	↑	↑
P	↓	55	88	96	86	112	134	171	202	239	277	332	↑	↑	↑	↑
Q	63	101	110	102	132	159	207	244	293	348	424	↑	↑	↑	↑	↑
R	116	127	120	155	189	247	298	362	438	541	↑	↑	↑	↑	↑	↑
	,02960	,04835	,06042	,1034	,1817	,2619	,4220	,5836	,8248	1,146	1,707	2,668	4,286	6,804	10,74	17,22
	,03011	,03762	,06433	0,1132	,1631	,2634	,3637	,5145	,7143	1,065	1,707	2,668	4,286	6,804	10,74	17,22

↓ There is no suitable plan in this area; use the first sampling plan below the arrow.  
If the sample size equals or exceeds the lot size, carry out 100 % inspection.

↑ There is no suitable plan in this area; use the first sampling plan above the arrow.

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Table N.1 Single Sampling S-method form p\*

## Tightened Inspection

Code letter	Acceptance quality limit (in percent nonconforming)															
	0,01	0,015	0,025	0,04	0,065	0,10	0,15	0,25	0,40	0,65	1,0	1,5	2,5	4,0	6,5	10,0
	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$
B	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	3 19,25	4 25,50
C	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	4 8,600	6 14,53
D	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	6 5,220	9 8,717	6 19,46
E	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	9 3,279	13 5,195	13 6,466	9 19,61
F	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	11 1,958	17 3,295	18 4,144	13 7,204	14 12,45	14 17,61
G	↓	↓	↓	↓	↓	↓	↓	↓	↓	15 1,245	22 2,011	23 2,518	18 4,381	20 7,627	21 10,85	21 17,29
H	↓	↓	↓	↓	↓	↓	↓	↓	18 7,546	28 1,266	30 1,592	24 2,751	27 4,799	30 6,857	32 10,94	33 17,03
J	↓	↓	↓	↓	↓	↓	↓	23 47 53	36 78 79	38 98 14	31 1,685	37 2,959	41 4,241	46 6,783	50 10,59	53 15,63
K	↓	↓	↓	↓	↓	↓	28 3027	44 4976	47 6222	40 1,071	48 1,876	54 2,687	63 4,313	71 6,738	78 9,963	82 14,80
L	↓	↓	↓	↓	↓	34 1,880	54 3,105	58 3,872	50 6,625	61 1,162	71 1,667	84 2,681	99 4,192	111 6,205	122 9,224	↑
M	↓	↓	↓	↓	40 1,180	64 1,954	69 2,436	60 4,150	76 7,336	89 1,052	108 1,694	131 2,654	150 3,936	170 5,851	↑	↑
N	↓	↓	↓	47 0,7418	75 1,218	82 1,524	73 2,605	93 4,595	110 6,602	137 1,063	169 1,666	201 2,470	233 3,679	↑	↑	↑
P	↓	↓	55 0,4641	88 0,7599	96 0,9473	86 1,614	112 2,852	134 4,100	171 6,611	214 1,039	260 1,540	312 2,292	↑	↑	↑	↑
Q	↓	63 0,2960	101 0,4835	110 0,6042	102 1,034	132 1,817	159 2,619	207 4,220	262 6,640	323 9,849	395 1,466	↑	↑	↑	↑	↑
R	90 0,2165	116 0,3011	127 0,3762	120 0,6433	155 1,132	189 1,631	247 2,634	320 4,141	398 6,152	498 9,152	↑	↑	↑	↑	↑	↑

↓ There is no suitable plan in this area; use the first sampling plan below the arrow.  
If the sample size equals or exceeds the lot size, carry out 100 % inspection.

↑ There is no suitable plan in this area; use the first sampling plan above the arrow.

Ref: ISO 3951-2:2013(E) Table D.2

# Table N.2 S-method form $p^*$

## Reduced Inspection

### Single sampling plans of Form $p^*$ for reduced inspection: s-method —

Code letter	Acceptance quality limit (in percent nonconforming)															
	0,01	0,015	0,025	0,04	0,065	0,10	0,15	0,25	0,40	0,65	1,0	1,5	2,5	4,0	6,5	10,0
	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$	$\frac{n}{100p^*}$
B – D	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	3 19,25	4 21,67	4 25,50	4 30,47	7 41,88
E	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	4 8,600	6 12,09	6 14,53	6 17,93	5 30,74	9 43,83
F	↓	↓	↓	↓	↓	↓	↓	↓	↓	6 5,220	8 7,090	9 8,717	9 10,82	6 19,46	7 31,49	8 41,30
G	↓	↓	↓	↓	↓	↓	↓	↓	9 3,279	11 4,162	13 5,195	13 6,466	9 11,43	9 19,61	9 27,43	12 40,88
H	↓	↓	↓	↓	↓	↓	↓	11 1,958	15 2,670	17 3,295	18 4,144	13 7,204	14 12,45	14 17,61	14 27,71	13 32,84
J	↓	↓	↓	↓	↓	↓	15 1,245	19 1,613	22 2,011	23 2,518	18 4,381	20 7,627	21 10,85	21 17,29	21 20,45	21 26,75
K	↓	↓	↓	↓	↓	18 ,7546	24 1,016	28 1,266	30 1,592	24 2,751	27 4,799	30 6,857	32 10,94	33 12,96	33 17,03	33 21,09
L	↓	↓	↓	↓	23 ,4753	30 ,6246	36 ,7878	38 ,9814	31 1,685	37 2,959	41 4,241	46 6,783	48 8,059	50 10,59	52 13,11	↑
M	↓	↓	↓	28 ,3027	37 ,3976	44 ,4976	47 ,6222	40 1,071	48 1,876	54 2,687	63 4,313	66 5,129	71 6,738	75 8,361	↑	↑
N	↓	↓	34 ,1880	44 ,2451	54 ,3105	58 ,3872	50 ,6625	61 1,162	71 1,667	84 2,681	90 3,182	99 4,192	105 5,204	↑	↑	↑
P	↓	40 ,1180	52 ,1540	64 ,1954	69 ,2436	60 ,4150	76 ,7336	89 1,052	108 1,694	117 2,012	131 2,654	143 3,290	↑	↑	↑	↑
Q	47 ,07418	61 ,09633	75 ,1217	82 ,1524	73 ,2605	93 ,4595	110 ,6602	137 1,063	149 1,264	169 1,666	186 2,069	↑	↑	↑	↑	↑
R	71 ,05982	88 ,07599	96 ,09473	86 ,1614	112 ,2852	134 ,4100	171 ,6611	187 ,7874	214 1,039	239 1,290	↑	↑	↑	↑	↑	↑

↓ There is no suitable plan in this area; use the first sampling plan below the arrow.  
If the sample size equals or exceeds the lot size, carry out 100 % inspection.

↑ There is no suitable plan in this area; use the first sampling plan above the arrow.

Reference ISO 3951-2:2013(E)

Table D.3

<b>Table O</b> Estimates of Nonconforming Percent $\hat{P}_U \& \hat{P}_L$ ISO3951 S Method form p																			
Qu or QL	n=5	n=7	n=10	n=15	n=20	n=30	n=40	n=50	n=100	Qu or QL	n=5	n=7	n=10	n=15	n=20	n=30	n=40	n=50	n=100
0	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	2.050	0	0.261	0.9398	1.3738	1.5590	1.7265	1.8045	1.8495	1.9361
0.05	48.2208	48.1290	48.0791	48.2208	48.0362	48.0250	48.0198	48.0169	48.0113	2.1000	0	0.1374	0.7421	1.1571	1.3369	1.5004	1.5766	1.6208	1.7057
0.10	46.4430	46.2607	46.1617	46.4430	46.0768	46.0545	46.0443	46.0385	46.0274	2.1500	0	0.0574	0.5755	0.9675	1.1405	1.2989	1.3730	1.4160	1.4989
0.15	44.6680	44.3978	44.2514	44.6680	44.1261	44.0931	44.0781	44.0695	44.0533	2.2000	0	0.0146	0.4371	0.8027	0.9677	1.1200	1.1917	1.2333	1.3137
0.20	42.8972	42.5431	42.3518	42.8972	42.1883	42.1454	42.1258	42.1147	42.0936	2.2500	0	0.0005	0.3240	0.6604	0.8164	0.9619	1.0307	1.0708	1.1483
0.25	41.1320	40.6993	40.4662	41.1320	40.2676	40.2156	40.1920	40.1785	40.1530	2.3000	0	0	0.2335	0.5385	0.6848	0.8227	0.8884	0.9267	1.0011
0.30	39.3738	38.8691	38.5981	39.3738	38.3682	38.3082	38.2809	38.2653	38.2360	2.3500	0	0	0.1626	0.4348	0.5708	0.7007	0.7630	0.7994	0.8704
0.35	37.6241	37.0551	36.7509	37.6241	36.4940	36.4272	36.3968	36.3795	36.3470	2.4000	0	0	0.1087	0.3475	0.4727	0.5941	0.6528	0.6873	0.7547
0.40	35.8843	35.2599	34.9279	35.8843	34.6488	34.5765	34.5438	34.5252	34.4901	2.4500	0	0	0.0691	0.2746	0.3887	0.5015	0.5565	0.5889	0.6525
0.45	34.1559	33.4863	33.1322	34.1559	32.8363	32.7601	32.7257	32.7061	32.6693	2.5000	0	0	0.0412	0.2144	0.3174	0.4213	0.4725	0.5029	0.5627
0.50	32.4404	31.7367	31.3669	32.4404	31.0600	30.9815	30.9461	30.9260	30.8883	2.5500	0	0	0.0226	0.1652	0.2572	0.3522	0.3997	0.4279	0.4838
0.55	30.7394	30.0137	29.6351	30.7394	29.3232	29.2440	29.2084	29.1882	29.1504	2.6000	0	0	0.0110	0.1254	0.2067	0.2930	0.3367	0.3628	0.4148
0.60	29.0545	28.3199	27.9395	29.0545	27.6289	27.5507	27.5157	27.4959	27.4589	2.6500	0	0	0.0045	0.0937	0.1647	0.2425	0.2824	0.3065	0.3547
0.65	27.3873	26.6576	26.2829	27.3873	25.9800	25.9045	25.8708	25.8518	25.8165	2.7000	0	0	0.0014	0.0688	0.1301	0.1996	0.2359	0.2580	0.3023
0.70	25.7395	25.0293	24.6678	25.7395	24.3790	24.3078	24.2763	24.2586	24.2258	2.7500	0	0	0.0003	0.0495	0.1017	0.1634	0.1962	0.2163	0.2570
0.75	24.1129	23.4374	23.0967	24.1129	22.8282	22.7631	22.7345	22.7185	22.6890	2.8000	0	0	0.0000	0.0348	0.0788	0.1330	0.1625	0.1807	0.2178
0.80	22.5092	21.8841	21.5718	22.5092	21.3297	21.2722	21.2472	21.2333	21.2079	2.8500	0	0	0	0.0239	0.0603	0.1076	0.1339	0.1503	0.1840
0.85	20.9305	20.3716	20.0952	20.9305	19.8853	19.8368	19.8161	19.8046	19.7840	2.9000	0	0	0	0.0160	0.0456	0.0865	0.1099	0.1245	0.1550
0.90	19.3785	18.9022	18.6687	19.3785	18.4965	18.4584	18.4424	18.4338	18.4186	2.9500	0	0	0	0.0103	0.0341	0.0691	0.0897	0.1028	0.1302
0.95	17.8556	17.4778	17.2942	17.8556	17.1645	17.1379	17.1273	17.1218	17.1126	3.0000	0	0	0	0.0064	0.0251	0.0549	0.0729	0.0844	0.1090
1.00	16.3638	16.1005	15.9729	16.3638	15.8902	15.8762	15.8714	15.8693	15.8664	3.0500	0	0	0	0.0038	0.0182	0.0432	0.0589	0.0691	0.0910
1.05	14.9056	14.7722	14.7064	14.9056	14.6743	14.6737	14.6751	14.6765	14.6805	3.1000	0	0	0	0.0022	0.0130	0.0338	0.0473	0.0563	0.0757
1.10	13.4834	13.4947	13.4956	13.4834	13.5172	13.5306	13.5385	13.5436	13.5546	3.1500	0	0	0	0.0012	0.0092	0.0263	0.0378	0.0456	0.0627
1.15	12.1001	12.2697	12.3414	12.1001	12.4191	12.4469	12.4614	12.4702	12.4884	3.2000	0	0	0	0.0006	0.0063	0.0202	0.0301	0.0368	0.0519
1.20	10.7585	11.0988	11.2445	10.7585	11.3796	11.4221	11.4433	11.4560	11.4814	3.2500	0	0	0	0.0003	0.0043	0.0154	0.0238	0.0296	0.0427
1.25	9.4619	9.9835	10.2053	9.4619	10.3986	10.4557	10.4836	10.5001	10.5327	3.3000	0	0	0	0.0001	0.0028	0.0117	0.0187	0.0236	0.0351
1.30	8.2140	8.9250	9.2240	8.2140	9.4752	9.5469	9.5813	9.6015	9.6412	3.3500	0	0	0	0.0000	0.0018	0.0088	0.0146	0.0188	0.0287
1.35	7.0187	7.9247	8.3007	7.0187	8.6088	8.6945	8.7352	8.7591	8.8056	3.4000	0	0	0	0.0000	0.0011	0.0065	0.0113	0.0149	0.0234
1.40	5.8807	6.9836	7.4350	5.8807	7.7981	7.8973	7.9441	7.9714	8.0245	3.4500	0	0	0	0.0000	0.0007	0.0048	0.0087	0.0117	0.0190
1.45	4.8053	6.1026	6.6266	4.8053	7.0419	7.1539	7.2064	7.2370	7.2961	3.5000	0	0	0	0.0000	0.0004	0.0035	0.0067	0.0092	0.0154
1.50	3.7988	5.2825	5.8747	3.7988	6.3387	6.4625	6.5204	6.5539	6.6187	3.5500	0	0	0	0.0000	0.0002	0.0025	0.0051	0.0071	0.0124
1.55	2.8691	4.5237	5.1785	2.8691	5.6870	5.8215	5.8842	5.9205	5.9904	3.6000	0	0	0	0.0000	0.0001	0.0018	0.0038	0.0055	0.0100
1.60	2.0260	3.8267	4.5369	2.0260	5.0849	5.2290	5.2959	5.3346	5.4091	3.6500	0	0	0	0	0.0001	0.0013	0.0029	0.0043	0.0080
1.65	1.2828	3.1916	3.9485	1.2828	4.5306	4.6829	4.7535	4.7943	4.8727	3.7000	0	0	0	0	0.0000	0.0009	0.0021	0.0033	0.0064
1.70	0.6595	2.6182	3.4118	0.6595	4.0220	4.1812	4.2549	4.2974	4.3791	3.7500	0	0	0	0	0.0000	0.0006	0.0016	0.0025	0.0051
1.75	0.1915	2.1062	2.9252	0.1915	3.5570	3.7217	3.7978	3.8418	3.9260	3.8000	0	0	0	0	0.0000	0.0004	0.0012	0.0019	0.0040
1.80	0	1.6549	2.4868	0	3.1335	3.3022	3.3801	3.4251	3.5113	3.8500	0	0	0	0	0.0000	0.0003	0.0008	0.0014	0.0032
1.85	0	1.2634	2.0944	0	2.7492	2.9204	2.9996	3.0453	3.1328	3.9000	0	0	0	0	0.0000	0.0002	0.0006	0.0010	0.0025
1.90	0	0.9302	1.7460	0	2.4019	2.5742	2.6540	2.7000	2.7881	3.9500	0	0	0	0	0.0000	0.0001	0.0004	0.0008	0.0020
1.95	0	0.6537	1.4390	0	2.0893	2.2613	2.3410	2.3870	2.4752	4.0000	0	0	0	0	0.0000	0.0001	0.0003	0.0006	0.0015
2.00	0	0.4314	1.1712	0	1.8091	1.9795	2.0586	2.1042	2.1919	Phat=(100*betacdf(.5*(1-Q*(1-Q^.5)/(n-1)),(n-2)/2,(n-2)/2)									

**Table P.1** Values of  $f_s$  in S--Method Double Specification

Normal

Code letter	Acceptance quality limit (in percent nonconforming)															
	0,010	0,015	0,025	0,040	0,065	0,10	0,15	0,25	0,40	0,65	1,0	1,5	2,5	4,0	6,5	10,0
	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$
B	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0,475	0,447	0,479
C	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0,365	0,366	0,484
D	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0,303	0,312	0,494
E	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0,265	0,274	0,285	0,333	0,395	0,458
F	↓	↓	↓	↓	↓	↓	↓	↓	↓	0,241	0,248	0,257	0,292	0,334	0,375	0,461
G	↓	↓	↓	↓	↓	↓	↓	↓	0,221	0,227	0,234	0,260	0,290	0,318	0,371	0,424
H	↓	↓	↓	↓	↓	↓	↓	0,206	0,211	0,216	0,237	0,260	0,280	0,316	0,350	0,401
I	↓	↓	↓	↓	↓	0,192	0,197	0,201	0,218	0,236	0,251	0,277	0,301	0,333	0,376	
K	↓	↓	↓	↓	0,182	0,185	0,189	0,203	0,218	0,230	0,250	0,268	0,291	0,319	0,367	
L	↓	↓	↓	0,172	0,175	0,179	0,190	0,203	0,212	0,229	0,242	0,259	0,279	0,312	↑	↑
M	↓	↓	0,164	0,167	0,170	0,180	0,190	0,199	0,212	0,222	0,236	0,251	0,275	↑	↑	↑
N	↓	0,157	0,160	0,162	0,171	0,180	0,187	0,198	0,206	0,217	0,230	0,248	↑	↑	↑	↑
P	0,145	0,151	0,153	0,155	0,163	0,171	0,177	0,186	0,193	0,202	0,212	0,226	↑	↑	↑	↑
Q	0,147	0,149	0,156	0,163	0,168	0,176	0,183	0,190	0,199	0,210	↑	↑	↑	↑	↑	↑
R	0,142	0,144	0,150	0,156	0,161	0,168	0,173	0,180	0,187	0,196	↑	↑	↑	↑	↑	↑

NOTE The MSSD is obtained by multiplying the standardized MSSD  $f_s$  by the difference between the upper specification limit,  $U$ , and the lower specification limit,  $L$ , i.e.  $MSSD = s_{\max} = (U - L)f_s$ .

Ref: ISO 3951-2:2013-E Table F.1

Table P.2 Values of  $f_s$  in S-Method Double Specification

Tightened

Code letter	Acceptance quality limit (in percent nonconforming)															
	0,010	0,015	0,025	0,040	0,065	0,10	0,15	0,25	0,40	0,65	1,0	1,5	2,5	4,0	6,5	10,0
	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$
B	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0,475	0,447
C	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0,365	0,388
D	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0,303	0,312	0,328
E	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0,265	0,274	0,285	0,333	0,395
F	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0,241	0,248	0,257	0,292	0,334	0,375
G	↓	↓	↓	↓	↓	↓	↓	↓	↓	0,221	0,227	0,234	0,260	0,290	0,318	0,371
H	↓	↓	↓	↓	↓	↓	↓	↓	0,206	0,211	0,216	0,237	0,260	0,280	0,316	0,367
J	↓	↓	↓	↓	↓	↓	↓	0,192	0,197	0,201	0,218	0,236	0,251	0,277	0,312	0,354
K	↓	↓	↓	↓	↓	↓	0,182	0,185	0,189	0,203	0,218	0,230	0,250	0,276	0,305	0,347
L	↓	↓	↓	↓	↓	0,172	0,175	0,179	0,190	0,203	0,212	0,229	0,248	0,269	0,298	↑
M	↓	↓	↓	↓	0,164	0,167	0,170	0,180	0,190	0,199	0,212	0,227	0,244	0,265	↑	↑
N	↓	↓	↓	0,157	0,160	0,162	0,171	0,180	0,187	0,198	0,210	0,224	0,240	↑	↑	↑
P	↓	↓	0,151	0,153	0,155	0,163	0,171	0,177	0,186	0,196	0,207	0,221	↑	↑	↑	↑
Q	↓	0,145	0,147	0,149	0,156	0,163	0,168	0,176	0,185	0,195	0,206	↑	↑	↑	↑	↑
R	0,140	0,142	0,144	0,150	0,156	0,161	0,168	0,175	0,183	0,192	↑	↑	↑	↑	↑	↑

Reference Table F.2 ISO 3951-2:2013(E)

# Table P.3 Values of $f_s$ in S--Method Double Specification Reduced

Code letter	Acceptance quality limit (in percent nonconforming)															
	0,010	0,015	0,025	0,040	0,065	0,10	0,15	0,25	0,40	0,65	1,0	1,5	2,5	4,0	6,5	10,0
	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$	$f_s$
B-D	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	▼	0,475	0,426	0,447	0,479	0,602
E	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	0,365	0,350	0,366	0,388	0,484	0,632
F	↓	↓	↓	↓	↓	↓	↓	↓	↓	0,303	0,303	0,312	0,328	0,399	0,494	0,598
G	↓	↓	↓	↓	↓	↓	↓	↓	0,265	0,267	0,274	0,285	0,333	0,395	0,458	0,599
H	↓	↓	↓	↓	↓	↓	↓	0,241	0,243	0,248	0,257	0,292	0,334	0,375	0,461	0,510
J	↓	↓	↓	↓	↓	↓	0,221	0,223	0,227	0,234	0,260	0,290	0,318	0,371	0,397	0,452
K	↓	↓	↓	↓	↓	0,206	0,207	0,211	0,216	0,237	0,260	0,280	0,316	0,333	0,367	0,401
L	↓	↓	↓	↓	0,192	0,194	0,197	0,202	0,218	0,233	0,251	0,277	0,289	0,312	0,333	↑
M	↓	↓	↓	0,182	0,183	0,185	0,189	0,203	0,218	0,230	0,250	0,259	0,276	0,291	↑	↑
N	↓	↓	0,172	0,173	0,175	0,179	0,190	0,203	0,212	0,229	0,235	0,248	0,259	↑	↑	↑
P	↓	0,164	0,165	0,167	0,170	0,180	0,190	0,199	0,212	0,217	0,227	0,236	↑	↑	↑	↑
Q	0,157	0,158	0,160	0,162	0,171	0,180	0,187	0,198	0,202	0,210	0,217	↑	↑	↑	↑	↑
R	0,151	0,153	0,155	0,163	0,171	0,177	0,186	0,190	0,196	0,202	↑	↑	↑	↑	↑	↑

Reference ISO 3951-  
Table F.3

Table Q.1 Single sampling  $\sigma$ -Method form k

## Normal

Code letter	Acceptance quality limit (in percent nonconforming)															
	0,01 $n$ $k$	0,015 $n$ $k$	0,025 $n$ $k$	0,04 $n$ $k$	0,065 $n$ $k$	0,10 $n$ $k$	0,15 $n$ $k$	0,25 $n$ $k$	0,40 $n$ $k$	0,65 $n$ $k$	1,0 $n$ $k$	1,5 $n$ $k$	2,5 $n$ $k$	4,0 $n$ $k$	6,5 $n$ $k$	10,0 $n$ $k$
B	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	3 0,709	4 0,571	3 0,417
C	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	3 1,115	5 0,945	5 0,821	4 0,436
D	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	4 1,406	6 1,240	6 1,128	5 0,770	5 0,431
E	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	4 1,595	7 1,506	8 1,419	7 1,115	7 0,792	7 0,555
F	↓	↓	↓	↓	↓	↓	↓	↓	↓	5 1,845	8 1,720	9 1,635	8 1,366	10 1,094	9 0,877	11 0,564
G	↓	↓	↓	↓	↓	↓	↓	↓	5 2,006	9 1,934	10 1,856	9 1,610	12 1,370	13 1,186	13 0,906	15 0,694
H	↓	↓	↓	↓	↓	↓	↓	6 2,218	10 2,122	11 2,046	10 1,820	13 1,599	16 1,439	16 1,191	19 1,009	23 0,786
J	↓	↓	↓	↓	↓	7 2,401	11 2,302	12 2,234	11 2,025	15 1,823	19 1,677	21 1,456	24 1,293	29 1,102	34 0,897	
K	↓	↓	↓	↓	7 2,541	12 2,468	13 2,401	13 2,210	17 2,018	21 1,882	27 1,683	29 1,533	35 1,361	42 1,182	53 0,937	
L	↓	↓	↓	8 2,710	13 2,629	15 2,573	14 2,387	19 2,209	24 2,083	32 1,900	34 1,761	42 1,606	52 1,446	66 1,231	↑	
M	↓	↓	8 2,844	14 2,780	16 2,726	15 2,550	21 2,382	27 2,264	36 2,092	39 1,963	50 1,821	61 1,674	79 1,481	↑	↑	
N	↓	10 3,141	17 3,069	19 3,023	19 2,865	26 2,711	33 2,603	45 2,450	51 2,337	65 2,212	82 2,086	110 1,923	↑	↑	↑	
P	↓	10 3,141	17 3,069	19 3,023	19 2,865	26 2,711	33 2,603	45 2,450	51 2,337	65 2,212	82 2,086	110 1,923	↑	↑	↑	
Q	11 3,275	18 3,207	20 3,155	20 3,002	28 2,856	35 2,752	49 2,607	57 2,500	72 2,381	92 2,262	125 2,110	↑	↑	↑	↑	
R	19 3,339	21 3,289	22 3,145	30 3,002	38 2,903	54 2,764	64 2,663	81 2,550	105 2,438	142 2,294	↑	↑	↑	↑	↑	



There is no suitable plan in this area; use the first sampling plan below the arrow.  
If the sample size equals or exceeds the lot size, carry out 100 % inspection.



There is no suitable plan in this area; use the first sampling plan above the arrow.

Reference ISO 3951-2:2013(E)

Table C.1

**Table Q.2** Single sampling  $\sigma$ -Method Form k  
Tightened

Code letter	Acceptance quality limit (in percent nonconforming)															
	0,01	0,015	0,025	0,04	0,065	0,10	0,15	0,25	0,40	0,65	1,0	1,5	2,5	4,0	6,5	10,0
	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>
B	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	3 0,709	4 0,571
C	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	3 1,115	5 0,945
D	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	4 1,406	6 1,240	5 0,770
E	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	4 1,595	7 1,506	8 1,419	7 1,115	7 0,792
F	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	5 1,845	8 1,720	9 1,635	8 1,366	10 1,094	9 0,877
G	↓	↓	↓	↓	↓	↓	↓	↓	↓	5 2,006	9 1,934	10 1,856	9 1,610	12 1,370	13 1,186	13 0,906
H	↓	↓	↓	↓	↓	↓	↓	↓	6 2,218	10 2,122	11 2,046	10 1,820	13 1,599	16 1,439	16 1,191	20 0,929
J	↓	↓	↓	↓	↓	↓	↓	7 2,401	11 2,302	12 2,234	11 2,025	15 1,823	19 1,677	21 1,456	25 1,223	32 0,994
K	↓	↓	↓	↓	↓	↓	7 2,541	12 2,468	13 2,401	13 2,210	17 2,018	21 1,882	27 1,683	31 1,471	39 1,267	49 1,035
L	↓	↓	↓	↓	↓	8 2,710	13 2,629	15 2,573	14 2,387	19 2,209	24 2,083	32 1,900	37 1,705	47 1,521	61 1,316	↑
M	↓	↓	↓	↓	8 2,844	14 2,780	16 2,726	15 2,550	21 2,382	27 2,264	36 2,092	43 1,912	55 1,742	72 1,556	↑	↑
N	↓	↓	↓	9 2,996	15 2,929	17 2,874	17 2,709	24 2,550	30 2,437	40 2,274	49 2,106	65 1,950	85 1,779	↑	↑	↑
P	↓	↓	10 3,142	17 3,076	19 3,023	19 2,865	26 2,711	33 2,603	45 2,450	55 2,291	74 2,145	99 1,987	↑	↑	↑	↑
Q	↓	11 3,275	18 3,207	20 3,155	20 3,002	28 2,856	35 2,752	49 2,607	61 2,456	83 2,318	112 2,169	↑	↑	↑	↑	↑
R	14 3,391	19 3,339	21 3,289	22 3,145	30 3,002	38 2,903	54 2,764	68 2,621	92 2,490	126 2,350	↑	↑	↑	↑	↑	↑

↓ There is no suitable plan in this area; use the first sampling plan below the arrow.  
If the sample size equals or exceeds the lot size, carry out 100 % inspection.

↑ There is no suitable plan in this area; use the first sampling plan above the arrow.

Reference **ISO 3951-2:2013(E)**  
**Table C.2**



# Table Q.3 Single sampling $\sigma$ -Method form k

## Reduced

Code letter	Acceptance quality limit (in percent nonconforming)															
	0,01	0,015	0,025	0,04	0,065	0,10	0,15	0,25	0,40	0,65	1,0	1,5	2,5	4,0	6,5	10,0
	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>
B – D	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	3 0,709	4 0,679	4 0,571	3 0,417	6 0,187
E	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	3 1,115	5 1,047	5 0,945	5 0,821	4 0,436	8 0,145
F	↓	↓	↓	↓	↓	↓	↓	↓	↓	4 1,406	5 1,314	6 1,240	6 1,128	5 0,770	5 0,431	7 0,204
G	↓	↓	↓	↓	↓	↓	↓	↓	4 1,595	6 1,581	7 1,506	8 1,419	7 1,115	7 0,792	7 0,555	11 0,220
H	↓	↓	↓	↓	↓	↓	↓	5 1,845	7 1,788	8 1,720	9 1,635	8 1,366	10 1,094	9 0,877	11 0,564	11 0,424
J	↓	↓	↓	↓	↓	↓	5 2,006	7 1,982	9 1,934	10 1,856	9 1,610	12 1,370	13 1,186	13 0,906	14 0,796	16 0,601
K	↓	↓	↓	↓	↓	6 2,218	8 2,171	10 2,122	11 2,046	10 1,820	13 1,599	16 1,439	16 1,191	18 1,096	20 0,929	23 0,786
L	↓	↓	↓	↓	7 2,401	9 2,355	11 2,302	12 2,234	11 2,025	15 1,823	19 1,677	21 1,456	22 1,369	25 1,223	29 1,102	↑
M	↓	↓	↓	7 2,541	10 2,518	12 2,468	13 2,401	13 2,210	17 2,018	21 1,882	27 1,683	26 1,601	31 1,471	35 1,361	↑	↑
N	↓	↓	8 2,710	10 2,669	13 2,629	15 2,573	14 2,387	19 2,209	24 2,083	32 1,900	31 1,825	37 1,705	42 1,606	↑	↑	↑
P	↓	8 2,844	11 2,822	14 2,780	16 2,726	15 2,550	21 2,382	27 2,264	36 2,092	38 2,024	43 1,912	50 1,821	↑	↑	↑	↑
Q	9 2,996	12 2,969	15 2,929	17 2,874	17 2,709	24 2,550	30 2,437	40 2,274	45 2,212	49 2,106	57 2,022	↑	↑	↑	↑	↑
R	13 3,113	17 3,076	19 3,023	19 2,865	26 2,711	33 2,603	45 2,450	50 2,390	55 2,291	65 2,212	↑	↑	↑	↑	↑	↑

↓ There is no suitable plan in this area; use the first sampling plan below the arrow.  
If the sample size equals or exceeds the lot size, carry out 100 % inspection.

↑ There is no suitable plan in this area; use the first sampling plan above the arrow.

Reference ISO 3951-2:2013(E)  
Table C.3

Table T Single Sampling S-Method form k

Normal

Code letter	Acceptance quality limit (in percent nonconforming)															
	0,01	0,015	0,025	0,04	0,065	0,10	0,15	0,25	0,40	0,65	1,0	1,5	2,5	4,0	6,5	10,0
	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>
B	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	3 0,950	4 0,735	4 0,586
C	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	4 1,242	6 1,061	5 0,939
D	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	6 1,476	9 1,323	9 1,218	7 0,887
E	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	9 1,696	13 1,569	13 1,475	9 1,190	9 0,869	9 0,618
F	↓	↓	↓	↓	↓	↓	↓	↓	↓	11 1,889	17 1,769	18 1,682	13 1,426	14 1,147	14 0,935	14 0,601
G	↓	↓	↓	↓	↓	↓	↓	↓	15 2,079	22 1,972	23 1,893	18 1,659	20 1,411	21 1,227	21 0,945	21 0,724
H	↓	↓	↓	↓	↓	↓	↓	18 2,254	28 2,153	30 2,079	24 1,862	27 1,636	30 1,471	32 1,225	33 1,036	33 0,806
J	↓	↓	↓	↓	↓	23 2,425	36 2,331	38 2,263	31 2,061	37 1,853	41 1,702	46 1,482	49 1,316	52 1,120	53 0,911	
K	↓	↓	↓	↓	28 2,580	44 2,493	47 2,428	40 2,237	48 2,043	54 1,904	63 1,702	69 1,552	75 1,377	79 1,195	82 0,946	
L	↓	↓	↓	↓	34 2,737	54 2,653	58 2,592	50 2,412	61 2,230	71 2,101	84 1,914	94 1,777	105 1,619	115 1,456	124 1,239	↑
M	↓	↓	↓	40 2,882	64 2,802	69 2,744	60 2,573	76 2,400	89 2,279	108 2,104	124 1,977	143 1,832	159 1,683	178 1,488	↑	↑
N	↓	↓	47 3,023	75 2,948	82 2,892	71 2,728	93 2,564	110 2,449	137 2,285	159 2,166	186 2,031	213 1,894	247 1,716	↑	↑	↑
P	63 3,288	101 3,219	110 3,167	102 3,016	132 2,867	159 2,762	207 2,615	244 2,508	293 2,388	348 2,268	424 2,114	↑	↑	↑	↑	↑
Q	116 3,351	127 3,301	120 3,156	155 3,012	189 2,912	247 2,771	298 2,670	362 2,556	438 2,443	541 2,298	↑	↑	↑	↑	↑	↑
R																

↓ There is no suitable plan in this area; use the first sampling plan below the arrow. If the sample size equals or exceeds the lot size, carry out 100 % inspection.

↑ There is no suitable plan in this area; use the first sampling plan above the arrow.

Reference ISO 3951-1

Table B.1

Table T.1 Single Sampling S-Method form K

Tightened

Code letter	Acceptance quality limit (in percent nonconforming)															
	0.01	0.015	0.025	0.04	0.065	0.10	0.15	0.25	0.40	0.65	1.0	1.5	2.5	4.0	6.5	10.0
	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>
B	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	3 0.950	4 0.735
C	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	4 1.242	6 0.939
D	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	6 1.476	9 1.323	6 0.887
E	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	9 1.696	13 1.569	13 1.475	9 1.190	9 0.869
F	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	11 1.889	17 1.769	18 1.682	13 1.426	14 1.147	14 0.935
G	↓	↓	↓	↓	↓	↓	↓	↓	↓	15 2.079	22 1.972	23 1.893	18 1.659	20 1.411	21 1.227	21 0.945
H	↓	↓	↓	↓	↓	↓	↓	↓	18 2.254	28 2.153	30 2.079	24 1.862	27 1.636	30 1.471	32 1.225	33 0.954
J	↓	↓	↓	↓	↓	↓	↓	23 2.425	36 2.331	38 2.263	31 2.061	37 1.853	41 1.702	46 1.482	50 1.245	53 1.010
K	↓	↓	↓	↓	↓	↓	28 2.580	44 2.493	47 2.428	40 2.237	48 2.043	54 1.904	63 1.702	71 1.489	78 1.281	82 1.045
L	↓	↓	↓	↓	40 2.737	64 2.653	69 2.592	60 2.412	76 2.230	89 2.101	108 1.914	131 1.720	150 1.533	170 1.325	122 1.325	↑
M	↓	↓	↓	47 2.882	75 2.802	82 2.744	73 2.573	93 2.400	110 2.279	137 2.104	169 1.924	201 1.752	233 1.564	↑	↑	↑
N	↓	↓	55 3.023	88 2.948	96 2.892	86 2.728	112 2.564	134 2.449	171 2.285	214 2.117	260 1.958	312 1.785	↑	↑	↑	↑
P	↓	63 3.161	101 3.089	110 3.036	102 2.879	132 2.723	159 2.614	207 2.459	262 2.300	323 2.152	395 1.992	↑	↑	↑	↑	↑
Q	↓	116 3.288	127 3.219	120 3.167	155 3.016	189 2.867	247 2.762	320 2.615	398 2.464	498 2.324	↑	↑	↑	↑	↑	↑
R	90 3.408	116 3.351	127 3.301	120 3.156	155 3.012	189 2.912	247 2.771	320 2.628	398 2.495	498 2.354	↑	↑	↑	↑	↑	↑

↓ There is no suitable plan in this area; use the first sampling plan below the arrow. If the sample size equals or exceeds the lot size, carry out 100 % inspection.

↑ There is no suitable plan in this area; use the first sampling plan above the arrow.

Reference ISO 3951-1

Table B.2

# Table T.2 Single Sampling S-Method form k

Reduced

Code letter	Acceptance quality limit (in percent nonconforming)															
	0,01	0,015	0,025	0,04	0,065	0,10	0,15	0,25	0,40	0,65	1,0	1,5	2,5	4,0	6,5	10,0
	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>	<i>n</i> <i>k</i>
B - D	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	3 0,950	4 0,850	4 0,735	4 0,586	7 0,218
E	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	4 1,242	6 1,155	6 1,061	6 0,939	5 0,550	9 0,162
F	↓	↓	↓	↓	↓	↓	↓	↓	↓	6 1,476	8 1,406	9 1,323	9 1,218	6 0,887	7 0,507	8 0,231
G	↓	↓	↓	↓	↓	↓	↓	↓	9 1,696	11 1,642	13 1,569	13 1,475	9 1,190	9 0,869	9 0,618	12 0,237
H	↓	↓	↓	↓	↓	↓	↓	11 1,889	15 1,835	17 1,769	18 1,682	13 1,426	14 1,147	14 0,935	14 0,601	13 0,454
J	↓	↓	↓	↓	↓	↓	15 2,079	19 2,033	22 1,972	23 1,893	18 1,659	20 1,411	21 1,227	21 0,945	21 0,830	21 0,626
K	↓	↓	↓	↓	↓	18 2,254	24 2,209	28 2,153	30 2,079	24 1,862	27 1,636	30 1,471	32 1,225	33 1,126	33 0,954	33 0,806
L	↓	↓	↓	↓	23 2,425	30 2,385	36 2,331	38 2,263	31 2,061	37 1,853	41 1,702	46 1,482	48 1,394	50 1,245	52 1,120	↑
M	↓	↓	↓	28 2,580	37 2,543	44 2,493	47 2,428	40 2,237	48 2,043	54 1,904	63 1,702	66 1,622	71 1,489	75 1,377	↑	↑
N	↓	↓	34 2,737	44 2,701	54 2,653	58 2,592	50 2,412	61 2,230	71 2,101	84 1,914	90 1,842	99 1,720	105 1,619	↑	↑	↑
P	↓	40 2,882	52 2,848	64 2,802	69 2,744	60 2,573	76 2,400	89 2,279	108 2,104	117 2,037	131 1,924	143 1,832	↑	↑	↑	↑
Q	47 3,023	61 2,991	75 2,948	82 2,892	73 2,728	93 2,564	110 2,449	137 2,285	149 2,222	169 2,117	186 2,031	↑	↑	↑	↑	↑
R	71 3,131	88 3,089	96 3,036	86 2,879	112 2,723	134 2,614	171 2,459	187 2,399	214 2,300	239 2,220	↑	↑	↑	↑	↑	↑

↓ There is no suitable plan in this area; use the first sampling plan below the arrow. If the sample size equals or exceeds the lot size, carry out 100 % inspection.

↑ There is no suitable plan in this area; use the first sampling plan above the arrow.

Refrence **ISO 3951-1:2013(E)**

Table B.3

# Table V

## Supplementary acceptability constants for qualifying towards reduced inspection in S & $\sigma$ Methods

Sample size code letter	AQL (%)	Acceptability constant for AQL that is one step tighter		
		<i>s</i> -method		$\sigma$ -method
		<i>k</i>	<i>p</i> <sup>*</sup> (%)	<i>k</i>
B	4,0	1,114	8,502	0,918
C	2,5	1,409	3,041	1,325
D	1,5	1,601	<i>n/a</i>	1,562
E	1,0	1,825	<i>n/a</i>	1,752
F	0,65	2,029	<i>n/a</i>	2,013
G	0,40	2,209	<i>n/a</i>	2,161
H	0,25	2,390	<i>n/a</i>	2,379
J	0,15	2,530	<i>n/a</i>	2,523
K	0,10	2,689	<i>n/a</i>	2,667
L	0,065	2,857	<i>n/a</i>	2,847
M	0,040	2,995	<i>n/a</i>	2,972
N	0,025	3,143	<i>n/a</i>	3,131
P	0,015	3,254	<i>n/a</i>	3,246
Q	0,010	3,385	<i>n/a</i>	3,382
R	0,010	3,449	<i>n/a</i>	3,446
NOTE For this part of ISO 3951, n/a is “not applicable.”				

Reference ISO 3951-1:2013(E)  
Table I.1



**Table U** Coefficients for  $\bar{X}$ , S & R Control Charts

n	$\bar{X}$ Control Chart			S Control Chart							R Control Chart					n
	A	A <sub>2</sub>	A <sub>3</sub>	C <sub>4</sub>	B <sub>1</sub>	B <sub>2</sub>	B <sub>3</sub>	B <sub>4</sub>	B <sub>5</sub>	B <sub>6</sub>	d <sub>2</sub>	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	
2	2.121	1.880	2.659	0.7979	0	1.843	0	3.267	0	2.606	1.128	0	3.686	0	3.267	2
3	1.732	1.023	1.954	0.8862	0	1.858	0	2.568	0	2.276	1.693	0	4.358	0	2.575	3
4	1.500	0.729	1.628	0.9213	0	1.808	0	2.266	0	2.088	2.059	0	4.698	0	2.282	4
5	1.342	0.577	1.427	0.9400	0	1.756	0	2.089	0	1.964	2.326	0	4.918	0	2.114	5
6	1.225	0.483	1.287	0.9515	0.026	1.711	0.03	1.970	0.029	1.874	2.534	0	5.078	0	2.004	6
7	1.134	0.419	1.182	0.9594	0.105	1.672	0.118	1.882	0.113	1.806	2.704	0.204	5.204	0.076	1.924	7
8	1.061	0.373	1.099	0.9650	0.167	1.638	0.185	1.815	0.179	1.751	2.847	0.388	5.306	0.136	1.864	8
9	1.000	0.337	1.032	0.9693	0.219	1.609	0.239	1.761	0.232	1.707	2.970	0.547	5.393	0.184	1.816	9
10	0.949	0.308	0.975	0.9727	0.262	1.584	0.284	1.716	0.276	1.669	3.078	0.687	5.469	0.223	1.777	10
11	0.905	0.285	0.927	0.9754	0.299	1.561	0.321	1.676	0.313	1.637	3.173	0.811	5.535	0.256	1.744	11
12	0.866	0.266	0.886	0.9776	0.331	1.541	0.354	1.646	0.346	1.610	3.258	0.922	5.594	0.283	1.717	12
13	0.832	0.249	0.850	0.9794	0.359	1.523	0.382	1.618	0.374	1.585	3.336	1.025	5.647	0.307	1.692	13
14	0.802	0.235	0.817	0.9810	0.384	1.507	0.406	1.594	0.399	1.563	3.407	1.118	5.696	0.328	1.671	14
15	0.775	0.223	0.789	0.9823	0.406	1.492	0.428	1.572	0.421	1.544	3.472	1.203	5.741	0.347	1.652	15
16	0.750	0.212	0.763	0.9835	0.427	1.478	0.448	1.552	0.440	1.526	3.532	1.282	5.782	0.363	1.636	16
17	0.728	0.203	0.739	0.9845	0.445	1.465	0.466	1.534	0.458	1.511	3.588	1.356	5.820	0.378	1.621	17
18	0.707	0.194	0.718	0.9854	0.461	1.454	0.482	1.518	0.475	1.496	3.640	1.424	5.856	0.391	1.608	18
19	0.688	0.187	0.698	0.9862	0.477	1.443	0.497	1.503	0.490	1.483	3.689	1.487	5.891	0.403	1.596	19
20	0.671	0.180	0.680	0.9869	0.491	1.433	0.510	1.490	0.504	1.470	3.735	1.549	5.921	0.415	1.585	20
21	0.655	0.173	0.663	0.9876	0.504	1.424	0.523	1.477	0.516	1.459	3.778	1.605	5.951	0.425	1.575	21
22	0.640	0.167	0.647	0.9882	0.516	1.415	0.534	1.466	0.528	1.448	3.819	1.659	5.979	0.434	1.566	22
23	0.626	0.162	0.633	0.9887	0.527	1.407	0.545	1.455	0.539	1.438	3.858	1.710	6.006	0.443	1.557	23
24	0.612	0.157	0.619	0.9892	0.538	1.399	0.555	1.445	0.549	1.429	3.895	1.759	6.031	0.451	1.548	24
25	0.600	0.153	0.606	0.9896	0.548	1.392	0.565	1.435	0.559	1.420	3.931	1.806	6.056	0.459	1.541	25
40	0.474	0.110	0.477	0.9936	0.65	1.31	0.659	1.341	0.655	1.332	4.322					40
50	0.424	0.094	0.426	0.9949	0.68	1.28	0.696	1.304	0.693	1.297	4.498					50
60	0.387	0.084	0.389	0.9958	0.71	1.26	0.723	1.278	0.720	1.271	4.639					60
80	0.335	0.069	0.337	0.9968	0.75	1.23	0.761	1.239	0.759	1.235	4.854					80
100	0.300	0.060	0.300	0.9975	0.78	1.20	0.786	1.214	0.785	1.210	5.015					100