Hamid
Bazargan

Classical Topics

In

Inventory

Control

November 2021

Classical Topics   in
Inventory Control

A=High value items

B=Medium value items

C=Low value items
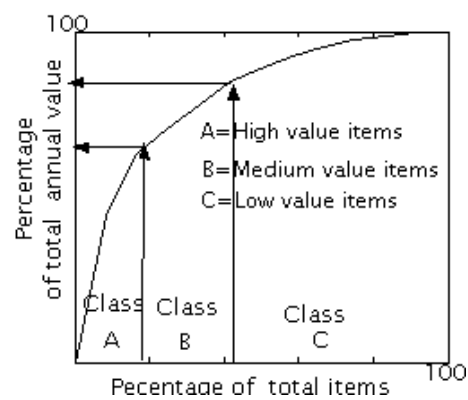
Hamid   Bazargan

To my parents

The late   Mohammad Ali Bazargan   (1905-1967)

&

The late   Robabeh   Eslampanah   (1921-1999)

CHAPTER   TITLES

C O N T E N T S

F O R E W O R D

This book[1] is the outcome of teaching a course titled "Inventory planning and control" for several years to B.S. students using many books especially the book written by Dr Tersine.

Thanks God for making me successful to present this work which I hope to be useful in both academic and industrial environments.

The book covers the classic topics in inventory control as well as some demand forecasting methods. The Persian version of the book has a chapter on MRP. But the author did not translate the chapter because there are many works available in the internet and in books.

 Mr Masoud Hajghani gets the credit for the last part of Chapter 6 i.e verifying the forecasts.   I would like to thank Mr Ali Bazargan who helped the author in some phases of editing.

The author would be pleased if the readers write him about any kind of deficiencies of the book.

> Hamid Bazargan
> College of Engineering,
> Shahid Bahonar University of Kerman , Iran
> bazargan@uk.ac.ir
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[1] The Persian version of book wriiten by the same author has been published by Shahid Bahonar Univerisity of Kerman, Iran in October 2021

Symbols and abbreviations	
A	
The increase in price from a future dats	a
The current level of inventory., the level of inventory before ordering at the period	A
Accumulated Part-Period	APP
Artificial Neural Networks	ANN
B	
Maximum back-ordered demand	B
Average shortage per unit time	$\bar{b}$
Optimal value of b	b^*
Amount of the shortage during the period	b(x)
Average of shortage in each cycle in (r , Q) model	$\bar{b}(r)$
Annual Average of shortage in (r , Q) model	$\bar{B}(r)$
Average of shortage in each cycle in (R , T) model	$\overline{b}(R)$
Annual Average of shortage in (R , T) model	$\overline{B}(R)$
C	
Cost of holding per unit product per unit time	Ch
Cost of Each Order or setup	C_O
Estimated cost of order	C_O'
Estimated Ch	C_h'
Cost of holding per unit product per unit time for ith product	$(C_h)_i$
Setup Cost for i^{th} product	$(C_O)_i$
Setup/order cost for period t	$(C_o)_t$
Per unit cost of holding for period t(at the end of period). $(C_h)_t$ for each t might be different	$(C_h)_t$

Symbols and abbreviations	
D	
a temporary special reduction of price d per unit.	d
Amount of demand or requirement	D
The average of deviation between observed and predicted values	$\overline{D}$
Demand for period t (t=1,2,…T)	D_t
Estimated amount of demand	D'
Annual demand for i^{th} product	Di
Daily rate of consumption for product i	$d_i = \frac{D_i}{N}$
consumption during lead time (T_L)	D_L
consumption during (T+L)	D_{T+L}
E	
Economic Order Quannty	EOQ
Economic Order Interval	EOI
Economic Production Quantity	EPQ
Economic Part Period	PP
Desired maximum of inventory	E
Forecast error of time t	e_t
F	
Fixed order size	FOS
A fraction of time(year) no shortage happens	$f_\circ$
Fixed order Interval	FOI
Fixed Order Period	FOP
Fixed order quantity	FOQ
Fixed Period Requirement	FPR
Demand probability density function	f(x) ↵ $f_D(x)$
Probability density of D_L	$f_{D_L}(x)$

Symbols and abbreviations	
Cum. dist func.	$F(x)$
Cum. dist func. Of variable X at point x	$F_X(t)$
G , H	
Saving in Sale Model	G
Optimal value of G	G^*
Normal Loss integral	$G_U(k)$
Stockout cost per outage	g
Per unit disposal cost at the end of period	H'
Actual cost of holding one unit available at the end of the period	H
I	
Holding cost rate, per unit cost of holding \$1in unit time	I
Average inventory in the warehouse	$\bar{I}$
The amount of inventory at the end of the period	t
Maximum of inventory	Max
The optimal value of Max	(I^*_{Max})
Average inventory in the warehouse for ith product	$\bar{I}_i$
Incremental Part- Pperiods	IPP
Incremental Part Period Algorithm	IPPA
K , L	
Average cost during period T' if a special order of size Q' is not placed.	K
average cost during period T' if a special order of size Q' is placed.	K'
Lead time	L
Salvage value of one unit	L
Lot for Lot	LFL
Least period cost	LPC
Least Total cost	LTC
Least Unit Cost	LUC

Symbols and abbreviations	
M , N	
Number of setups//orders or cycles per unit time(usually one year	m
Optimal value of m	m^*
Safety stock	M
1)Number working days in a year 2) Total number of periods in time horizon (dynamic lot sizing) 3) number periods used in moving average method 4) number periods in a cycle in ratio-to- trend method	N
Annual average number of cycles having shortage	N_b
P	
Probability of shortage, service level	p
Unit price/cost	P
Cost of producing 1 unit of ith product	Pi
Purchase cost of 1 unit in period t	P_t
Period order Quantity	POQ
Periods Of Supply	POS
Part-Period	PP
Part Period Algorithm	PPA
Part Period Balancing	PPB
Q	
Amount of each order	Q
Optimal amount of order	Q^*
Optimal amount of ordering product no. j each time	Q^*_j
Economic order quantity in Wilson Model	Q_W
Amount of ordering at time of temporary reduction of price	Q'
stock position on the expiration date in special	q

Symbols and abbreviations	
sale price Model	
Optimum Q'	Q'^*
The economic order quantity with unit price P+a in known increase in price Model	Q_a^*
The amount ordered at the beginning of period t	Q_t
Desired maximum of inventory	Q_m
The sum of demands for Period t through e in Wagner_wittin Algorithm	Q_{te}
R	
1)the inventory at reorder point in terms of on-hand and on-order quantities 2)reorder point in FOS model	r
1)production rate in EPQ model 2)Maximum of inventory in periodic review model	R
Annual production rate for i^{th} product	R_i
Reorder Level	RL
The ratio between estimated and actual C_O	r_O
1)The ratio between estimated and actual C_h 2)on-hand inventory at the time ordering	r_h
The ratio between estimated and actual demand	r_D
Reorder point	ROP
Optimal value of the maximum of inventory in periodic review model	R^*
The ratio of the observed value (y_i) to the predicted value $(\hat{y}_t)$ Period t in Rato-to-trend Forecasting method	Rt
Root Measn Squared Error	

Symbols and abbreviations	
$=\sqrt{\dfrac{\sum\limits_{i=1}^{n}\left(y_i-\hat{y}_i\right)^2}{n}}$	RMSE
S	
Standard deviation of a sample	S
Safety Stock	SS
Machine setup time required for producing ith product in multiple-item EPQ model	Si
standard error of estimate	SEE
Sum of Squared Errors	SSE
T	
Time interval between 2 successive orders, the time interval between 2 order arrivals ,The time for consumption in classic EOQ model, number of periods (month, day, week,,,)in the time horizon considered for dynamic lot sizing	T
Optimal value of T	T^*
The time required to consume $Q'=\dfrac{Q'}{D}$	T'
The optimal value of the time required to produce ith product in each run	$t_{P_i}^*$
The time required to produce ith product in each cycle	$(tp)i$
Lead Time	TL
Total Cost of inventory system	TC
Total Variable Cost	TVC
Optimal value of TVC	TVC*
Optimal value of TVC in Classic EOQ Model	TCw
the cycle time when the setup times are negligible in multiple-item EPQ model	T_0^*
The total cost for ith product in inventory system	TC_i

Symbols and abbreviations		Symbols and abbreviations	
$\dfrac{\sum_{j=1}^{n} S_j}{1 - \sum_{j=1}^{n} \frac{D_j}{R_j}} =$	T_{min}	The parameter in Poisson and exponential distributions	λ
The time interval between 2 successive cycles in which shortage happen	T_b	The fixed cost of shortage per unit	π
U , V		Mean of demand	μ_D
Income during the period (in Single period model)	U	Mean of lead time	μ_L
Income during the period (in Single period model)	VNS	Mean of T+L	μ_{L+T}
Variable Neighborhood Search	Var(D)	The cost of one unit shortage in 1 unit of time say 1 year	$\hat{\pi}$
Thee variance of demand	V	The cost of one unit shortage	$\pi_\circ$
X ,Y , Z		Total cost of one unit shortage	π
demand	X		
observed value for the ith element of the data	y_i	The cost of one unit shortage (except the lost profit)	π_0
Predicted value for the ith element of the data	$\hat{y}_i$	Variance of demand‹	σ_D^2
Cost(of production /purchase , holding , shortage)during the period (in Single period model)	Y	Variance of the lead time	σ_L^2
profit during the period (in Single period model)	Z	Mean of the demand	μ_D
Coefficient of confidence	Z1-p=k	Variance of the lead time plus the cycle time	σ_{L+T}^2
α, β, … ..		Standard deviation of consumption during lead time +	$\sigma_{D_{L+T}}$
1)$\frac{TVC}{TC_w} = \alpha$, 2)The idle time of the station in multiple EPQ model$\alpha = 1 - \sum_{i=1}^{n} \frac{D_i}{R_i}$ 3) a coefficient in exponential smoothing	α	Standard deviation of consumption during lead time	σ_{D_L}
		End of example	▲
The ratio of the amount ordered to the Q_w = $\frac{Q}{Q_w}$	β	End of example or proof	■

Prayer is the meeting
between
God as such
and
man as such

Chapter 1
Introduction
&
Basic Concepts

Chapter 1

Introduction and Basic Concepts

Aims of the chapter

This chapter deals with definitions and basic concepts needed in inventory control. The chapter also describes ABC analysis.

1.1 Definition of inventory control systems

A system of inventory control is compromised of people, devices, softwares and procedures for controlling inventories and orders in an institution. There are several items in an institution and each item has several units. The system is designed to decide which items (i), how much (Q_i) and when to place the orders.

1.2 The purpose of holding inventory

The purpose of holding inventory in an organization could be the followings:
a)For finished products:
    To cope with demand fluctuations,
    To satisfy customers demand immediately,
    To cope with production variations and halt
b)For In-Process Goods
    To cope with production halt,
c)For raw materials
    To cope with production halt,
    Using the vendor's discount.

1.3 Inventory costs

Inventory costs are associated with the operation of an inventory system and result from action or lack of action on the part of management in establishing the system (Tersine,1994 p13). The costs are classified as fixed and variable. The former class is independent of the level of output and the latter changes in proportion to *production* output. The costs could be itemized as follows (Tersine,1994 p13):

1.Cost of ordering goods from outside or cost of machine setup for internal production.

2.The holding(carrying)cost which subsumes the costs associated with investing the inventory and maintaining the physical investment in storage. This costs includes such ones as insurance, tax, theft, fire, rent, heating, cooling and lighting. Carrying(holding) costs are expressed as a proportion (I) of the total value of inventory. The cost of holding one unit per unit time (usually 1 year), denoted by Ch, is obtained by multiplying I times the unit price (P). Sometimes a fixed cost (C) is added to $I \times P$, therefore:

$$C_h = IP+C, \qquad 0 < I < 1 \qquad (1\text{-}1)$$

where
  C_h Cost of holding one unit per unit time (usually 1 year)
  P   unit price
  C   Fixed cost of holding for one unit per unit time
  I   Holding cost rate, cost of carrying \$1 of inventory for one unit of time.(e.g. 1 year)

For example, if the annual fixed cost for unit product is \$30 is incurred as well as the holding cost rate of 2% and the price is \$400 per ton then $C_h = 30 + 0.02 \times 400 = 38$.

It is worth knowing that: Depreciation and salvage values are frequently incorporated in the insurance cost. However, if important they may be modeled mathematically . Moreover holding cost sometime is incorporated in C_h as a function of the stored inventory and not as $I \times P$.

3.The purchase cost(P) is either the cost of purchase from external sources or the cost of production internally plus any freight cost (Terine,1994, page 13).

4.The stockout or depletion cost occurs when a customer's order is not filled. In some models presented for inventory systems the stockout is not allowed and in some it is allowed as backorder or lost sale.

5. The cost of data processing and updating the information
It should be added that some textbooks itemize the cost as follows (Hajji, 2012):
a. costs related to the warehouse(Electricity, heating, cooling, rent, depreciation),
b. handling and transportation cost,
c. deterioration cost in inventory
d. cost of the obsolete inventory
e. The cost of money or capital held by the inventory
f. cost of insurance and tax
g. shortage cost
h. cost of purchase of materials
i. order/setup cost

1.4 Calculation of inventory average

Average inventory level in a warehouse and the average amount of shortages play important roles in mathematical models developed for inventory systems. Here a way to calculate the average amount of inventory is described. Suppose the function $I(t)$ describe the inventory of an item in a warehouse in terms of time (Fig 1.1). The inventory average during time interval (0,T) , $\bar{I}$,is given by:

$$\bar{I} \quad or \quad \overline{Inv} = \frac{1}{T} \int_0^T I(t)dt \qquad\qquad (1\text{-}2)$$
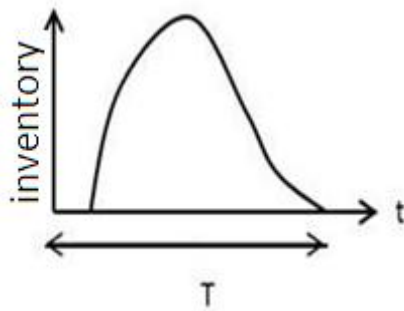
Fig. 1.1 A time-related function of   Fig.1.2 The average of inventory
inventory
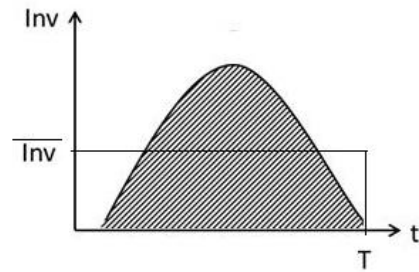
Figure 1.2 shows the average as the width of a rectangle having the same area as the function inv(t) has from 0 to T.

The calculation of the average amount of shortages during a period is calculated in a similar way.

Example 1.1

If the amount of the inventory of an item in a store in terms of time (in month)is described by the function e^t, calculate the average inventory for the interval (0-4) months

Solution $\bar{I} = \frac{1}{4} \int_0^4 e^t \, dt = 13.4$. End of example▲

Example 1.2

The inventory of an item changes as shown in the following figure.

Calculate the average of inventory during of the cycles i.e from 0 to T.
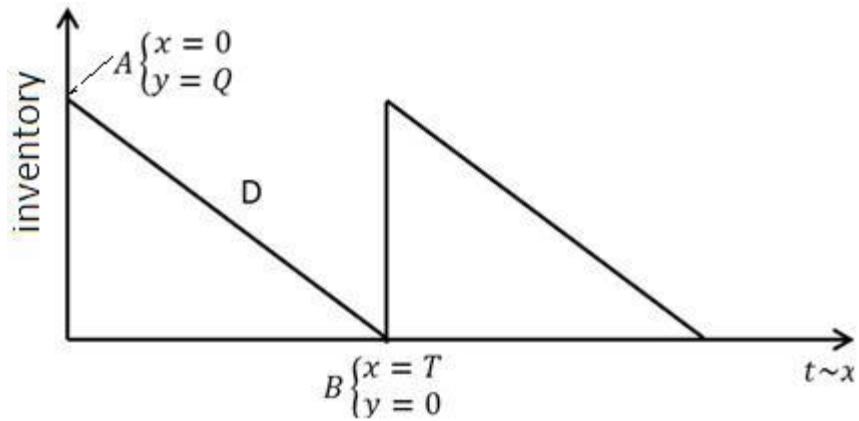
Solution

Let variable y denote the inventory and x denote the time, then the equation of line AB could be written as:

$$\frac{y - y_B}{x - x_B} = \frac{y_A - y_B}{x_A - x_B} \qquad \Rightarrow \qquad \frac{y - 0}{x - T} = \frac{Q - 0}{0 - T}$$

$$\Rightarrow y = \frac{-1}{T} Q(x - T)$$

Therefore the equation of line AB is $y = Q - \frac{Q}{T}x$, and the inventory average is calculated as follows:

$$\bar{I} = \frac{1}{T} \int_0^T \left(Q - \frac{Q}{T}x \right) dx = \frac{1}{T}(Qx) \Big|_0^T - \frac{Q}{2T^2} x^2 \Big|_0^T = \frac{Q}{2} \bar{I} = \frac{Q}{2}.$$

A simple way to calculate the average in this example is to note that the average is to divide the surface of the triangle by T i.e. $Q\frac{T}{2} \times \frac{1}{T} = \frac{Q}{2}$. The answer is equivalent to the calculation of the average of the maximum and minimum of inventory i.e. $\frac{0+Q}{2} = \frac{Q}{2}$. ▲

Example 1.3

If the holding cost of one dollar of an item as inventory is I dollars per year, the unit price of the item is P and $G(t)$ in the following figure is a function that gives the inventory stock level awaiting for use or marketing,
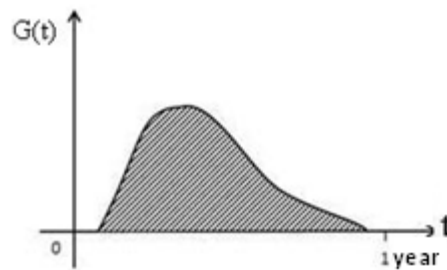
Find the average inventory in one year , annual holding cost and the holding cost for some finite time period like T (in year).

Solution

$$\text{annual average inventory} = \int_0^1 G(t)dt,$$

annual holding cost$= IP \int_0^1 G(t)dt,$

The average holding cost for a time T is: $T \times (IP \int_0^1 G(t)dt)$ ▲

Example 1.4

The following figure shows the inventory and shortage of an item (in tons). Find the annual average inventory and the related cost if the holding cost of one tone is $100.
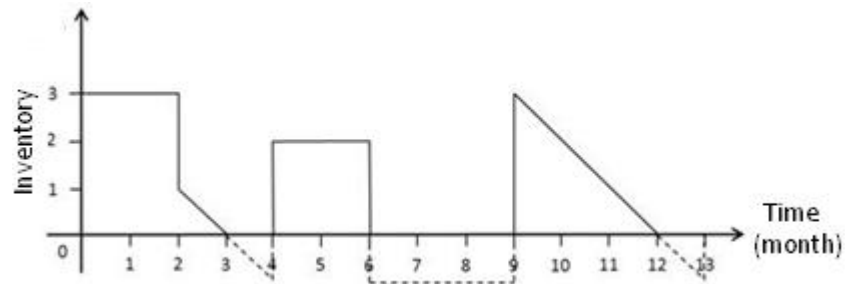
Solution

$$\text{annual average inventory} = \frac{3\left(\frac{2}{12}\right) + 0.5\left(\frac{1}{12}\right) + 2\left(\frac{2}{12}\right) + \frac{3}{2}\left(\frac{3}{12}\right)}{1} = \frac{\frac{15}{12}}{1} = \frac{15}{12}$$

$$\text{Holding cost} == \frac{15}{12} \times 100 \times 12 = 1500 \ \blacktriangle$$

1. 5 Calculation of shortage average

Shortage average is needed to calculate shortage cost. Suppose b(t) is a function of time denoting the shortage of an item at time t. The average amount of shortage during the time interval (0 T) is given by

$$\bar{b} = \frac{1}{T} \int_0^T b(t)dt \qquad\qquad (1-3)$$

In Fig. 1-3 the negative inventory is indicative of shortage.
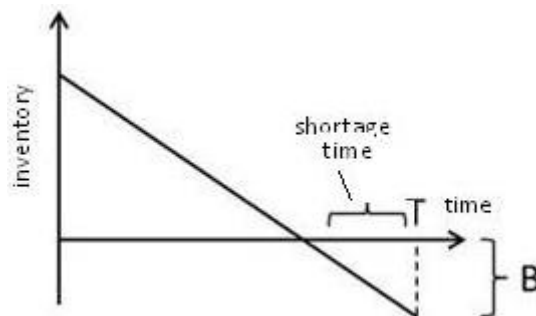
Fig.1.3 shortage during a time period T

Note that $I(t) - b(t)$ is sometimes called the net inventory, where $I(t)$ is the level of inventory at time t.

Example 1.5

In Example 1.4 find the average shortage per year

Solution

$$\text{annual average inventory } = \frac{3\left(\frac{2}{12}\right) + 1\left(\frac{4}{12}\right)}{1} = \frac{\frac{10}{12}}{1} = \frac{10}{12}$$

End of example▲

1.5.1 Unit normal loss integral

Since the calculation of the average shortages in some stochastic inventory models discussed in chapter 5 uses the so-called unit normal loss integral; this integral is introduced below

Let $S = \int_{x=a}^{\infty}(x - a) f(x) dx$ where a is a constant and f is the probability density function of a normal distribution with mean μ and standard deviation σ, then:

$$S = \int_{x=a}^{\infty} (x - a) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx;$$

S is easily computed by the loss integral developed by Robert Schlaifer, described below.

Let $u = \frac{x-\mu}{\sigma} \Longrightarrow x = \mu + u\sigma$ و $dx = \sigma du$. For x=a, the value of u would be $\frac{a-\mu}{\sigma}$, which is denoted here it by k then:

$$\frac{a-\mu}{\sigma} = k \Longrightarrow a = \mu + k\sigma \quad \Longrightarrow x - a = (u - k)\sigma.$$

Since $u = k$ is equivalent to $x = a$ then

$$S = \int_{u=k}^{\infty} (u - k)(\sigma) \frac{1}{\sigma\sqrt{2\pi}} e^{- \frac{u^2}{2}} (\sigma du) \quad \Longrightarrow$$

$$S = \sigma \underbrace{\int_{u=k}^{\infty} (u - k) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du}_{G_U(k)}$$

Let $G_U(k) = \int_{k}^{\infty} (u - k) \frac{1}{\sqrt{2\pi}} e^{- \frac{u^2}{2}} du$ then

$$S = \sigma G_U(k) \qquad\qquad k = \frac{a-\mu}{\sigma} \qquad (1\text{-}4)$$

$G_U(k)$ as given above is called the unit normal loss integral and its values are given in Table A at the end of the book. It is worth knowing that it can also be calculated using the MATLAB command:

exp(-k^2/2)/sqrt(2*pi)-k*(1-normcdf(k))

1.6 Some points on statistical distributions used in inventory control

Normal or Gaussian distribution is frequently used in inventory control for demand, lead time,...; however some other such as Poisson, uniform, lognormal and empirical distributions are also used.
It is worth mentioning that
    The distribution of the sum of several independent Poisson distributions is Poisson, however the product of a constant and a Poisson random variable does not have a Poisson distribution
    The product of a constant and an exponential random variable has an exponential distribution, however the distribution of the sum of several exponential distribution is not exponential

1.6.1 The distribution of the sum and the product of two independent normal distribution

In probability theory, it is proved that the sum of two normally distributed independent random variables is normally distributed.

Distribution of the product

The product of two normally distributed independent random variables X &Y is not normally distributed, however, using Taylor series of $f(x, y) = xy$ expanded about the mean of the variables i.e. μ_X و μ_Y we have:

$W = f(x, y) \cong f(\mu_x, \mu_Y) + [(x- \mu_{X)}\mu_Y + (Y - \mu_{X)}\mu_X]$

$f(\mu_x, \mu_Y) = \mu_X \times \mu_Y \implies W \cong (\mu_Y)X + (\mu_X)Y - \mu_X \times \mu_Y$

Now W has been approximated by a linear combination of X and$Y$ .When X and Y are independent normal variables, this combination follows a normal distribution; that is why in some inventory books the product of 2 independent normal variables is assumed normal.

Seijas-Mac´ıas &Oliveira(2012) showed that for two uncorrelated normally distributed X&Y, the more $\frac{\mu_X}{\sigma_X}$ and $\frac{\mu_Y}{\sigma_Y}$, the better fits the normal approximation to the distribution of $X \times Y$.

As an illustration, if annual demand(D) for a product is normally distributed variable with mean 1000 and standard deviation 40, and variable the time needed for an order of the product to receive (L) is a variable which has normal distribution with mean 1 week and standard deviation $\frac{1}{4}$ week , the product $D \times$L is the demand during time L.

The following figure shows the histogram of the product of 100 random number from $N(1000,40)$ and 100 random number from $N(\frac{1}{52}$yr, $\frac{0.25}{52}$yr) prepared using the following MATLAB commands:

```
D=normrnd(1000,40,100,1);L=normrnd(1/52,.25/52,100,1);W=D.*L;hist(W)
```
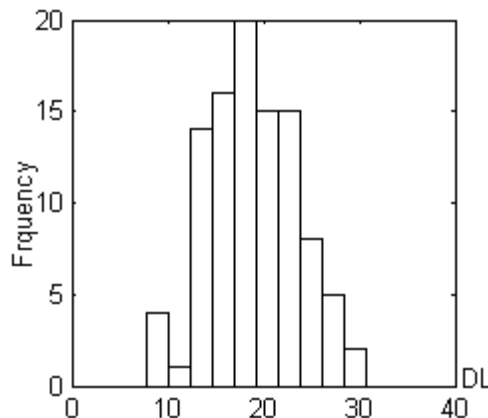

Fig. 1.4 The histogram of the product of 2 normal distributions

The histogram indicates that the consumption during time L is well approximated by a normal distribution.

1.7 Pareto Principle and ABC Analysis

Since the so-called ABC analysis is ABC is useful for analyzing the inventories in an institution, it is deal with below.

ABC analysis is a categorization method in which inventory is classified into A, B and C category with A being the lowest quantity, highest value. C being the highest quantity and lowest value. The purpose of this analysis is to help the managers identify those items that represent the large segment of the inventory costs in order to better manage these resources. It allows different inventory management techniques to be applied to different segments of the inventory in order to increase revenue and decrease costs.

Although the ABC analysis has had some modifications from the date it was developed, but the steps of the conventional method is described here after stating a related principle i.e. the Pareto principle.

Pareto Principle

The Pareto principle, named after esteemed scientist Vilfredo Pareto (1848-1923) specifies that within any system or organization a small portion of input has the highest value and output. Actually ABC analysis could be considered an application of this principle. The criterion for categorization might be such things as delivery time as well as dollar value. The following is a sample categorization in a company:

'A' items include the materials or components are necessary for production and have a long delivery time or a high value. The lot containing these items is delivered to the warehouse from which they are delivered to the production and repair departments with sealed or signed official sheets.

'B' items include the production materials or components which have a medium delivery time or value. The control of these items is done by the direct supervisor of the department.

'C' items include regular standard components or materials whose delivery time is short or their value is low. When the order is received, they are submitted to the warehouse or the department depending on the type. When the inventory of the item reaches a small amount, an order is placed. A very low control is applied in these items

In the ABC analysis described below, the inventory items are listed and the annual consumption value of each item (Annual unit usage × unit cost/price). Very important items(A) items, medium important items and relatively unimportant items(C) could be identified after prepar ing the table and the graph for the ABC analysis.

The proportion of A, B and C items can be identified from a graph similar to Fig 1-4 and more control and energy applied on important items.

1.7.1 Steps in conduction ABC Analysis

1.Enlist items.
2. Estimate annual consumption – Unit wise.
3.Determine unit price of each item – .
4. Multiply the results of steps(2) &(3) to obtain annual usage value
5. Arrange in descending order.
6. Calculate cumulative usage value percentages.
7. Graph cumulative usage value percentage against cumulative item percentage.

Note that A,B and C categories are identified according the higher-level management viewpoints. For example one manager may place the items with 8o% of value in category A, the items with 15% of value in category B, the items with 5% of value in category C; the other one might choose the percents 70, 20 and 10 for this purpose.
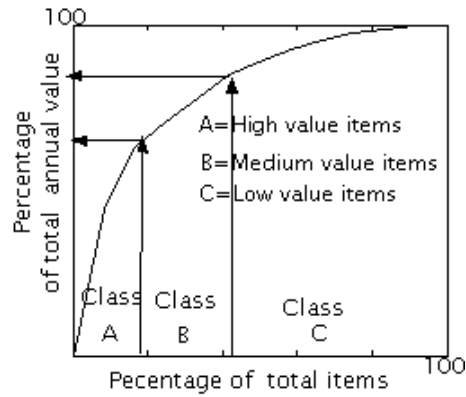
Fig 1-4 item classification in ABS analysis

Example 1.6

Perform an ABC analysis on the products given following table

Product no.	4837	9261	4395	3521	5223	5294	6081	4321	8046
Annual usage	685	371	129.2	62	1266.7	962.5	1822.6	5100	25.8
Unit price	12	8.6	131.8	91.8	32	101.8	4.8846	0.88	622.5

Product no.	9555	2926	1293
Annual usage	862	1940	967
Unit price	18.1	0.38	2.2

Solution

The result of performing Step 4 of ABC analysis is seen in Row 4 of Table 1.1.

Table 1.1 The inventory items of a company									
	1	2	3	4	5	6	7	8	9
Product no.	4837	9261	4395	3521	5223	5294	6081	4321	8046
Annual usage	685	371	129.2	62	1266.7	962.5	1822.6	5100	25.8
Unit price	12	8.6	131.8	91.8	32	101.8	4.8846	0.88	622.5
Annual value	8220	3190.6	17028.56	5691.6	40534.4	97982.5	8902.7	4488	16060.5

Table 1.1 continued			
	10	11	12
Product no.	9555	2926	1293
Annual usage	862	1940	967
Unit price	18.1	0.38	2.2
Annual value	15602.2	737.2	2127.4

Table 1-2 shows the result of performing Steps 5&6:

Table 1-2						
Rank (J)	NO. in Table 1-1	Product NO.	Annual Value ($)	Cum. Annual Value($)	Cum. Annual Value(%)	Cum Item NO.(%) $\frac{j}{12} \times 100$
1	6	5294	97982.5	97982.5	$\frac{97982.5}{220565.66} \times 100$ $= 44.423$	$\frac{1}{12} = 8.3$
2	5	5223	40534.4	138516.9	$\frac{138516.9}{220565.66} \times 100$ $= 62.801$	$\frac{2}{12} = 16.6$
3	3	4395	17028.56	155545.46	70.521	$\frac{3}{12} = 25$
4	9	8046	16060.5	171605.96	77.803	33.3
5	10	9555	15602.2	187208.16	84.876	41.7
6	7	6081	8902.7	196110.86	88.913	50
7	1	4837	8220	204330.86	92.639	58.3
8	4	3521	5691.6	210022.46	95.22	66.7
9	8	4321	4488	214510.46	97.255	75
10	2	9261	3190.6	217701.06	98.701	83.3
11	12	1293	2127.4	219828.46	99.666	91.7
12	11	2926	737.2	220565.66	100	100
Sum			220565.66			

The management of the company decides to place the first 2 items of Table 1-2 with cumulative annual value 63% in Category A, the next 3 others with cumulative annual value 22% in Category B and the rest in Category C with cumulative annual value 15%.   Table 1-3 and Fig. 1-6 shows the categories A , B, and C .

Table 1-3 the categories A , B, and C for Example 1-6.			
Category	Product No from Table1- 1	x-axis number of products in the category /12(%)	Y-axis Annual value(%)
A	5223 ,5294	$\frac{2}{12} \times 100 = 16.6$	62.801
B	4395 , 8046 ,9555	$\frac{3}{12} \times 100 = 25$	84.876- 62.801= 22.07
C	4321 ,3521 , 4837, 6081,9261, 1293,2926	$\frac{7}{12} \times 100 = 58.3$	100-84.876= 15.12
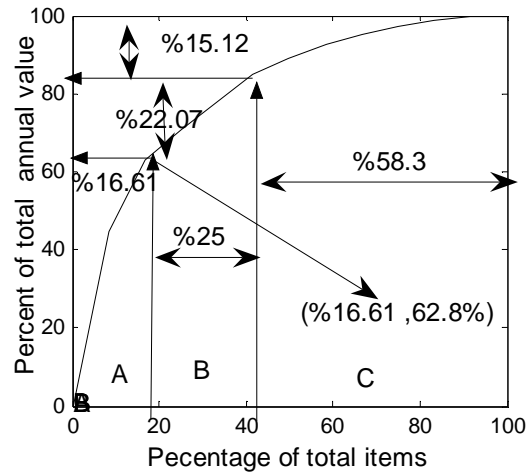
Fig.1-5 Cumulative percentage of inventory products for
Example 1-6

Therefore

   16.66% of the 12 items (Products   No. 5223 and 5294) having
62.8% of the annual value constitutes category A.

   25% of the 12 items (3 products i.e. 4395, 8046 and 9555)having
22.07% of the annual value constitutes category B

   58.3% of the 12 items (seven products i.e. 4321 ,3521 , 4837,
6081,9261, 1293و2926)   having 15.12% of the annual value

constitutes category C. End of example▲

1.7.1 Control activities on different categories

   Some of the control activities on the 3 categories are listed below:
Control on Category A
   Evaluation of forecasting methods and improving forecasts
accuracy,
   Updating the inventories of the items,
   Frequent reviewing of demand, order quantity, safety stock to
reduce the order quantity,
   .attempt to reduce lead times.

Control on Category B

The activities needed to perform on the items of Category B are similar to those applied on the previous group, but with with less frequent review and less accuracy.

Control on Category C

Keeping a relatively large number of units on hand,
Simple inventory recordof the items or periodic review of the items
Making the inventory of the items easily accessible to the operators.

At the end of this section, it is worth knowing that recent researchers on inventory control analysis, do not necessary categorize the inventory of a firm into 3 categories. For example Ameri (2016) performed the analysis in a copper steel mill and suggested a four-category inventory control.

1-8 Inventory models classification

Many models have been developed for controlling the inventories in firms and organizations. These models are classified based on the decision conditions governing the inventory systems i.e. a) complete certainty b) uncertainty including complete uncertainty and risk.

In certainty conditions, parameters such as the amount of product demand, the waiting time to receive the ordered goods(lead time) are approximately constant; in other words regardless of small fluctuations, the parameters are almost constant and independent of time.

In day-today conversation, usually the two terms 'risk' and 'uncertainty' are used synonymously meaning 'a lack of certainty'. Let us divide the uncertainty conditions into complete uncertainty conditions and risk condition:

In complete uncertainty conditions there in no record of past data; therefore calculation of the occurrence probabilities for model

parameters is not possible. The decision under this condition is done using criteria such as Minmin and Mini-max.

In risk conditions, we have a record of past data which make it possible to calculate the probability of occurrence of alternative values of parameters such as demand and lead time.

Models such as Wilson-Harris model, safety stock, total discount model are used for certainty conditions.

Models such as single period model, periodic review model are used for uncertainty conditions.

It is worth mentioning that sometimes the inventory models are classified in two categories: deterministic and probabilistic inventory models.

It is advised, now at the end of this chapter, to make the data of a problem, when using a formula, have the same dimension; e.g. if the amount of daily demand and the annual holding cost are given, make both of them have the same time interval, say calculate annual demand to the same dimension as the holding cost has.

Exercises

1-1 The following figure shows the amount of the inventory of a product in a warehouse. The per unit monthly shortage cost is $ 10. Find the average shortage during the past 6 months, and the shortage cost during this period.
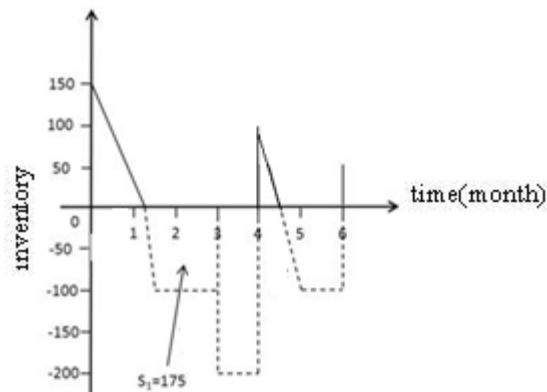
1-2. If the functions I(t) and D(t) denote the inventory and the demand for a product at time t.   (answer choice a.)

a) $I(t) = 0$ و $D(t) > 0$ b) $I(t) > 0$ و $D(t) > 0$
c) $I(t) < 0$ و $D(t) < 0$ d) $I(t) > 0$ و $D(t) < 0$

1-3 If the inventory of an item follows the following function,

$$I(t) = [0.2 \times \ln(0.1t) + 0.2]e^{2(0.1t)\ln(0.1t)}$$

Find the average inventory from t=3 through t=6.

1-4 Regarding the ABC analysis which of the following 4 choices is correct?

a) The items in Category A has the largest percent of items.

b) The items in Category C has the lowest percent of items.c) The items in Category C has the largest percent of total annual consumption (in dollar).

d) The items in Category A has the largest percent of the total annual consumption (in dollar).

1-5 For which of the following cases, inventory control and planning is performed?

a)production equipment and tooling, raw materials, final products, in-process products

b)production equipment and tooling, raw materials, final products, tooling for services

c)tooling for services, raw materials, final products

If the doors of heavens and earth are closed to someone, then he chooses piety, God shall relieve him

Chapter2

Deterministic

Inventory

Models

Chapter 2

Deterministic Inventory Models

Aims of the chapter

This chapter introduces several models for inventory management under conditions of certainty: Economic order quantity model, Safety stock model Back -order model, lost sale model,…

2-1Economic Order Quantity(EOQ) model

Here the classic Economic Order Quantity (EOQ) model, which is the best known and the most ideal and fundamental inventory decision model, is described.

2-1-1Assumptions of Classic EOQ model

The following assumptions are present in the formulation of the classic EOQ model, in other words without these assumptions, the EOQ model cannot work to its optimal potential.

Assumptions
-The conditions of certainty governs our inventory system. This mean that parameters such as demand rate (D), the lead time(T_L), price(P) are constants and not random variables.
-Orders placed arrive all at once.
-Price(P) is fixed and does not change with the order quantity(Q),

   -No shortage occurs(replenishments arrive   when the inventory level reaches zero.

   -There is no constraint and restriction on capital, order quantity, warehouse space,…

   -The goods have a largish lifetime and could be stored for a long time without deterioration, or the rate of deterioration is ignorable.

   It is worth knowing that the purpose of inventory model is to plan the orders in such a way that the total cost of the inventory system is minimized. The output of the planning is to answer the following questions:

   What is the quantity of each order ?

   When to place an order? Every T-time period ?or   when the inventory reaches a specified amount?

List of Symbols

C_h Carrying (holding) costs, the cost of holding one unit per unit
          time (usually 1 year)

C_O Cost of placing an order

D   Demand rate, demand per unit time

EOQ   Economic Order Quantity, amount of economic order

I   a proportion of the total value of inventory, the cost of holding
        one dollar per unit time (usually 1 year)

$\overline{I}$ Average inventory

m   Number or orders placed per unit time

P   Price

Q   Order quantity

 Q^* Optimal quantity for orders

 Q_W Optimal quantity for orders derived from Wilson formula

ROP   Re-order Point

T   Order interval, time between placing 2 successive orders or
        between arrival of 2 successive orders, the time required for

$L=T_L$ Lead time

TC   Total cost of inventory system per unit of time

TVC   Total variable cost of inventory system per unit of time

   The total cost of inventory systems is the sum of the ordering, holding, and purchase costs.   By multiplying the average annual inventory and the annual holding cost per unit product (C_h) , the annual holding cost is calculated on the average. Figure 1-2 shows the

level of inventory based on the above assumptions for the   EOQ model.
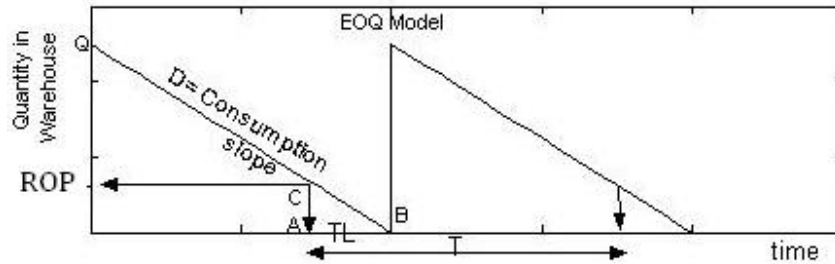
It could easily be shown (see Example 1-2) that the average inventory per cycle is the quotient of the area of the triangle and its base leg ;here it will be equal to:

$$\frac{Q}{2} = \frac{Q + 0}{2} = \frac{\text{Max inventory} + \text{Min inventory}}{2}.$$

Given the annual demand (D)for a product with unit price P, order quantity(Q) and cost of placing each order(C_O), the annual order cost would be $\frac{D}{Q} C_O$ and the annual total cost of the inventory system is:

$$\text{TC} = C_O \frac{D}{Q} + C_h \frac{(Q+0)}{2} + \text{PD} \qquad (2\text{-}1)$$

Note that stockout cost is not included here, because it was assumed that stockouts are not permitted in this model. If the order quantity (Q) is a continuous variable, since $\frac{d^2 TC}{dQ^2} = \frac{C_O D}{Q^3} > \cdot$, the function TC has a minimum. The optimal Q, is derived from $\frac{dTC}{dQ} = 0$.

$$TC = C_O \frac{D}{Q} + C_h \frac{Q}{2} + PD \, , \qquad \frac{dTC}{dQ} = 0 \Rightarrow Q^* = \sqrt{\frac{2DC_O}{C_h}}$$

Then in the classic Inventory model, the optimal order quantity (Q^*) which is also called economic order quantity and denoted by Q_W or EOQ is equal to:

$$Q_W = EOQ = \sqrt{\frac{2DC_O}{C_h}} \qquad (2\text{-}2)$$

Thi is also called Wilson inventory formula or Wilson-Harris formula.
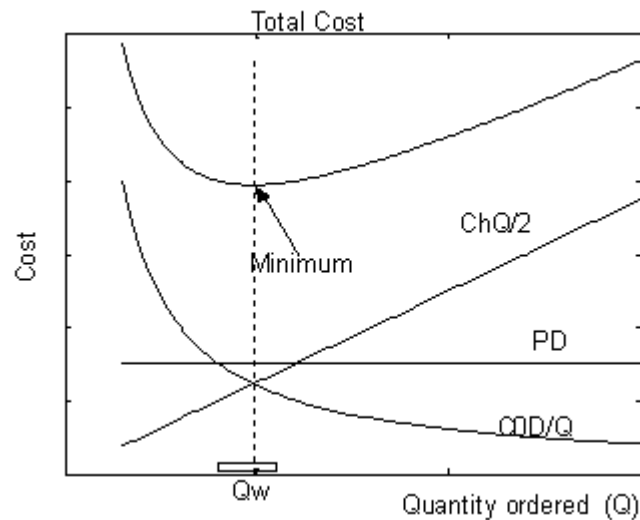
Fig.2-2 The components of annual total cost of an inventory system

The total inventory cost per year (TC) and its 3 components are depicted by Fig. 2-2 . As the figure shows the minimum of TC occurs at the intersection of the holding cost and the order cost i.e. at the intersection of $C_O \frac{D}{Q}$ and $C_h \frac{Q}{2}$:

$$C_O \frac{D}{Q} = C_h \frac{Q}{2} \implies Q = \sqrt{\frac{2DC_O}{C_h}} = Q_W.$$

Note that,

1)As it is evident from Fig. 2-2,

when $Q < Q_W$, the annual order cost(i. e. $C_O \frac{D}{Q}$) will exceed the annual holding cost (i. e. $C_h \frac{Q}{2}$)

when Q $= Q_W$,the order cost will be equal to the annual holding cost.

when $Q > Q_W$,the order cost will be greater than the annual carrying cost (i. e. $C_h \frac{Q}{2}$).

2)The optimal order quantity in this model i.e. Q_W is the point where the annual holding cost and the annual order cost are equal.

Now substituting Q=Q_W in relationship 2-2 results in:

$$TC^* = \sqrt{2DC_O C_h} + PD = C_h{\times}Q_W + PD.$$

Denoting the first part of this relationship by TC_W, we would have

$$TC_W = \sqrt{2DC_O C_h} = C_h \times Q_W \qquad (2\text{-}3)$$

Optimal number of orders placed each year (m) would be:

$$m^* = \frac{D}{Q^*} \qquad\qquad (2 - 4)$$

and the interval time between successive orders (T) in its optimum state is $\quad T^* = \frac{1}{m^*} = \frac{Q^*}{D} \Longrightarrow$

$$T^* = \sqrt{\frac{2C_O}{DC_h}} \qquad\qquad (2 - 5)$$

In this model T^* is also one cycle time in the optimum state and also the time required for consumption of Q*.

2-1-2 The maximum and the average of Inventory in EOQ model

In the EOQ model when the quantity of each order is Q_W, the maximum inventory in the warehouse (I_{max}) would be Q_W and the average inventory($\bar{I}$)would be $\frac{Q_W}{2}$.

$$\text{optimal } \bar{I} = \bar{I}^* = \frac{Q_w}{2} = \sqrt{\frac{C_0 D}{2C_h}} \qquad (2\text{-}6)$$

2-1-3 The reorder point(ROP) in EOQ model

If the time interval between placing an order until receipt of the products by the customer , known as the lead time and denoted here by T_L, is less than cycle time T(see Fig 2-3), the reorder point (ROP)
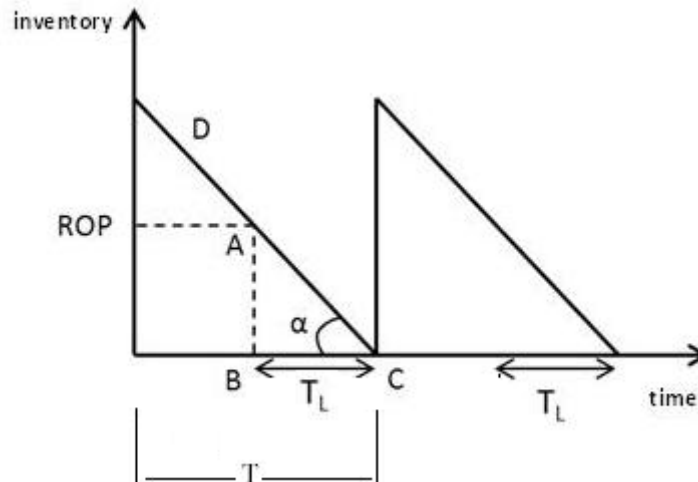
Fig. 2-3 Reorder point in the classic EOQ model ($T_L < T$)

is calculated as follows:

$$\tan \alpha = D = \frac{AB}{BC} \implies D = \frac{ROP}{T_L} \implies ROP = DT_L \ .$$

When $T_L \geq T$,as shown in Fig. 2-4, the orders arrive at points A, B, C,…
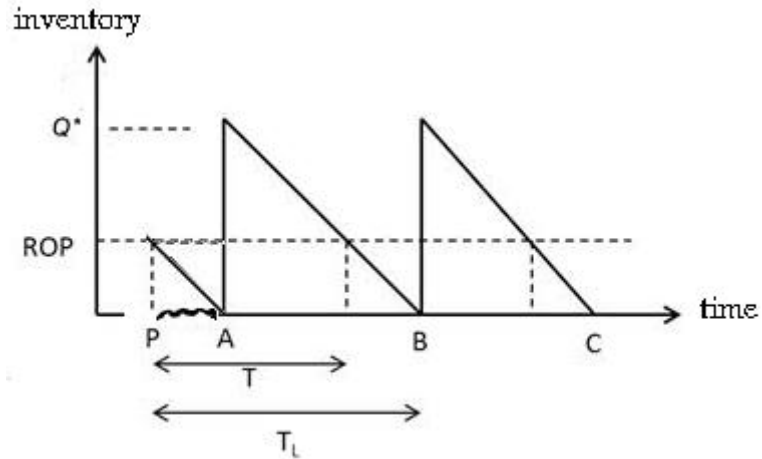
Fig. 2-4 Reorder point in the classic EOQ model ($T_L \geq T$)

When the order arrives at B, then ROP equals demand times the time interval PA which is equal to $T_L - T$; therefore

$$ROP = D(T_L - T) = DT_L - DT = DT_L - Q$$

and in the optimal case $DT^* = Q^*$ and $ROP = DT_L - Q^*$.

Generally $ROP = DT_L - KQ^*$

where K is $K = [\frac{T_L}{T}]$ i.e. the biggest integer number equal to or less than $\frac{T_L}{T}$ (Patel,1986).

Then in the classic EQQ model:

$$ROP = \begin{cases} DT_L & T_L < T^* \\ DT_L - KQ_W & \left(K = \left[\frac{T_L}{T^*}\right] \leq T_L\right) & T_L \geq T^* \end{cases} \qquad (2\text{-}7)$$

where

K is the integer part of the ratio of lead time and cycle time.

Note that:

-When replacing the parameters in formulas, their dimensions must agree; e.g. if D is given per month and C_h is given in year, both must have the same time interval.

-If the amount of D in is dollars, the amount of Q will be in dollars.

-In this model the cycle time T, which is equal to the time interval between placing two successive orders, is equal to the time required to consume the amount ordered Q.

Example 2-1

An item may be purchased for $20 per unit. The order cost is $100. The annual holding cost fraction is 10% and the monthly demand for the item is 500. There is 265 working days and 12 month in a year.

 a)Calculate the economic order quantity , total annual cost, the time interval between 2 successive orders, the annual number of orders and also the reorder point if the lead time is 25 days.

b) Calculate the reorder point if the lead time is 40 days.

Solution

a)

$$EOQ \ or \ Q_W = \sqrt{\frac{2DC_O}{C_h}} = \sqrt{\frac{2 * 500 * 12 * 100}{0.1 * 20}} \approx 775$$

$$TC^* = PD + \sqrt{2DC_OC_h} = PD + C_h \times Q_W = 20 \times 6000 + 2 \times 775 = 121550$$

$$T^* = \frac{775}{12 \times 500} = 0/13 \text{yr} = 0/13 \times 265 \text{days} = 35 \text{days}$$

$$m^* = \frac{1}{T^*} = \frac{12 \times 500}{775} \cong 8$$

$$T^* > T_L \Rightarrow ROP = DT_L = \frac{6000}{265} * 25 = 566;$$

That is when the inventory reaches 566 units, an order off 775 has to be placed.

$$b) \ T^* < T_L \Rightarrow ROP = DT_L - KQ_W = \frac{6000}{265} * 40 - \left[\frac{T_L}{T^*}\right] Q_W \Rightarrow$$

$$ROP = 906 - \left[\frac{40}{35}\right] * 775 \cong 130. \text{End of example}\blacktriangle$$

2-1-4   Sensitivity Analysis for EOQ Model

Sensitivity analysis in a model determines how target variables are affected by changes or errors in input variables. It is a way to predict the outcome of a decision given a certain range of input variables. If while keeping the rest of inputs constant, a vast range of an input variable does not change the amount of output variable significantly, it is said the model is not insensitive to the input variable. If any change in the   input variable changes the amount of output variable significantly, it is said the model not sensitive to the variable.

The EOQ model assumes that annual demand D, holding cost Ch and order cost C_o are deterministic and without variation; however this section will analyze the impact of errors in determining the parameters D, C_h and C_o in EOQ model.

2-1-4-1 Impact of Errors in C_O، C_h and D on Q and total cost

Definition

The quotient of estimated Co (C_O') to actual Co is denoted by r_O and called the error factor of order cost:

$$r_O = \frac{\text{estimated } C_O}{\text{actual } C_O} = \frac{C_O'}{C_O}$$

Similarly

$$r_h = \frac{\text{estimated } C_h}{\text{actual } C_h} = \frac{C_h'}{C_h} \qquad r_D = \frac{\text{estimated } D}{\text{actual } D} = \frac{D'}{D}.$$

If error occurs in estimating or determining D , C_O and C_h then to determine the order quantity D', C_O ' and c_h' replace D, C_O and C_h:

$$Q = \sqrt{\frac{2\, D'C_O'}{C_h'}} = \sqrt{\frac{2(D\, r_D)(C_O\, r_O)}{C_h\, r_h}} = Q_W \sqrt{\frac{r_D r_O}{r_h}} \qquad (2\text{-}8)$$

If no error occurs in estimating, then $r_O = r_h = r_D = 1$.

The error fraction in order quantity is as follows

$$\text{Error fraction in } Q_W = \frac{Q - Q_W}{Q_W} = \sqrt{\frac{r_D r_O}{r_h}} - 1 \qquad (2\text{-}9)$$

When the order quantity in this model is as much as Q_W, the variable cost totally is denoted here by TC_w .   When the order quantity is less or more than the economic order quantity ($Q \neq Q_W$) , the total variable cost denoted by $TVC(Q)$ could be calculated from

$$TVC(Q) = TC_w \sqrt{r_D r_O r_h} \qquad (2\text{-}10)$$

The error in the total variable cost is equal to $TVC_{(Q)} - TC_w$ and the error fraction in then the optimal cost is as follows:

$$\text{Error fraction in } TC_W = \frac{TVC_{(Q)} - TC_w}{TC_w} = \sqrt{r_D r_O r_h} - 1 \qquad (2\text{-}11)$$

Note that:

The error in only one parameter results in the same error fraction in TC_W.

Example 2-2   If 90% of the actual holding cost is inserted in the Wilson formula for order quantity, calculate the fraction of error in Q_W and TC_W.

Solution

$$\text{Error fraction in } Q_W = \sqrt{\frac{r_D r_O}{r_h}} - 1 = \sqrt{(1*1)/0.9} - 1 = -0.0541$$

That is inserting 90% of the holding cost in the Wilson formula will cause 5.41% reduction in optimal order quantity. This will cause the error fraction in TCw to be:

$$\text{Error fraction in } TC_W = \sqrt{r_D r_O r_h} - 1 = \sqrt{1*1*0.9} - 1 = 0.949 - 1 = -0.051$$

End of example▲

The following table, shows the error fractions calculated for several error factors .   (Error has occurred in only one and only one parameter: in D   or C_h or C_O). According to this table, if for example $r_D = 0.9, \quad r_O = r_h = 1$, the error   fraction in TC_w would be -5.1% which coincides with Eq.2-11:

$$\text{Error fraction in } TC_W = \sqrt{r_D r_O r_h} - 1 = \sqrt{0.9} - 1 = -0.051 \ = \text{-} 5.1\%$$

According to the table if the error fraction in only one of the parameters D or C_h or C_O occurs between 0.9-1.1% and the 2 others are free of error, the error fraction in TVC* would be as small as -5.1 % to + 4.9 %.

Error factor (related to D or C_h or C_o)	Error fraction (%)in TC_W
0.1	-68.4
0.2	-55.3
0.3	-45.2
.	.
.	.
.	.
0.9	-5.1
1	0
1.1	4.9
1.2	9.5
1.4	18.3
1.6	26.5
2	41.4
3	73.2
4	100
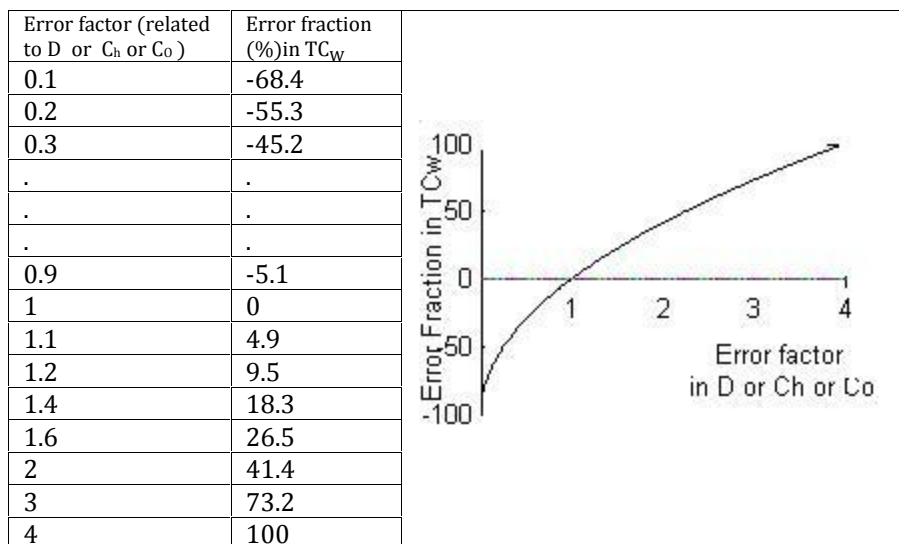
2-1-4-2   Impact of Errors in Q_W on total variable cost

To deal with the impact of error in Q_W on total variable cost notice that

$$TC = C_O \frac{D}{Q} + C_h \frac{Q}{2} \quad \text{and} \quad Q^* = Q_w = \sqrt{\frac{2DC_O}{C_h}}$$

$$TVC^* = TC_w = \sqrt{2DC_OC_h}$$

Now let $\beta = \frac{Q}{Q_w}$ then it could be shown that

relative increase in TVC $= \frac{TVC(Q)-TC_w}{TC_w} = \frac{TVC(Q)}{TC_w} - 1 \Longrightarrow$

relative increase in TVC $= \frac{1}{2}\left(\frac{1}{\beta}+\beta\right) - 1 > 0$ (2-12)

Proof:

$$\alpha = \frac{TVC(Q)}{TC_w} = \frac{\frac{C_O D}{Q} + \frac{C_h Q}{2}}{\sqrt{2DC_O C_h}} = \sqrt{\frac{\frac{D^2 C_O^2}{Q^2}}{2DC_O C_h}} + \frac{Q}{2}\sqrt{\frac{C_h^2}{2DC_O C_h}}$$

$$= \frac{\sqrt{\frac{2DC_O}{C_h}}}{2Q} + \frac{Q}{2\sqrt{\frac{2DC_O}{C_h}}} = \frac{Q_w}{2Q} + \frac{Q}{2Q_w} = \frac{1}{2}\left(\frac{Q_w}{Q} + \frac{Q}{Q_w}\right) \Longrightarrow$$

$$\Longrightarrow \alpha = \frac{TVC(Q)}{TC_w} = \frac{1}{2}\left(\frac{1}{\beta}+\beta\right) \quad \text{then} \quad \frac{TVC(Q)-TC_w}{TC_w} = \frac{1}{2}\left(\frac{1}{\beta}+\beta\right) - 1.$$

Since TC_w is the minimum of TVC then $TVC(Q) - TC_w > 0$ and hence the relative increase in TVC i.e. $\frac{TVC(Q)-TC_w}{TC_w} > 0$ for $Q \neq Q_w$.

End of proof. ∎

The following table shows some Q_w error factors and their corresponding relative increase in TVC. According to this table the error factors in the range $0.5\ Q_w$ to 2 times Q_w ,cause at most 25% increase in TVC.

Q_w Error factor	0.1	0.2	0.3	0.4	0.5	...	1	1.2	1.4	...	2
Relative increase in TVC(%)	405	160	81	45	25		0	1.7	5.7	...	25

As a sample computation, suppose $\beta = \frac{Q}{Q_w} = 2$, then relative increase in the total variable cost is equal to $\frac{1}{2}\left(\frac{1}{\beta}+\beta\right) - 1 = \frac{1}{4} = \%25$.

Example 2-3

Using the data of the following table, find

a)Simultaneous effects of error in D and C_O on Q_w and simultaneous effects of error in the 3 parameters on it,

b)The effect of error in holding cost C_h on Q_w and TC_w.

parameter	Actual value	Estimated value
D	2000	1000
C_h	20	10
C_o	25	50

Solution

a)

$$r_h = \frac{C_h'}{C_h} = \frac{10}{20} = \frac{1}{2} \qquad r_o = \frac{C_o'}{C_o} = \frac{50}{25} = 2 \qquad r_D = \frac{C_D'}{C_D} = \frac{1000}{2000} = \frac{1}{2}$$

Simultaneous effect of the errors in D& C_O on Q_w

$$= \sqrt{\frac{r_o r_D}{r_h}} - 1 = \sqrt{\frac{2*0.5}{1}} - 1 = 0$$

Simultaneous effect of the errors in all 3 parameters on Q_w=

$$= \sqrt{\frac{r_o r_D}{r_h}} - 1 = \sqrt{\frac{2 * 0.5}{0.5}} - 1 = 0.414 \text{ or } 41.4\%.$$

b)

The effect of error in all parameters   on TVC= $\sqrt{r_D r_o r_h} - 1$

The effect of error in C_h on TVC =

$$\sqrt{1 \times 1 \times r_h} - 1 = \sqrt{1 * 1 * 0.5} - 1 = -0.293$$

or 29.3% reduction on TVC. End of example ▲

Example 2-4

If an order quantity equal to one half or 2 times the optimal value Q_W is placed, what will be the effect on the total variable cost?

$$\alpha = \frac{TVC(Q)}{TC_w} = \frac{1}{2}(\frac{1}{\beta} + \beta)$$

$$\begin{cases} \frac{Q}{Q_w} = \beta = 2 & \longrightarrow & \alpha = \frac{5}{4} \\ \frac{Q}{Q_w} = \beta = \frac{1}{2} & \longrightarrow & \alpha = \frac{5}{4} \end{cases}$$

Figure 2-5 shows the relation between $\alpha = \frac{TVC(Q)}{TC_w}$ and $\beta = \frac{Q}{Q_w}$. .
According to the figure, $\frac{TVC(Q)}{TC_w}$ Is slightly sensitive to $\frac{Q}{Q_w}$ when $0.5 \leq \frac{Q}{Q_w} \leq 2$.
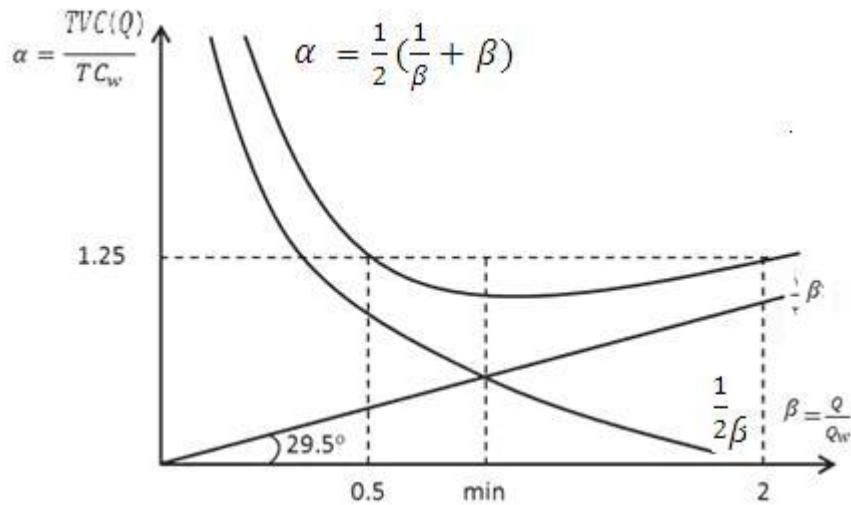
Fig. 2-5 The relationship between α and β.

Example 2-5

An item is purchased for $2000 per unit. The order cost is $4000. The annual holding cost fraction is 20% and the annual demand for the item is 20000. What order cost incur+5% in total variable cost compared to the optimum TVC?

Solution

$$\frac{TVC}{TC_w} = \frac{1}{2}\left(\frac{Q}{Q_w} + \frac{Q_w}{Q}\right) \qquad \frac{TVC}{TC_w} = \alpha \qquad \frac{Q}{Q_w} = \beta$$

$$\alpha = \frac{1}{2}\left(\beta + \frac{1}{\beta}\right) \qquad \Rightarrow \beta = \alpha \pm \sqrt{\alpha^2 - 1}$$

$$TVC = TC_w + 0.05TC_w \qquad \alpha = \frac{TVC}{TC_w} = 1 + 0.05 = 1.05$$

$$\beta = 1.051 \pm \sqrt{1.05^2 - 1} \quad \Rightarrow \quad \beta = 1.37 \ or \ 0.73$$

$$Q_w = \sqrt{\frac{2DC_O}{C_h}} = sqrt\left(\frac{2*20000*4000}{0.2*2000}\right) = 632$$

$$Q = \beta\, Q_w \cong 462 \ or \ 867.$$

Therefore placing an order of $Q = 462 \ or \ 867$ units will have a total variable inventory cost equal to $1.05TC_w$. This fact is shown in the figure below where the minimum occurs at $Q_w = 632$. ▲
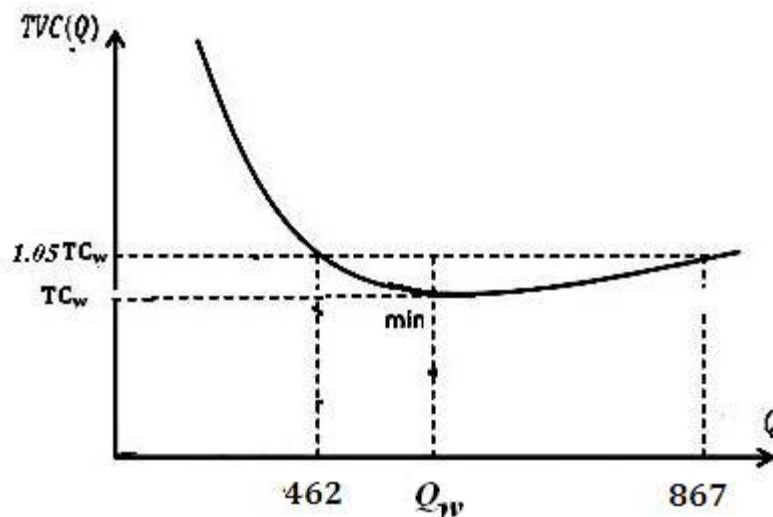
Note that TVC=(1-0.05) TC_w cannot be considered in this problem. (Why?)

EOQ model for items with discrete order quantity(Q)

When the order quantity is discrete rather than continuous and hence Q is a discrete variable, you cannot use differentiation approach to determine Q. Instead, the following approach could be used :

We know that $TC(Q) = \frac{C_O D}{Q} + C_h \frac{Q}{2} + PD$. Suppose the vendor supply an item in lots of size n only; therefore the order quantity Q has to be an integer multiple of n i.e. Q=K×n   where K=1,2,3,.. Let the optimum order quantity is Q^* and the minimum cost is TC(Q^*); if one n is added to or deducted from Q^*, the corresponding total cost would be greater or equal to $TC(Q^*)$

$TC(Q^*) \leq TC(Q^* + 1n)$

$TC(Q^*) \leq TC(Q^* - 1n)$

$$\Rightarrow \begin{cases} \dfrac{C_O D}{Q^*} + C_h \dfrac{Q^*}{2} \leq \dfrac{C_O D}{Q^* + n} + \dfrac{C_h(Q^* + n)}{2} & (I) \\[4mm] \dfrac{C_O D}{Q^*} + C_h \dfrac{Q^*}{2} \leq \dfrac{C_O D}{Q^* - n} + \dfrac{C_h(Q^* - n)}{2} & (II) \end{cases}$$

The following inequalities are derived from EQ. I &II:

$$\begin{cases} Q_W^2 = \dfrac{2C_O D}{C_h} \leq Q^*(Q^* + n) \\[4mm] Q_W^2 = \dfrac{2C_O D}{C_h} \geq Q^*(Q^* - n) \end{cases}$$

Proof for $\frac{2C_O D}{C_h} \leq Q^*(Q^* + n)$:

By multiplying $2Q^*(Q^* + n)$ to both sides of inequality $\mathbf{I}$:

$$2C_O D(Q^* + n) + C_h Q^{*2}(Q^* + n) \leq 2Q^* C_O D + C_h Q^*(Q^* + n)^2 \Rightarrow$$

$$2C_O DQ^* + 2C_O Dn + C_h Q^{*3} + C_h Q^{*2}n$$
$$\leq 2Q^* C_O D + C_h Q^{*3} + C_h Q^* n^2 + 2C_h Q^{*2}n \Rightarrow$$

$$2C_O Dn - C_h Q^* n^2 - C_h Q^{*2}n \leq 0 \Rightarrow \frac{2C_O D}{C_h} \leq Q^*(Q^* + n)$$

In a similar manner $\frac{2C_O D}{C_h} \geq Q^*(Q^* - n)$ is derived. Therefore:

$$Q^*(Q^* - n) \leq Q_W^2 = \frac{2DC_O}{C_h} \leq Q^*(Q^* + n) \qquad (2\text{-}13)$$

Notice that when n approaches zero, $Q^* = \sqrt{\frac{2DC_O}{C_h}} = Q_W$.

2-2-1 Calculation of order quantity

Solution No.1

Q^* is obtained by solving the inequality 2-13 and noting that it is a multiple of n   i. e. $Q^* = Kn$. $K = 1,2,3, \ldots$

Solution No.2

It can be shown mathematically that the best integer value is one of the two integers surrounding Q_w(Peterson & Silver , 1991, P187).

In other words from the two integer multiples of n surrounding Q_w (immediate value less than Q_w or greater than Q_w), the one with less TVC is the solution to 2-13 (adopted from page 123,Smith ,1989 as referenced by Ericson,1996 page 31)

Example 2-6

An item is purchased for $100 per unit. The order cost is $11. The annual holding cost fraction is 10% and the annual demand for the

item is 1200 units. If the vendor provides lots of 50 units only, How many lots do we buy in each order to minimize te inventory total cost?

Solution no.1:

The optimal order quantity Q^* satisfies

$$Q^*(Q^* - n) \leq \frac{2DC_o}{C_h} \leq Q^*(Q^* + n) \text{ and } Q^*(Q^* - 50) \leq \frac{2 \times 1200 \times 11}{10} \leq Q^*(Q^* + 50)$$

The following 2 inequalities have to be solved

$$Q^{*2} - 50Q^* - 2640 \leq 0 \qquad Q^{*2} + 50Q^* - 2640 \geq 0$$

Solving Inequality $Q^{*2} - 50Q^* - 2640 \leq 0$

$Q^{*2} - 50Q^* - 2640 = 0$ has two answers -32.14 and 82.4

The sign of the inequality in different sub-interval is as follows

Subinterval on Q	$-\infty$		-32.14	82.14	∞
sign		+		-	+

Q cannot be negative therefore $0 \leq Q^* \leq 82.14$.satisfies the inequality

Solving inequalit $Q^{*2} + 50Q^* - 2640 \geq 0$

$Q^{*2} + 50Q^* - 2640 = 0$ has two answers 32.14 and -82.4

The sign of the inequality in different sub-intervals is as follows

Subinterval on Q	$-\infty$		-82.14	32.14	∞
sign	+			-	+

Q cannot be negative; therefore $Q^* \geq 32.1$. satisfies the inequality.

The answer lies in $0 \leq Q^* \leq 82.14$ & $Q^* \geq 32.14$ that is

Q* lies In the interval $32.14 \leq Q^* \leq 82.14$ and is also a multiple on 50, therefore $Q^* = 50$.

   If the inequality were such that either 50 or 100 could have been the answer, we had to choose the one with less TVC.

Solution no.2:

$$Qw = \sqrt{\frac{2DC_O}{C_h}} = \sqrt{\frac{2 \times 1200 \times 11}{10}} = 51.4$$

$Q*$

The immediate value less than Q_w is 50 and the immediate value greater than Q_w is 100,

$$TVC(Q = 50) = 11 * \frac{1200}{50} + 10 * \left(\frac{50}{2}\right) = 514,$$
$$TVC(Q = 100) = 11 * \frac{1200}{100} + 10 * (\frac{100}{2}) = 632$$

The one with less TVC is the answer i.e. $Q^* = 50$. ▲

2-3 Safety stock model

   The difference between the classic EOQ model and the safety stock model is keeping an extra inventory known as safety stock(SS=M)in the warehouse of this system to cope with variations of D and T_L(Fig. 6.2)
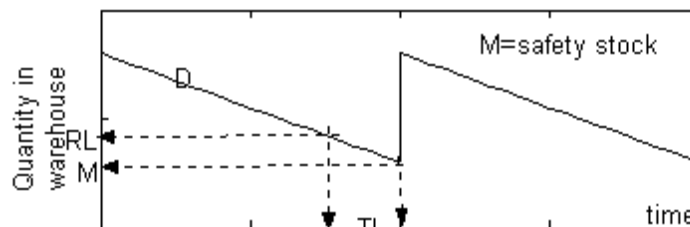
Fig. 2-6 Safety Stock Model

If the order quantity is Q, the total inventory cost would be:

$$TC = \frac{C_O D}{Q} + C_h \frac{M + M + Q}{2} + PD + PM \Rightarrow$$

$$TC = \frac{C_O D}{Q} + \frac{C_h Q}{2} + MC_h + P(D + M) \qquad (2-15)$$

The second derivative of TC with respect to variable Q ($= \frac{C_O D}{Q^3}$)is positive then TC has a minimum which satisfies $\frac{dTC}{dQ} = 0$.

$$\frac{dTC}{dQ} = 0 \Rightarrow Q^* = Q_W = \sqrt{\frac{2DC_O}{C_h}}.$$

$$TVC(Q^*) = \sqrt{2DC_O C_h} + MC_h = C_h(Q^* + M).$$

The reorder point in safety stock model is

$$ROP = \begin{cases} SS + DT_L & T_L < T^* \\ SS + DT_L - KQ^* & \left(K = \left[\frac{T_L}{T^*}\right] \leq T_L\right) & T_L \geq T^* \end{cases} \qquad (2-16)$$

Where $\left[\frac{T_L}{T^*}\right]$ denotes the integer part of $\frac{T_L}{T^*}$.

Note that when replacing the parameters in formulas, their dimensions must agree; e.g. if D is given per day and T_L is given in year, both must have the same time unit; e.g. multiply D by N (the number working days in a year).

In this model the following formulae might be useful:

$$Max\ inventory = Q_W + SS \qquad (2\text{-}17\text{-}1)$$

$$Min\ inventory = SS \qquad (2\text{-}17\text{-}2)$$

$$\text{Inventory Average} = SS + \frac{Qw}{2} \qquad (2\text{-}17\text{-}3)$$

2-4 Economic Order Interval(EOI) Model-Single item[1]

In this model the time interval between successive orders are the same and the main problem is determining the optimal interval(T) and the desired maximum inventory(Imax). Economic order interval is calculated by maximizing the total cost function. Under no stockout assumption, the annual total cost TC is:

Annual holding cost =

$$C_h \times \text{average annual inventoty} = C_h \frac{Q}{2} = C_h \frac{DT}{2}$$

If T is given in year, the annual number of orders is $m = \frac{1}{T}$ and therefore;

$$TC = C_O \frac{1}{T} + C_h \frac{DT}{2} + PD \qquad (2\text{-}18)$$

taking the derivative of the function with respect to T: $\frac{dTC}{dT} = 0 \implies$

$$T^* = \sqrt{\frac{2C_O}{DC_h}} \qquad (2-19)$$

Replacing T with T^* in Eq. 2-18 gives optimal annual cost:

$$TC^* = DC_h T^* + PD = C_h Q^* + PD \qquad (2\text{-}20)$$

where (Tersine,1994 page136)

$$Q^* = DT^* \qquad (2-21)$$

---

[1]Tersine(1985)page 596

Noting that the optimum occurs where the annual order cost equals annual holding cost, TC^* could also be calculated as follows:

$$TC^* = \frac{DC_hT^*}{2} + \frac{DC_hT^*}{2} + PD = DC_hT^* + PD \,.$$

The maximum inventory in this model must be large enough to satisfy demand during subsequent interval T and also during the lead time (Tersine,1994,page 136, Tersine, 1985,596)

$$E = I^*_{max} = DT^* + DT_L = D(T^* + T_L) \qquad (2-22)$$

or

$$,I^*_{max} = Q^* + DT_L \qquad (2-23)$$

Note that

-When replacing the parameters in formulas, their dimensions must agree; e.g. if D is given per month and TL is given in year, both must have the same time unit.

- In this model, there is no need to give a separate formula for reorder points(why?)

-If certainty conditions hold, there is no difference between the optimal T&Q of classic EOQ model and those of EOI model.

- In probabilistic models there are models titled fixed order size and fixed order interval in which D and TL might be random variables. As will be dealt in the related chapter, in this case to determine T and I_{Max}, the mean of D could be inserted in the above formulae. Further more, when placing an order, if the available inventory is A, then

$$Q = I_{Max} - A \qquad (2\text{-}24)$$

Example 2-6

An item is purchased for $10 per unit. The order cost is $30. The annual holding cost per unit is $3 and the annual demand for the item is 8000 units. If the lead time is 10 working days and there is 260 working days in a year, Find the time interval between 2 successive orders, the maximum inventory level and the annual total cost in the optimal state.

Solution

$$T^* = \sqrt{\frac{2C_O}{DC_h}} = \sqrt{\frac{2*30}{8000*3}} = 0.05 \text{ yr} = 0.05 * 260 = 13 \text{ days}$$

$$_{Max}^* = D(T^* + T_L) = 8000(\frac{13 + 10}{260}) \cong 708$$

$$Q^* = DT^* = 8000 * 0.05 = 400 \qquad \text{or}$$

$$Q^* = I_{Max} - DT_L = 708 - 8000(\frac{10}{260}) \cong 400$$

In this inventory system, every 13 working days an order of 400 units is placed. $TC^* = C_h Q^* + PD = 400 \times 3 + 10 \times 8000 = 81200\$ per yr. ▲

2-5 OQ Model -Back Order

In this model, any demand, when out of stock, is backordered and filled as soon as an adequate sized replenishment arrives (Peterson&Silver, 1991 p 209). It is assumed that when we are out of stock the demand arrives with the same rate(see Fig. 2-7)

Symbols

π fixed stockout cost per unit
$\hat{\pi}$ stockout cost per unit per year ($\hat{\pi} \neq 0$)
b   maximum backordering (stockout) quantity
$\bar{b}$ average backordering (stockout) quantity
s   maximum inventory in units
Q   Order quantity
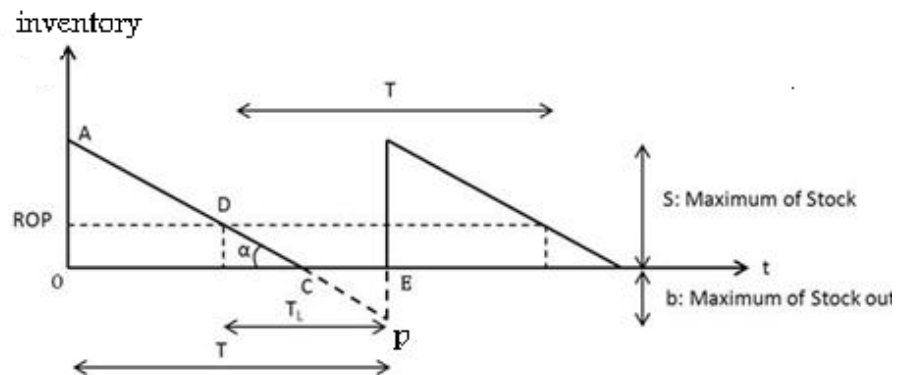
Fig. 2-7 EOQ Model with Back Order

2-5-1 Average inventory and stockout level

Below it is shown that :

Average inventory level per year ($\bar{I}$) is given by:

$$\bar{} = \frac{(Q-b)^2}{2Q} \qquad (2-25-1)$$

Average stockout per year ($\bar{b}$) is given by:

$$\bar{b} = \frac{b^2}{2Q} \qquad (2-25-2)$$

Proof

Assuming the rate of demand and the rate of stockout are the same, in Fig. 2-7 we have:

$$\frac{AO}{OC} = \tan\alpha = D, \qquad Q = b + s \Rightarrow s = Q - b = AO \Rightarrow OC = \frac{Q - b}{D}$$

$$\bar{I} = \frac{\text{Area of Triangle OAC}}{\text{time T}} = \frac{\frac{1}{2}(AO)(OC)}{OC + CE} = \frac{\frac{1}{2}(Q - b)\frac{(Q - b)}{D}}{T} = \frac{(Q - b)^2}{2TD}$$

Q=DT   then $\bar{I} = \frac{(Q-b)^2}{2Q}$,

Average stockout per year ($\bar{b}$):

$$\bar{b} = \frac{\text{Area of Triangle CEP}}{\text{time T}} = \frac{\frac{CE \times b}{2}}{T}$$

$CE = \frac{b}{\tan\alpha} = \frac{b}{D}$ then $\bar{b} = \frac{b \times b}{2TD} \Longrightarrow \bar{b} = \frac{b^2}{2Q}$. End of proof. ∎

Costs:

 Total cost includes total variable cost +PD

   Total variable cost(TVC) is comprised of order cost, carrying cost and stockout cost.

Variable cost for one period $= C_O + C_h \bar{I}T + \hat{\pi}\bar{b}T + \pi b$

Total annual cost $= C_O \frac{D}{Q} + C_h \bar{I}T \frac{D}{Q} + \frac{D}{Q}\hat{\pi}\bar{b}T + \pi b \frac{D}{Q} + PD$

 Since $\frac{DT}{Q} = 1$, $\bar{b} = \frac{b^2}{2Q}$ and $\bar{I} = \frac{(Q-b)^2}{2Q}$ then

$$TC(Q, b) = \frac{C_O D}{Q} + C_h \frac{(Q-b)^2}{2Q} + \hat{\pi}\frac{b^2}{2Q} + \frac{\pi bD}{Q} + PD \qquad (2\text{-}26)$$

2-5-2 Optimal order quantity(Q) and
maximum stockout (b) in EOQ model with backorder

Differentiating from Eq. 2-26 with respect to Q and b and solving the following simultaneous equations, yields the optimal answers:

$$
\begin{cases}
\dfrac{\partial TC}{\partial Q} = 0 \\[2mm]
\dfrac{\partial TC}{\partial b} = 0
\end{cases}
\Rightarrow
$$

$$
\begin{cases}
Q^* = \sqrt{\dfrac{\hat{\pi} + C_h}{\hat{\pi}}} \sqrt{\dfrac{2DC_O}{C_h} - \dfrac{(\pi D)^2}{C_h(\hat{\pi} + C_{h)}}} \;\; = \dfrac{\pi D}{C_h} + \left(1 + \dfrac{\hat{\pi}}{C_h}\right)b^* \quad \hat{\pi} \neq 0 \quad (2-27) \\[4mm]
b^* = \dfrac{1}{\hat{\pi} + C_h}\left(-\pi D + \sqrt{2DC_O C_h\left(1 + \dfrac{C_h}{\hat{\pi}}\right) - \dfrac{C_h(\pi D)^2}{\hat{\pi}}}\right) \hspace{2.5cm} (2-28)
\end{cases}
$$

or
$$
b^* = \dfrac{1}{\hat{\pi} + C_h}(C_h Q^* - \pi D) \quad (2-29)
$$

If Eq. 2-29 gives a negative or complex value then b=0. However, this does not mean that an optimal value for b^* is zero in this case, and therefore we cannot use $b^* = 0$ in the formulas that contain b^*.

2-5-3 Reorder level in EOQ with backorder model

The reorder point in this model is calculated from:

$$
ROP = \begin{cases}
DT_L - b^* & & T_L < T \\
DT_L - b^* - KQ^* & \left(K = \left[\dfrac{T_L}{T}\right] \leq T_L\right) & T_L \geq T
\end{cases} \quad (2-30)
$$

Note that in this model Imax ,$\bar{I}$ and TC are less than the corresponding quantities in classic EOQ model, and $Q^* > Q_W$. The following theorem is useful regarding determining the optimal value of the two-parameter function used in this model.

Theorem 2-1[1]

Second Derivative maximum-minimum test for functions of two variables. Let f(x,y) be of class C^3 on an open set U in R2. A point $(X_0.Y_0)$ is a (strict) local minimum of $f(x,y)$ provided the following three conditions hold:

(i) $\dfrac{\partial f}{\partial x}(x_0, y_0) = \dfrac{\partial f}{\partial y}(x_0, y_0) = 0$

(ii) $\dfrac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$

(iii) $D = \left(\dfrac{\partial^2 f}{\partial x^2}\right)\left(\dfrac{\partial^2 f}{\partial y^2}\right) - \left(\dfrac{\partial^2 f}{\partial x\, \partial y}\right)^2 > 0$ at (x_0, y_0)

D is called the discriminant of the Hessian. If in (ii) we have <0 instead of > 0 and condition (iii) is unchanged, then we have a (strict) local maximum.

Question : what is the criterion for a point to be a global optimum of function *f*.

Answer   : There is no simple answer; however if *f* is continuous and $\nabla f\,(x\,,y\,) = 0$ has only one answer, it is the global.

2-5-4 Optimal (Q) and (b) when $\hat{\pi} \neq 0\,\&\pi = 0$:

If the stockout cost per unit time for each unit is not zero $(\hat{\pi} \neq 0)$ and fixed stockout cost per unit is zero$(\pi = 0)$;   substituting $\pi = 0$ in Eq. 2-26 to2-29 yields the following results:

---

[1] Marsden,J.&. Trombaa (2003)page 216

$$Q^* = \sqrt{\frac{\widehat{\pi}+C_h}{\widehat{\pi}}}\sqrt{\frac{2DC_O}{C_h}} \qquad b^* > 0 \qquad\qquad (2\text{-}31)$$

$$b^* = \sqrt{\frac{2DC_OC_h}{\widehat{\pi}(\widehat{\pi}+C_h)}} = \sqrt{\frac{2DC_O}{C_h}}\left(\sqrt{\frac{\widehat{\pi}+C_h}{\widehat{\pi}}} - \sqrt{\frac{\widehat{\pi}}{\widehat{\pi}+C_h}}\right)$$

$$= Q^*\left(\frac{C_h}{\widehat{\pi}+C_h}\right) \qquad (2-32)$$

$$TC^* = \sqrt{2DC_OC_h}\sqrt{\frac{\widehat{\pi}}{\widehat{\pi}+C_h}} + PD = C_hs^* + PD = \widehat{\pi}b^* + PD \qquad (2\text{-}33)$$

$$Q = b + s \Rightarrow s = Q - b \Rightarrow$$

$$s^* = Q^* - b^* \qquad\qquad (2-33)$$

$$s^* = I^*_{max} = \sqrt{\frac{2DC_O}{C_h}}\sqrt{\frac{\widehat{\pi}}{\widehat{\pi}+C_h}} = Q^*\frac{\widehat{\pi}}{\widehat{\pi}+C_h} \qquad (2\text{-}34)$$

$$\overline{b}^* = \frac{b^{*2}}{2Q^*} = \frac{Q^*}{2} \times \left(\frac{C_h}{\widehat{\pi}+C_h}\right)^2 \qquad\qquad (2\text{-}35)$$

Also note that in this model if $\widehat{\pi} \neq 0$ and $\pi = 0$; we have:

$$Q^*_{backorder} = Q_W\sqrt{\frac{\widehat{\pi}+C_h}{\widehat{\pi}}} \qquad\qquad (2\text{-}36)$$

$$TVC^* = TC_W\sqrt{\frac{\widehat{\pi}}{\widehat{\pi}+C_h}} \qquad\qquad (2\text{-}3\vee)$$

$$b^*_{(\pi=0)} = \frac{TC_W}{\sqrt{\widehat{\pi}(\widehat{\pi}+C_h)}} = Q^*\frac{C_h}{\widehat{\pi}+C_h} \qquad \widehat{\pi} \neq 0 \quad (2\text{-}38)$$

and if the cost of holding one unit per unit time(Ch)is largish then :

$$Q^* = \sqrt{\frac{2DC_O}{\widehat{\pi}}}, \quad TC^* = \sqrt{2DC_O\widehat{\pi}}, \ b^* = Q^*.$$

Example 2-8

An item is purchased for $10 per unit. The order cost is $117.5. The daily holding cost per unit is 1% of the price and the monthly demand for the item is 125 units. The lead time is 10 working days and there is 200working days in a year. If back ordering is possible and the stockout cost per unit per day is $0.2.

Find the optimal order quantity, maximum of inventory, maximum of stockout, reorder point, the cycle time and the annual total cost in the optimal state. Also calculate the carrying cost and the stockout cost during a cycle time.

Solution

$$I_{daily} = 0.01 \ , P = \$10 \ \ , D = 125 \text{per month} , C_O = \$117.5 \ \ , T_L = 10 \text{ days}$$

$$\hat{\pi} = \$0.2 \ \text{ per day} \qquad \pi = 0$$

$$Q^* = \sqrt{\frac{\hat{\pi} + C_h}{\hat{\pi}}} \sqrt{\frac{2DC_O}{C_h}}$$

$$C_h = IP = 0.01 * 10 \text{per day} = .01 * 10 * 365 = 36.5 \ \$ \text{ per year}$$

$$\hat{\pi} = 0.2 * 200 = 40\$ \text{ per year}$$

Using MATLAB:

$$Q^* =$$

$$\text{sqrt}((.2 * 200 + 36.5)/(.2 * 200)) * \text{sqrt}((2 * 125 * 12 \\ * 117.5)/(36.5)) \cong 136$$

$$b^*_{(\pi=0)} = Q^* \frac{C_h}{\hat{\pi} + C_h} = 65$$

The maximum inventory is:

$$s^* = I^*_{max} = Q^* \frac{\hat{\pi}}{\hat{\pi} + C_h} = 70$$

$$\text{ROP} = \text{DT}_L - b^* = \frac{125*12*10}{200} - 65 = 10,$$

$$\text{TC}^* = \hat{\pi}b^* + \text{PD} = 0.2*200*65 + 10*125*200 = 252600\$$$

$$\text{T}^* = \frac{Q^*}{D} = \frac{136}{125 \times 12} = 0.09\text{yr} \rightarrow \text{T}^* = 0.09*200 = 18\text{days}$$

The carrying cost for a cycle time (T) is equal to $\hat{\pi} \times T = \$3.6$,

the stockout cost during T equals ChT = $0.01*10*18 = \$1.8$.

The reader should verify that $b^* = 65$ and $Q^* = 136$ satisfy

theorem 2-1▲

2-2-5 Some comments on backordering

In this section some comments are provided on the backordered EOQ model. Most of these comment could be verified using the following relationships especially Eq. (I).

$$\text{TC}(Q, b) = \frac{C_O D}{Q} + C_h \frac{(Q - b)^2}{2Q} + \hat{\pi} \frac{b^2}{2Q} + \frac{\pi b D}{Q} + \text{PD}$$

Differentiating with respect to b&Q :

$$\frac{\partial \text{TC}}{\partial b} = 0 \implies -\text{Ch}(Q - b) + \hat{\pi}b + \pi D = 0$$

$$\frac{\partial \text{TC}}{\partial Q} = 0 \implies \frac{1}{Q^2}(DC_O + \pi Db + \frac{\hat{\pi} + C_h}{2}b^2) = \frac{C_h}{2} \implies$$

$$\frac{1}{2}Q^2 = \frac{1}{C_h}(DC_O + \pi Db + \frac{\hat{\pi}}{2}b^2) + \frac{b^2}{2}$$

$$\begin{array}{l} \frac{\partial \text{TC}}{\partial \text{Q}} = 0 \\ \frac{\partial \text{TC}}{\partial \text{b}} = 0 \end{array} \Rightarrow (\hat{\pi}^2 + \hat{\pi}C_h)b^2 + 2\pi\hat{\pi}Db + (\pi D)^2 - 2DC_OC_h = 0 \quad \textbf{(I)}$$

$$\textbf{(I)} \Rightarrow$$

$$b^* = \frac{-\pi D + \sqrt{(\pi D)^2 + \frac{\hat{\pi}+C_h}{\hat{\pi}}(2DC_oC_h - \pi^2D^2)}}{\hat{\pi}+C_h} \qquad \hat{\pi} \neq 0$$

or Eq. 2-28 i.e.

$$b^* = \frac{1}{\hat{\pi}+C_h}\left(-\pi D + \sqrt{2DC_OC_h\left(1+\frac{C_h}{\hat{\pi}}\right) - \frac{C_h(\pi D)^2}{\hat{\pi}}}\right)$$

Comments on the model when $\hat{\pi} = 0$:
  a)If $b^*=0$
  As mentioned above

$$\frac{\partial TC}{\partial Q} = 0 \Rightarrow \frac{1}{Q^2}(DC_O + \pi Db + \frac{\hat{\pi}+C_h}{2}b^2) = \frac{C_h}{2}$$

$$\Rightarrow \frac{1}{Q^2}(DC_O + 0 + 0) = \frac{C_h}{2} \Rightarrow Q^* = \sqrt{\frac{2DC_o}{C_h}} = Q_W$$

i.e. the model would be the classic EOQ model in which stockout is not permitted.

b) If $b^*=\infty$

When b^* is largish it is preferred to place no order. In fact there would be no inventory system and an optimal back ordered cost of πD is incurred.

c) If $\pi D = TC_W = \sqrt{2DC_OC_h}$ or $\pi D = C_hQ_W$ or $\pi = \frac{\sqrt{2C_OC_h}}{\sqrt{D}}$

In this case from Eq. (I) it would concluded   optimal b could be any value ≥ 0. Q^* is dependent on the selected b^*.

d) If $\pi D \neq TC_W$ and $\hat{\pi} = 0$

In this case From Eq. (I) it is concluded that there is no positive solution for b . An also

if $\pi D \neq TC_W$ according to case f and e of this section, optimizing TC would result in either b=0 or b=∞

e)if $\pi D > TC_W = \sqrt{2DC_O C_h}$ or $\pi D > C_h Q_W$ or $\pi > \frac{\sqrt{2C_O C_h}}{\sqrt{D}}$

when $\hat{\pi}$ is very small, Eq.2-28 yields a complex number, and we have to use b=0 and according to the following equation derived above :

$$\frac{1}{Q^2}(DC_o + \pi Db + \frac{\hat{\pi} + C_h}{2}b^2) = \frac{C_h}{2}$$

$$b = 0 \quad \Rightarrow \quad \frac{1}{Q^2}(DC_o + 0 + 0) = \frac{C_h}{2} \quad \Rightarrow \quad Q = Q_W.$$

f) if $\pi D < TC_W$ & $\hat{\pi}$=0

if $\hat{\pi}$=0 then $b^* = \infty$. Because according to Eq.2-28 or its equivalent i.e.

$$b^* = \frac{-\pi D + \sqrt{(\pi D)^2 + (1 + \frac{C_h}{\hat{\pi}})(2DC_o C_h - \pi^2 D^2)}}{\hat{\pi} + C_h}, \hat{\pi} = 0 \Rightarrow$$

$$b^* = \frac{-\pi D + \sqrt{(\pi D)^2 + (1 + \frac{C_h}{0})(TC_w - \pi^2 D^2)}}{0 + C_h} = \infty$$

This means we do not have an inventory systems.

Some other comments

g) if $\pi = 0$ & $\widehat{\pi} \neq 0$ and finite

If the fixed cost of stockout is negligible and $(0 < \hat{\pi} < \infty$, then in this model b^* is always positive and it will be never zero or negative $(b^*>0)$.

h) if $\pi \neq 0$& $\widehat{\pi} \neq 0$ and finite

In this case if Eq.2-28 return a negative b* (b*< 0), let b=0 and order as much as   Q=Q_W. note that this does not mean that the optimal values for b and Q are respectively zero and Q_W($b^* \neq$ **0** and $Q^* \neq Q_w$).

i) if $\widehat{\pi} \neq 0$

if $\widehat{\pi} \neq 0$, b^* would be finite

j) if $\widehat{\pi} \neq 0$

if $\widehat{\pi} \neq 0$,use Eq.2-27 i.e. $Q^* = \sqrt{\frac{\hat{\pi}+C_h}{\hat{\pi}}}\sqrt{\frac{2DC_O}{C_h} - \frac{(\pi D)^2}{C_h(\hat{\pi}+C_h)}}$ when b^*>0, other wise when b^*<0 choose the Q_w as the order quantity; however it is not meant the optimal value is Q_w.

k)when b^* =0

If Eq. 2-28 returns b^*=0,let $Q^* = Q_w$ i.e the backordered model converts to classic EOQ model. However, when b=0 , and we let Q=Q_w if $\pi D \neq TC_W$ then $\frac{\partial TC}{\partial b} \neq 0$ and therefore b=0 in this case could not be optimal:

$$TC(Q,b) = \frac{C_O D}{Q} + C_h \frac{(Q-b)^2}{2Q} + \hat{\pi}\frac{b^2}{2Q} + \frac{\pi bD}{Q} + PD$$

$$\frac{\partial \text{TC}}{\partial \text{b}} = \frac{-\text{Ch}(Q-b)+\pi D+\hat{\pi}b}{Q} \qquad Q = \boldsymbol{Q_w} \ . \ b{=}0$$

$$\frac{\partial \text{TC}}{\partial \text{b}} = \frac{-\text{Ch}(\boldsymbol{Q_w}-0)+\pi D+0}{\boldsymbol{Q_w}} = \frac{-\text{Ch}\boldsymbol{Q_w}+\pi D}{\boldsymbol{Q_w}} = \frac{-\boldsymbol{TC_w}+\pi D}{\boldsymbol{Q_w}} \neq 0$$

Therefore in this case when $\pi D \neq TC_W$, b=0 cannot be the optimal value for b.

2-6 On-hand inventory and on-order inventory

Since in inventory books you may encounter the terms " on-hand inventory " and " on-order inventory " and also symbols r $\&r_h$, a short description of them is followed.

A firm's inventory position consists of the on-hand inventory plus on-order inventory. On-hand inventory is the amount of stock items available to be sold. Quantity on order is the amount ordered from a supplier/vendor but not yet received. This also includes quantities of items being made in a work order. r is the inventory on hand + the inventory on order and r_h is the available inventory.

For example for both classic EOQ (Wilson) model and back-order model if $T_L < $ T then $r_h = r$ is:

$$r = r_h = \begin{cases} DT_L & \text{Wilson EOQ Model} \\ DT_L - b^* & \text{Back} - \text{ordered EQO} \end{cases} \qquad T_L \ < \ T \quad (2\text{-}39)$$

If $T_L \geq T$

$r =$ on-order inventory is :

$$r_h = \begin{cases} DT_L - KQ^* & \text{EOQ Model} \\ DT_L - b^* - KQ^* & \text{Back} - \text{ordered} \end{cases} \quad K = \left[\frac{T_L}{T}\right] \quad T_L \geq T \quad (2-41)$$

Fig 2-8 EOQ model: On-hand +on-order inventory (r) when $T_L < T$	Fig 2-9 EOQ model : On-hand +on-order inventory (r) when $T_L > T$
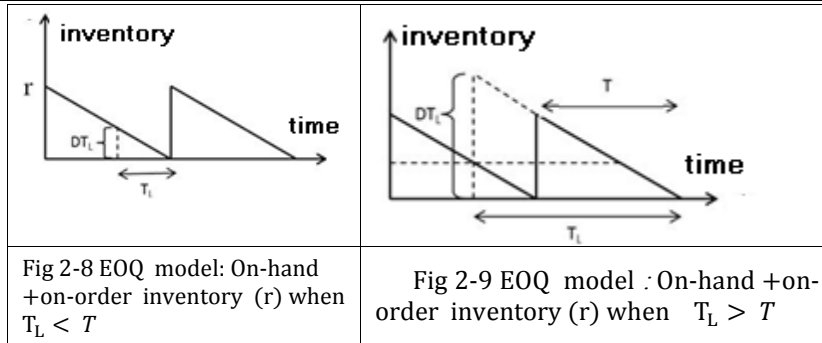
At the time point just before the arrival an order, the sum of on-hand inventory and on-order inventory is equal to the consumption during lead time i.e. $D \times T_L$[1];because r_h =ROP=DT_L –KQ $K=[\frac{T_L}{T}]$ is the on hand inventory (r_h) at this point and the on-order inventory is KQ, where K is the integral part of $\frac{T_L}{T}$. At point in time just after the arrival of an order quantity, DT_L is increased by Q, then

$$DT_L \leq \quad \text{On-hand+on-order inventory} \quad \leq DT_L + Q \qquad (2\text{-}43)$$

2-7 EOQ Model –lost sale case

In the previous models, there was either no stockout in the system, or the stockout was backordered and later compensated. Now we would like to analyze a case in which for a time say T_2 (see Fig 2-10)the demand is not satisfied and is lost (or is backordered without compensation). In this case the aim is to find the optimal value of T_2 and Q .

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[1] Hajji,1391,p37
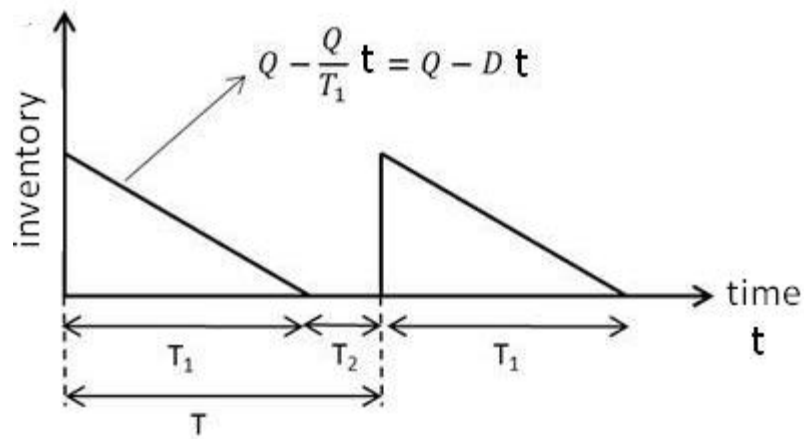
Fig. 2-10 Lost –sale Model

Now considering an inventory system in which there is stockout, but is not compensated and is lost, let us calculate its total cost(TC), which is actually an average annual cost.
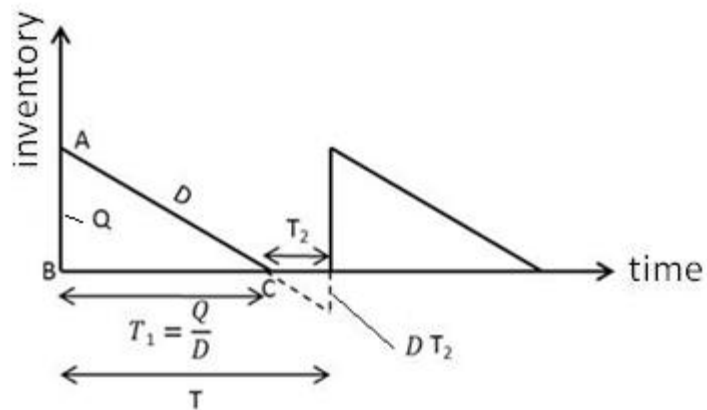
Fig. 2-11 Maximum stockout and inventory in lost –sale model

$$T = T_1 + T_2 \implies T = \frac{Q}{D} + T_2 = \frac{Q + DT_2}{D},$$

Number of annual cycles (m) and the average annual inventory($\bar{I}$) are:

$$m = \frac{1}{T} = \frac{D}{Q+DT_2},$$

$$\bar{I} = \frac{m \times \text{area of one triangle in Fig } 2-11}{1 \text{ year}} = \frac{1}{T}(\frac{1}{2}Q\frac{Q}{D}) \Longrightarrow$$

$$\bar{I} = \frac{D}{Q+DT_2}\left(\frac{1}{2}Q\frac{Q}{D}\right) = \frac{1}{2} \times \frac{Q^2}{Q+DT_2}.$$

If the annual carrying cost of unit product is C_h and the cycle time in year is T . then

annual average carrying cost in the system=$C_h\bar{I}$,

average carrying cost per cycle time= $C_h\bar{I}T = \frac{1}{2}\frac{Q^2}{D}C_h$.

Stockout cost per cycle $= \pi DT_2$

Number of cycles per year= $\frac{D}{Q+DT_2}$,

Annual stockout cost $= \pi DT_2(\frac{D}{Q+DT_2})$.

$$TC = C_O\left(\frac{1}{T}\right) + C_h(\bar{I}) + \pi DT_2\left(\frac{1}{T}\right),$$

$$TC = C_O\left(\frac{D}{Q+DT_2}\right) + \frac{C_h}{2}\left(\frac{Q^2}{Q+DT_2}\right) + \pi DT_2\left(\frac{D}{Q+DT_2}\right).$$

TC is a bivariate function, to find its optimum, its partial derivatives are set equal to zero:

$$\begin{cases} \dfrac{\partial TC}{\partial Q} = 0 \Longrightarrow -C_OD + \dfrac{C_hQ^2}{2} - \pi D^2T_2 + C_hQDT_2 = 0 & (I) \\[4mm] \dfrac{\partial TC}{\partial T_2} = 0 \Longrightarrow \pi D = \dfrac{C_OD}{Q} + C_h\dfrac{Q}{2} \quad 0 < T_2 < \infty, \ 0 < Q < \infty & (II) \end{cases}$$

$$(II) \Longrightarrow \pi D = \frac{2C_OD + C_hQ^2}{2Q} \quad \Rightarrow C_hQ^2 - 2\pi DQ + 2C_OD = 0 \Longrightarrow$$

$$Q = \frac{\pi D \pm \sqrt{(\pi D)^2 - 2C_O D C_h}}{C_h} = \quad Q = \frac{\pi D \pm \sqrt{(\pi D)^2 - TC_w^2}}{C_h} \Longrightarrow$$

$$Q = \frac{\pi D}{C_h} \pm \sqrt{\left(\frac{\pi D}{C_h}\right)^2 - \frac{2C_O D}{C_h}} \quad (III)$$

Now let us talk about the optimal value of Q and T_2 when the result of the radical in Eq.(III) is a complex number, zero, a real number or equivalently πD is less than, equal or greater than $TC_W = \sqrt{2C_O D C_h}$ in this model.

The value of Q^*

1) $\pi D < TC_W$

In Eq. (III), if $\pi D < TC_W$, then there would be no real answer for Q there is no inventory system i.e. Q=0. Later it will be shown that $T_2^* = \infty$. Substituting $T_2^* = +\infty$ & $Q = 0$ in annual average cost i.e.

$$TC = C_O \left(\frac{D}{Q + DT_2}\right) + \frac{C_h}{2}\left(\frac{Q^2}{Q + DT_2}\right) + \pi D \left(\frac{D}{\frac{Q}{T_2} + D}\right)$$

results $TC = \pi D$. Note there in no income in this case.

2) $\pi D = TC_W$

Eq. (III), If $\pi D = TC_W$, Eq. (III) has double root of $Q^* = \frac{\pi D}{C_h}$. It will be shown that T_2 could be any positive number.

3) $\pi D > TC_W$

Although Eq. III gives 2 answers for Q; but It will be shown that $T_2^* = 0$ and the order quantity is necessarily equal to Q_W

The value of T^*

There is a discussion about the optimum value of the cycle time(T) in some books including Bazargan (2021). The summary of the discussion is:

1) $\pi D < TC_W$

It is proved that $T_2 = \infty$

2) $\pi D = TC_W$

It is proved T_2 could be any positive number.

3) $\pi D > TC_W$

In this case $T_2^* = 0$

We summary the above discussion is as follows:

Case 1) $\pi D < TC_W$

In this case it is proved that $T_2 = \infty$ & $Q^* = 0$ i.e. there is no inventory system.

Case 2) $\pi D = TC_W$

In this case it is proved $Q^* = \frac{\pi D}{C_h}$ and T_2 could be any positive number.

Case 3) $\pi D > TC_W$

In this case $T_2^* = 0$ & $Q^* = Q_W$ i.e. the model converts to the classic model .

Note that:

-the product $\pi \times D$ is the cost of lost sale for the whole demand.

-the case in which $\pi D < TC_W$ is similar to one of the cases in backordered classic EOQ model where $\pi D < TC_W$ & $\hat{\pi} = 0$ and consequently we did not an inventory system.

-Some researches has been done to combine backordering with lost sales in EOQ model.

Example 2-9

Consider an EOQ model where lost sale is possible and

$C_h = 0.8 per\ unit\ per\ year, C_O = 0.2, T_L = 0.1\ year, \pi = 0.2$

Deter mine which the above cases is applicable here? And what should be done?

Solution

$\pi D = 0.2 \times 104000 = 20800, \ TC_W = \sqrt{2 \times 104000 \times 0.2 \times 0.8} = 182.4$

$\pi D = 20800 > TC_W = 182.4$

Then Case 3 is applicable here : $T_2^* = 0$ and $Q^* = Q_W = 228$.

Quantity Discount Models

The preceding models have assumed that the unit price of an item is the same regardless of the quantity in the batch; however, It is common for suppliers to give price discounts when order quantities are high. When discounts are factored into the calculation, the economic order quantity may change. In this section we deal with two types of discount models in inventory systems:

$$\text{Discount Model}\begin{cases} \text{Total Discount Model} \begin{cases} C_h \ \text{changing with price} \\ \text{Fixed} \ \ C_h \end{cases} \\ \text{Incremental Discount Model} \end{cases}$$

2-8 Total Discount Model

In this type of discount model, the unit price changes with order quantity in a manner similar to what the following table shows:

Price	Order quantity(Q)
P1=max(Pi)	$Q < Q1$
P2	$Q1 \leq Q < Q2$
P3	$Q2 \leq Q < Q3$
P4=min(Pi)	$Q \geq Q3$

Let R(Q) denote the purchase cost. In this type R(Q)= PQ. Figure 2-13 shows the function R(Q) in terms of Q. Q_1, $Q_{2,...}$ are called price break points.
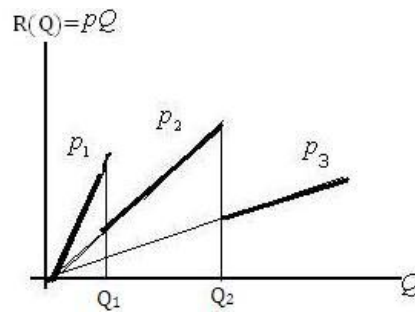
Fig. 2-12 Purchase cost of an order in a

Total discount model with 2 break points

Remember that $TC = C_O \frac{D}{Q} + \frac{IP}{2} Q + PD$ gives, total cost for each price. The graphical description of the components of the total cost is shown in figure 2-13
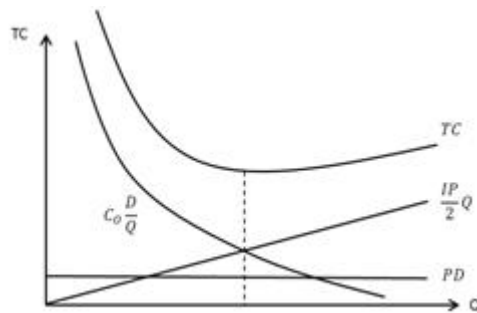
Fig. 2-13 The components of the total cost for one price

This model has two types i.e. either Ch changes with unit price or does not change with price.

2-8-1 Quantity Discount Model –Ch variable

If Ch changes with unit price and the price is similar to those given in the table above, the carrying cost reduces as the order quantity increases. The optimal order quantity in this type of the model could be determined using an algorithm described below.

2-8-1-1 The algorithm for finding optimal Q - Case 1: C_h variable

Figure 2-14 shows the curves of total cost for an all-unit-discount model where there are 3 price break points.
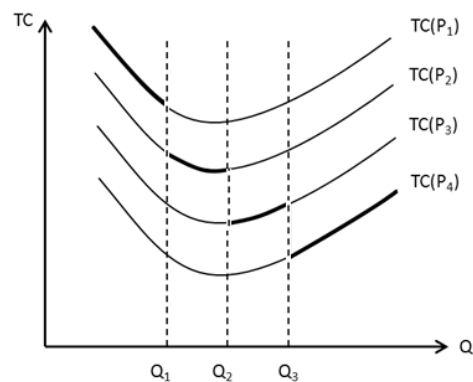
Fig.2-14 Total cost curves for a total- discount model

The steps of the algorithm of finding optimal order quantity Q* is (Dilworth,1989, page 263):

Step 1:

Calculate $Q_W = \sqrt{\dfrac{2DC_O}{IP}}$ for P=min(P_i). If Q_W is feasible i.e. satisfies the corresponding interval of this price, it is the answer to our problem, otherwise go to step2.

Step 2:

Calculate Q_W for the immediate higher price, if it is feasible calculate annual total cost $TC = \frac{C_o D}{Q} + \frac{C_h Q}{2} + PD$ for this price and the break points which are greater than it; the value with least TC is the optimal Q. Other- wise if the Q_W is not feasible go to step 3.

Step 3

Repeat Step 2 until a feasible Q_W is obtained.

The reorder point is $ROP = DT_L$ for $T_L < T$.

Example 2-10

The annual demand for a product is 2500, the yearly carrying cost of unit product is \$ 0.10 and the order cost is \$100. The supplier offers discount according the following Table:

Ro	Q	Pi
1	$0 \leq Q < 500$	5
2	$500 \leq Q < 2500$	4.75
3	$2500 \leq Q < 5000$	4.6
4	$Q \geq 5000$	4.5

Find the optimal order quantity, the cycle time T^*. There are 300 working days in a year and the lead time in 10 working days.

Solution

The minimum price is 4.5; $Q_W = \sqrt{\frac{2 \times 2500 \times 100}{0.1 \times 4.5}} \cong 1054$.

The amount does not satisfy the corresponding interval i.e. $Q \geq 5000$.

For the price P=4.6 $Q_{wP=4.6} \cong 1043$ is not feasible;

For P=4.75 $Q_{wP=4.6} \cong 1026$ is feasible.

We calculate TC for this value and the break points which are greater:

$$TC(Q = 1026, P = 4.75) = \frac{C_o D}{Q} + \frac{C_h Q}{2} + PD = 12362$$

$$TC(Q = 2500, P = 4.6) = 12175$$

$$TC(Q = 5000, P = 4.5) = 12425$$

There fore $Q^* = 2500$. $T^* = \frac{Q^*}{D} = \frac{2500}{2500} = 1$

There is no reorder point in 1 year▲

2-8-2 Quantity Discount Model –Case II:C_h Fixed

This type of discount model is similar to the previous one described in Sec 2-8-1 except that the carrying cost per unit product (C_h) does not depend on the price and is a fixed value. In this type Q_w is the same for all intervals. If Q_w satisfies the interval related to the minimum price, it is the optimal order quantity; otherwise calculate the total cost for Q_w and the price break points greater than it; the value with less TC is the answer.

Example 2-11

A supplier offers all- unit discount according to the following table for a product whose annual C_h is \$100, $C_o = \$100$ and annual D=1000. Find the

optimal order quantity.

Q	0 – 99	100-199	200 and more
price	500	400	300

Solution

The curves of total cost for the 3 prices are shown below.
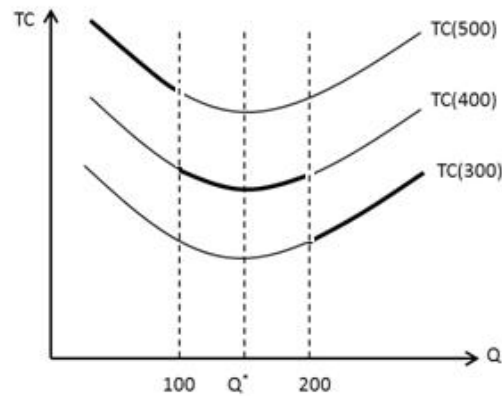
$$Q_w = \sqrt{\frac{2DC_O}{C_h}} = \sqrt{\frac{2 \times 1000 \times 100 \times 12}{100}} = 154.9 \approx 155$$

$TC(Q_W = 155, P = 400) = \sqrt{2 \times 12000 \times 100 \times 100} + 400 \times 12000 = 4815492$.

Q_w does not satisfy the interval related to the minimum price i.e. (200 and more). Therefore the total cost for the price break points more than Q_w has to be calculated.

$$TC(Q, P) = C_O \frac{D}{Q} + C_h \frac{Q}{2} + PD,$$

$TC(Q = 200, P = 300) = 3616000$, therefore $Q^* = 200$.

2-9 Converse of Discount Model (rate increase with quantity increase)

Here the purpose is to deal with the cases where a rate and the holding cost increases as the quantity increases. An example follows:

Suppose in a deterministic inventory system, stockout is not permitted and the rent of a warehouse is to paid as well as C_h=IP for each unit hold in warehouse. The rent is not included in the holding cost C_h and changes with the increase of order quantity. The total cost(TC) component of annual rent cost is determined based on the maximum inventory. Now we would like to calculate the economic order quantity. If

the annual rent per unit product is h_1, then:

$$TC(Q) = PD + \frac{C_O D}{Q} + IP \frac{Q}{2} + h_1 Q \qquad \frac{dTC}{dQ} = 0 \qquad \Rightarrow$$

$$Q^* = \sqrt{\frac{2DC_O}{2\,h_1 + IP}} \qquad\qquad (2-47)$$

The algorithm for determining the economic order quantity is similar to the previous algorithm described in Sec 2-8-1-1 and is illustrated below. In this model the break point located at left side of Q^* could also be the answer.

Example 2-12

The annual demand for a product is 10000, the order cost is $64, the unit price is $4 the annual cost of holding 1 unit product in warehouse is $ 0.25.

Find the economic order quantity. No stockout is permitted and as well as this cost, for each unit product a separate annual cost(h_1) has to be paid for holding the products in warehouse. The annual rent cost

per unit product depends on the quantity ordered(Q) as given in the following table:

Q	$0 \leq Q < 500$	$500 \leq Q < 750$	750& more
h_1	1	1.5	2

Solution

Starting with the least rate $h_1=1$

$$h_1 = 1 \quad \rightarrow Q_1 = \sqrt{\frac{2DC_O}{2\,h_1 + IP}} = \sqrt{\frac{2 \times 10000 \times 64}{2 \times 1 + 0.25 \times 4}} \cong 653 \quad \text{infeasible,}$$

Q_1 is not reasible because it does not satisfy 0< Q $\leq$500.

$$h_1 = 1.5 \quad \rightarrow Q_2 = \sqrt{\frac{2 \times 64 \times 10000}{2 \times 1.5 + 0.25 \times 4}} \cong 566 \quad \text{feasible}$$

$$h_1 = 2 \quad \rightarrow Q_3 = \sqrt{\frac{2 \times 64 \times 10000}{2 \times 2 + 0.25 \times 4}} \cong 506 \quad \text{infeasible}$$

Now we compare the total cost feasible $Q = 566$ and the break points 500 &750.

$$TC(Q) = PD + \frac{C_O D}{Q} + IP\frac{Q}{2} + h_1 Q$$

$$TC(Q = 566) = 4 \times 10000 + 64 \times \frac{10000}{566} + 0.25 \times 4 \times \left(\frac{566}{2}\right)$$
$$+ 1.5 \times (566) \cong 42263,$$

$$TC(Q = 500) = 4 \times 10000 + 64 \times \frac{10000}{500} + \frac{0}{25} \times 4 \times \left(\frac{500}{2}\right) + 1.5$$
$$\times (500) = 42280,$$

$$TC(Q = 750) = 4 \times 10000 + 64 \times \frac{10000}{750} + 0.25 \times 4 \times \left(\frac{750}{2}\right) +$$
$$2 \times (750) = 42728.$$

The minimum TC belongs to $Q = 566$; then it is the optimum.

Before dealing with another type of discount model , note that

$$PD = \frac{D}{Q}(PQ) = m \times R(Q) \qquad (2 - 48)$$

Where

$m = \frac{1}{T} = \frac{D}{Q}$: is the number of orders per unit time (year,…),

$R(Q) = PQ$ is the purchase cost per order. ▲

2-10 Incremental discount model

In all-unit discount model, the reduced price is valid for each unit in the order quantity, whereas in this variation of discount models that is called incremental discount, only the quantity exceeding the price break quantity is available at lower price. The goal is to determine the economic order quantity and the optimal order point with minimizing costs.

Purchase cost of order quantity Q

In this model the following recursive relationship is used to calculate the amount of money for buying the order quantity Q., R(Q) is given by the following relationship and show in Fig 2-15.

purchase cost per order $=$

$$R(Q)$$
$$= \begin{cases} R(q_j) + (P_j)(Q - q_j), & q_j < Q \leq q_{j+1} \quad j = 0,1,2,\dots,n \\ (P_0)(Q) & q_0 < Q \leq q_1 \end{cases}$$

$$R(q_0) = 0 \quad , q_0 = 0 \quad , q_{n+1} = \infty$$

$R(q_j)$ is the purchase cost of quantity q_j.
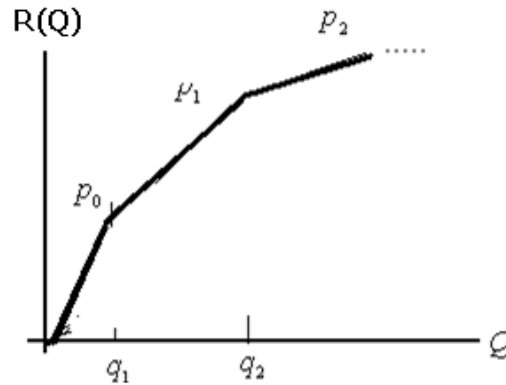
Fig. 2-15 Purchase cost of order quantity Q

2-10 Incremental Discount Model

Let $TC_j(Q)$ denote the total cost of order quantity Q when $q_j < Q \le q_{j+1}$. Using the relationship $TC = C_O \frac{D}{Q} + \frac{IP}{2}Q + PD$ we could write:

$$TC_j(Q) = C_O \frac{D}{Q} + \frac{I}{2}R(Q) + \frac{D}{Q}R(Q) \quad \Rightarrow$$

$$TC_j(Q) = C_O \frac{D}{Q} + \frac{I}{2}\left[R(q_j) + P_jQ - P_jq_j\right]$$
$$+ \frac{D}{Q}\left[R(q_j) + P_jQ - P_jq_j\right] \quad \Rightarrow$$

$$TC_j(Q) = \frac{D}{Q}\left[C_O + R(q_j) + P_jQ - P_jq_j\right] + \frac{I}{2}\left[R(q_j) + P_jQ - P_jq_j\right]$$
$$\Rightarrow$$

$$TC_j(Q) = \frac{D}{Q}\left[C_O + R(q_j) - P_jq_j\right] + \frac{I}{2}\left[P_jQ + R(q_j) - P_jq_j\right] + P_jD$$

$$q_j < Q \leq q_j + 1 \qquad j = 0,1,2, \dots, n$$

And therefore:

$$\frac{dTC_j(Q)}{dQ} = 0 \implies Q_j^* = \sqrt{\frac{2D[C_O + R(q_j) - P_j q_j]}{IP_j}} \qquad j = 0,1,2, \dots, n.$$

Plotting the $TC_j(Q)$ for $j = 0,1,2, \dots$ results in a figure such as Fig. 2-16
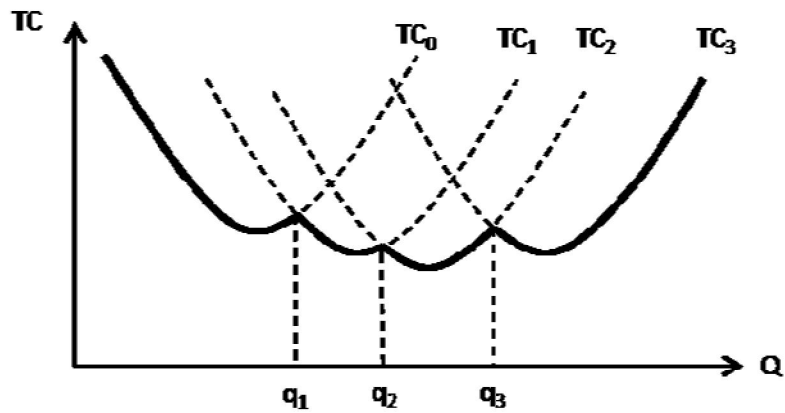
Fig. 2-16 Total Cost in incremental discount model

2-10-1 The algorithm for finding optimal Q - incremental model

The following steps determine the order quantity.

Step1: Calculate R(Q) for all break points:

$$q_0 = 0, \qquad R(q_0) = 0 \qquad\qquad R(q_1) = P_0 q_1$$

$$R(q_{j+1}) = R(q_j) + (P_j)(q_{j+1} - q_j), \qquad j = 0,1,2, \dots, n \quad (2-49)$$

Step2: Calculate $Q_j^* = \sqrt{\dfrac{2D[C_O + R(q_j) - P_j q_j]}{IP_j}}$ $\qquad (2-50),$

for $j = 0,1,2,\dots,n$,   and determine which of them are feasible.

Step3:

Calculate the total cost for the feasible Q_j^*'s   using the following relationship:

$$TC(Q_j^*) = \frac{D}{Q_j^*}[C_O + R(q_j) - P_j q_j] + \frac{I}{2}[P_j Q_j^* + R(q_j) - P_j q_j] + P_j D,$$
$$q_j < Q_j^* \le q_{j+1} \qquad\qquad (2\text{-}51)$$

The feasible Q_j^* with least total cost is the optimum.

Note:

It could be proved that a break point q_j could not be the local or global optimum of the total cost curves shown in Fig 2-16.

Example 2-13

The annual demand for a product is D=2500, annual I=0.1 and the order cost is \$100.   Find the optimal order quantity if the price per unit is as follows:

P_j	Q	comments
5	$q_0=0$,$q_1=500$	for the 1st 500 units
4.75	$q_1=500$,$q_2=2500$	for 501,502, ... 2500
4.6	$q_2=2500$,$q_3=5000$	for2501,2502, ... 5000
4.5	Quantities exceeding $q_3=5000$	for 5001 ‚5002 ‚...

Solution

Step 1: Calculation of R(Q) for break points q_j:

$$R(q_{j+1}) = R(q_j) + P_j(q_{j+1} - q_j)$$

$$R(q_0) = 0$$

$$R(q_1) = R(q_0) + P_0 q_1 = 0 + 5(500 - 0) = 2500$$

$$R(q_2) = R(q_1) + P_1(q_2 - q_1) = 2500 + 4.75(2500 - 500)$$
$$= 12000$$

$$R(q_3) = R(q_2) + P_2(q_3 - q_2) = 12000 + 4.6(5000 - 2500)$$
$$= 23500$$

Step 2:

$$Q_0^* = \sqrt{\frac{2D[C_O + R(q_0) - P_0 q_0]}{IP_j}} = \sqrt{\frac{2*2500[100 + 0 - 5 \times 0]}{0.1 \times 5}} = 1000 \text{ infeasible}$$

$$Q_1^* = \sqrt{\frac{2D[C_O + R(q_1) - P_1 q_1]}{IP_j}} = \sqrt{\frac{2*2500[100 + 2500 - 4.75 \times 500]}{0.1 \times 4.75}} = 1539$$
feasible

$$Q_2^* = \sqrt{\frac{2D[C_O + R(q_2) - P_2 q_2]}{IP_j}} = \sqrt{\frac{2*2500[100 + 12000 - 4.6 \times 2500]}{(0.1)(4.6)}} =$$
2554 feasible

$$Q_3^* = \sqrt{\frac{2*2500(100 + 23500 - 4.5 \times 5000)}{(0.1)(4.5)}} = 3496 \qquad \text{infeasible}$$

Step 3:

Calculation the total cost of feasible values $Q_1^* \& Q_2^*$ obtained in step 2

$$TC(Q_1^*) = \frac{D}{Q_1^*}[C_O + R(q_1) - P_1 q_1] + \frac{I}{2}[P_1 Q_1^* + R(q_1) - P_1 q_1] + P_1 D$$

$$TC(Q_1^* = 1539) = \frac{2500}{1539}[100 + 2500 - 4.75*500] + \frac{0.1}{2}[4.75*1539 + 2500 - 4.75*500] + 4.75*2500$$

$$TC(Q_1^* = 1539) = 12612$$

$$TC(Q_2^*) = \frac{D}{Q_2^*}[C_O + R(q_2) - P_2 q_2] + \frac{I}{2}[P_2 Q_2^* + R(q_2) - P_2 q_2] + P_2 D$$

$$TC(Q_2^* = 2554) = \frac{2500}{2554}[100 + 12000 - 4.6 * 2500]$$
$$+ \frac{0.1}{2}[4.6 * 2554 + 12000 - 4.6 * 2500] + 4.6 * 2500$$

$$TC(Q_2^* = 2554) = 12700$$

$$TC(Q_1^* = 1539) < TC\left(Q_2^* = 2554\right) \implies \quad Q^* = Q_1^* = 1539. \quad \blacktriangle$$

Example 2-14

Calculate the purchase cost per unit product for $q_j < Q \le q_{j+1}$.

Solution

The purchase cost of Q units in $q_j < Q \le q_{j+1}$ is:

$$R(Q) = R\left(q_j\right) + (P_j)\left(Q - q_j\right) =$$

$$P_j\left(Q - q_j\right) + \sum_{i=1}^{j} P_{i-1}\left(q_j - q_{j-1}\right) \implies$$

$\bar{P_j}$,The cost per unit is:

$$= \bar{P_j} = \frac{R(Q)}{Q} = P_j\left(\frac{Q-q_j}{Q}\right) + \sum_{i=1}^{j} P_{i-1}\left(\frac{q_j-q_{j-1}}{Q}\right) \blacktriangle$$

The inventory models for price change

A number of inventory models have been proposed to gain insight into the relationship between price changes including temporary

discounts, increase of price and order policy. Two models of these kinds are described below.

2-11 EOQ Model with sale price(temporary discount)[1]

Suppose a supplier discounts the unit price of one of his goods during a limited time in a regular replenishment period. The customer can buy once, as much as he wants with a temporary special reduction of price *d* per unit.

The aim is to take the advantage of the short-lived discount and determine the optimum size of a special order. Consider Fig. 2-17. At point there is 2 options : 1) To continue ordering the regular quantity Q; the first lot arrives with unit price p-d.
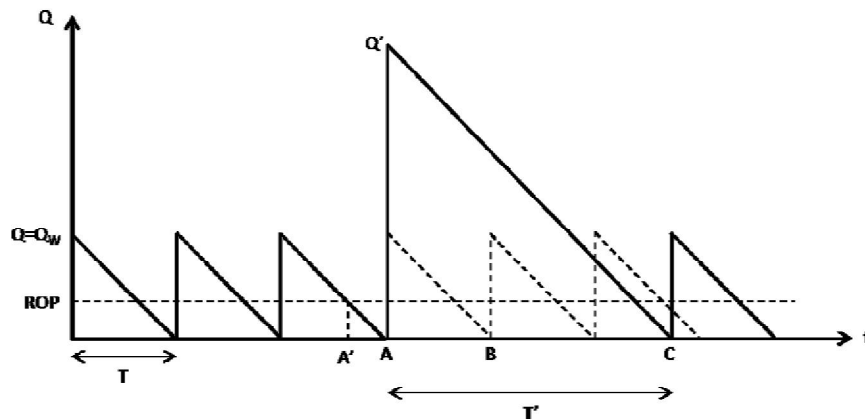
Fig. 2-17 Special sale price model $\quad A'A = T_L \quad (q = 0)$

2) To place a special order of size Q' with unit price p-d; when this amount is consumed, lots of regular size Q and unit price p arrive from point C. $T'' = \frac{Q'}{D}$ is the time needed to consume the special order. Saving in this model is equal to the difference between the cost during the time period T′ with and without the special order Q'. Now we would like to find that value of Q' which maximize the saving.

---

[1] Tersine(1994) page 113-116

Let K' denote the average cost during period T' if a special order of size Q' is placed. K' has three components i.e. order cost (C_O) , purchase cost : $(P - d)Q'$ and average carrying cost during the time period which is derived as following using Fig. 2-17:

Average inventory during time T'$=\dfrac{\frac{1}{2}Q'\text{T}''}{\text{T}'}=\dfrac{1}{2}Q'$

C_h is the holding cost of one unit product in 1 year

$C_h\,T'$ is the holding cost of one unit product during time period T'=

$$I(P - d) \times \frac{(Q')}{D}$$

average carrying cost during the time period

$$\text{T}'=I(P - d) \times \frac{(Q')}{D} \times \frac{1}{2}Q' = I(P - d)\frac{(Q')^2}{2D}. \quad \text{then :}$$

$$K' = C_O + I(P - d)\frac{(Q')^2}{2D} + (P - d)Q' \qquad (2-52)$$

Let K denote the average cost during period T' if a special order of size Q' is not placed. Noting that only the unit price of the first order *is P-d* and that of the other orders is p, we could write:

$$K = C_O\frac{Q'}{Q} + \begin{cases} I(P - d)\frac{Q}{2}\overbrace{\left(\frac{Q}{D}\right)}^{time\ AB} \\ +IP\frac{Q}{2}\underbrace{\left(\frac{Q'-Q}{D}\right)}_{time\ BC} \end{cases} +(P - d)Q + (Q' - Q)P.$$

To find the optimal one-time special order (Q'), the saving i.e. the difference in the above 2 cost must be maximized:

$$G = K - K' =$$

$$d \times (Q' - Q) - \frac{dIQ^2 - IPQQ'}{2D} - \frac{I(P-d)Q'^2}{2D} + \frac{C_O Q'}{Q} - C_O \quad (2-52)$$

where $Q = Q_W$.

The second derivative of G with respect to Q' is $-\frac{I(P-d)}{D} < 0$; then G has a maximum. To find the optimal Q', the first derivative is set equal to zero(tersine,1994 page 116):

$$\frac{dG}{dQ'} = 0 \Rightarrow d + \frac{IPQ}{2D} - \frac{I(P-d)Q'}{D} + \frac{C_O}{Q} = 0$$

$$\Rightarrow Q'^* = \frac{2dDQ + IPQ^2 + 2C_O D}{IQ(P-d)} = \frac{2dDQ + IPQ^2 + IPQ^2}{IQ(P-d)}$$

The above formula is valid when the stock position is zero (q=0)on the expiration date:

$$Q'^* = \frac{dD}{I(P-d)} + \frac{PQ}{P-d} = \frac{dD + IPQ}{I(P-d)} \qquad q = 0 \quad (2\text{-}53)$$

The saving due to placing this amount of order is((Tersine,1994 page 116) :

$$G^* = \frac{C_O(P-d)}{P}\left(\frac{Q'^*}{Q_W} - 1\right)^2 \qquad q = 0 \quad (2\text{-}54)$$

If the special order must be placed before the regular replenishment time and the stock position is q units on the expiration date, the optimizing formulations are(Tersine, 1994, page116):

$$Q'^* = \frac{dD + IPQ_W}{I(P-d)} - q \qquad\qquad q \neq 0 \qquad (2\text{-}55)$$

$$G^* = C_O \left[\left(\frac{Q'^*}{Q_W \sqrt{\frac{P}{P-d}}} \right)^2 - 1 \right] \qquad q \neq 0 \qquad (2\text{-}56)$$

$$\text{or} \quad G^* = C_O \left[\frac{P-d}{P} \left(\frac{Q'^*}{Q_W} \right)^2 - 1 \right]$$

Note that in this case

-we must have $Q'^* > Q_W \sqrt{\frac{P}{P-d}}$, if we want $G^* > 0$.

- when $d = 0$ and $q \neq 0 \implies Q'^* = Q_W - q$.

2-11-1 Summary : EOQ Model with sale

$$Q'^* = \frac{dD + IPQ_W}{I(P-d)} - q,$$

$$G^* = \begin{cases} \dfrac{C_O(P-d)}{P} \left(\dfrac{Q'^*}{Q_W} - 1 \right)^2 & q = 0 \\[4em] C_O \left[\left(\dfrac{Q'^*}{Q_W \sqrt{\dfrac{P}{P-d}}} \right)^2 - 1 \right] & q \neq 0 \end{cases}$$

Example 2-15

   The annual demand for a product of unit price $10 is 8000; the annual carrying cost of $1 is $ 0.30 and the cost order is $30. The supplier is offering a special discount during regular replenishment. He has temporarily reduced the unit price from $10 to $9.   There are 330 working days in a year.

a)The amount of the special discount that should be purchased.

b)The time interval between 2 consecutive order

c)The time in which the special order is consumed

d)The optimal saving due to ordering the special order

Solution

Annual D=8000 ,P=\$10,   d=1,   annual I=0.3 ,C_O=\$30

a)

$$Q'^* = \frac{dD}{I(P-d)} + \frac{PQ}{P-d} \qquad Q = Q_W = \sqrt{\frac{2DC_O}{IP}} = 400 \quad Q'^* = 3407$$

b)

$$T^* = \frac{Q^*}{D} = \frac{400}{8000} = \frac{1}{20}\text{yr} = \frac{1}{20}(200) = 10\text{days}$$

c)

$$T'^* = \frac{Q'^*}{D} = \frac{3407}{8000} = 0.43yr = 0.43 \times 200 \cong 86 \; days$$

d)

$$G^* = \frac{C_O(P-d)}{P}\left(\frac{Q'^*}{Q_W} - 1\right)^2 = 30\left(1 - \frac{1}{10}\right)*\left(\frac{3407}{400} - 1\right)^2 = \$1525.8$$

or G^* could be calculated by substituting $Q'^* = 3407$in the relationship which gives G:

$$G^* = d(Q' - Q_W) + \frac{I}{2D}[-dQ_W{}^2 + PQ_WQ'^* - (P-d)Q'^{*2}] \; + \frac{C_OQ'^*}{Q_W} - C_O$$

$$=1(3407\text{-}400) + \frac{-1\times\frac{3}{10}\times400^2 + \frac{3}{10}\times10\times400\times3407 - \frac{3}{10}\times(10-1)\times3407^2}{2\times8000} + \frac{30\times3407}{400} - 30$$

$$G^* = \$1526.22$$

The difference in the 2 values obtained for G^* could be due to the approximation used for fraction numbers. ▲

It is worth knowing that Martin(1994) gives a more accurate formula fo the average inventory in this model; however if the discount per unit product is small the above formulae from Tersine (1194) gives acceptable answers. Based on Martin's modifications Q'^* and G^* would be 43401 and 1533.75 respevtively.

2-12 EOQ Model -permanent reduction price

If we know a permanent decrease in the price will occur, no special order will be placed.

2-13 EOQ Model –known increase price

Suppose a supplier inform us that in the early future, the unit price increases from P to P'=P+a. Now We would like to know how much should we order with current price P before the new prices is applied(Tersine, 1994, page,117).

Symbols

Q'	The special order quantity before the higher price
q	The stock position at time when Q' is placed
Q_a^*	The economic order quantity with unit price P+a
Q'^*	The optimal value of Q'
a	The increase in unit price
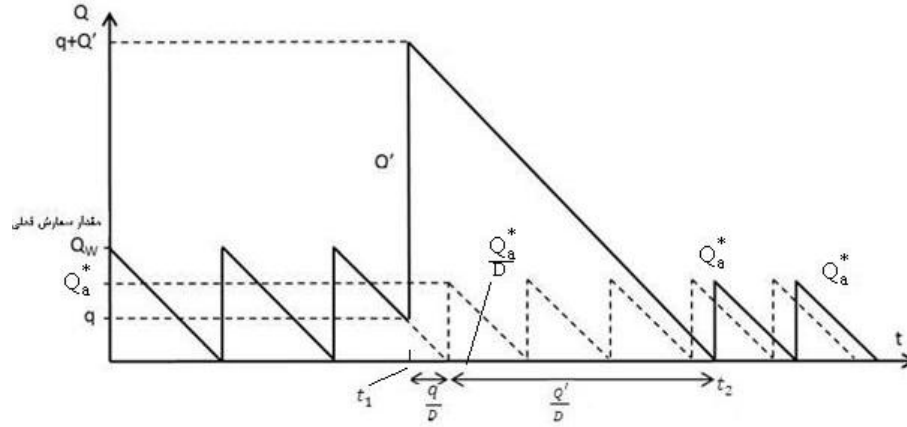
Fig. 2-18 Known increase price model ($T_L \cong 0$)

Suppose at time t_1 when the stock position is q units, an order Q' of unit price P is placed. At first, suppose $T_L \cong 0$ i.e. the lead time is ignorable and Q' arrives at time t_1(Fig. 2-18. The special order of Q' and the q units are enough for time $\frac{q+Q'}{D}$, after the time $t_2 = t_1 + \frac{q+Q'}{D}$ the new price becomes effective and the optimal order quantity will become:

$$Q_a^* = \sqrt{\frac{2DC_O}{I(P+a)}} \qquad (2\text{-}57)$$

The total cost in period t_2-t_1 if Q' is placed equals:

$$K' = C_O + C_h\left(\frac{q + Q'}{2}\right)(t_2 - t_1) + P(Q' + q) \qquad C_h = IP$$

If no special order is placed and all orders are purchased at unit price P+a, the total cost during t_2-t_1 is as follows

$$\text{K} = C_O \frac{Q'}{Q_a^*} + IP\frac{q}{2}\frac{q}{D} + I(P + a)\frac{Q_a^*}{2}\frac{Q'}{D} + (P + a)Q' + Pq.$$

To determine the optimal Q', $G = K - K'$ i.e. the saving in total cost must be maximized:

If $T_L \cong 0$, $\frac{dG}{dQ'} = 0$ $\Rightarrow$

$$Q'^* = Q_a^* + \frac{a(IQ_a^* + D)}{IP} - q \qquad\qquad (2-58)$$
$$Q'^* = Q_a^* + \frac{a}{P}(Q_a^* + \frac{D}{I}) - q \qquad\text{Equivalent formulae}$$

$$Q'^* = Q_a^*(1 + \frac{a}{P}) + \frac{aD}{IP} - q \qquad\qquad :$$

$$Q'^* = (P + a)\frac{Q_a^*}{P} + \frac{aD}{IP} - q$$

The optimum cost saving is(Tersine,1994, page 119):

$$G^* = C_O[\left(\frac{Q'^*}{Q_W}\right)^2 - 1] \qquad\qquad (2\text{-}59)$$

If the lead time is considerable then q is reduced to $q - DT_L$ Q' arrive and we have

$$Q'^* = (P + a)\frac{Q_a^*}{P} + \frac{a}{IP}D - \left(q - DT_L\right) \qquad (2-60)$$

If the Q' could be placed when the stock position reaches reorder point i.e. $q = ROP$, then (Tersine, 1994,page 120)

$$Q'^* = (P + a)\frac{Q_a^*}{P} + \qquad \text{If } q = ROP \qquad\qquad (2-61)$$
$$\frac{aD}{IP}$$

$$G^* = C_O\left(\frac{Q'^*}{Q_W} - 1\right)^2 \qquad \text{If } q = ROP \qquad\qquad (2-62)$$

Example 2-16

The annual demand for a product is 8000, the supplier is going to increase the current price $10 to $11 from the beginning of the next year. The cost of each order is $30, the lead time is 2 weeks, and the carrying cost of $1 per year is $.03. What amount should be purchased on the last day of this year before the price increase if the stock position is $q = 346$. What is the saving with this action? There are 52 working in a year?

Solution

$$a=1 \cdot P=10 \qquad Q_W = \sqrt{\frac{2DC_O}{IP}} = 400 \qquad Q_a^* = \sqrt{\frac{2DC_O}{I(P+a)}} = 381$$

$$Q'^* = Q_a^* + \frac{a}{IP}(IQ_a^* + D) - q + DT_L$$
$$= 381 + \frac{1}{0.3 \times 10}(0.3 \times 381 + 8000) - 346 + 8000 \times \frac{2}{52} = 3048$$

$$ROP = DT_L = 8000 \times \frac{2}{50} = 307$$

Since $\neq ROP$, G^* has to be calculated using Eq. 2-62:

$$G^* = C_O\left[\left(\frac{Q'^*}{Q_W}\right)^2 - 1\right] = 30 \times \left[\left(\frac{3048}{400}\right)^2 - 1\right] = 1712.$$

The above calculations show that at the end of the year an order of 3048 units with price $10 has to placed; this amount is consumed in $\frac{3048}{8000} = 0.381 year$; bringing $1712 saving. The next orders would be of amount 381 units and unit price $11. ▲

Economic Production Quantity(EPQ) Models

This model, which is also called finite production rate model or manufacturing model has the following types:

$$\begin{cases} a - \text{single item model} \begin{cases} 1 - \text{stockout not permitted} \\ 2 - \text{stockout permitted} \end{cases} \\ b - \text{Multiple item model} \begin{cases} 1 - \text{n items on n machines} \\ 2 - \text{machine for all items} \begin{cases} 1 - C_o \cong 0 \\ 2 - C_o \neq 0 \end{cases} \end{cases} \end{cases}$$

The models are described below.

2-14 Economic Production Quantity–single item

To deal with EPQ model, when we have single item, two cases are distinguished: either stockout is permitted or it is not permitted.

2-14-1 EPQ –single item,stockout unpermitted

In this model, it is assumed that a product is consumed with annual rate D at the same time it is produced gradually with annual rate R>D and therefore the remaining is stored with annual rate R-D in the ware- house simultaneously. No stockout is permitted. Needlesas to say that this model exists if R>D.
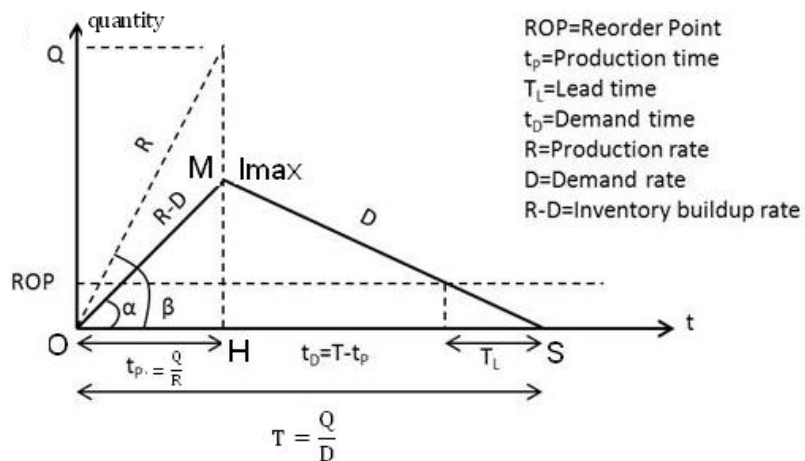
Fig.2-19 EPQ model   or Gradual arrival model

The annual total cost is:

$$TC = C_O \frac{D}{Q} + C_h \bar{I} + PD,$$

Where Q is the order quantity and $\bar{I}$ is the average inventory.

Referring to Fig 2-19 :

$$T = \frac{Q}{D}, \quad t_P = \frac{I_{Max}}{R-D}$$

$$\bar{I} = \frac{I_{max} \times T}{2T} = \frac{I_{max}}{2} \qquad I_{Max} = \frac{Q}{R}(R-D) = Q\left(1-\frac{D}{R}\right) \Longrightarrow$$

$$\bar{I} = \frac{Q(1-\frac{D}{R})}{2}$$

$$TC = \frac{C_O D}{Q} + C_h \frac{Q(1-\frac{D}{R})}{2} + PD$$

If Q is continuous, since $\frac{d^2 TC}{dQ^2} > 0$, then the   function TC has minimum which satisfies $\frac{d\,TC}{dQ} = 0$. This equation yields:

$$Q^* = EPQ = \sqrt{\frac{2DC_O}{IP\left(1-\frac{D}{R}\right)}} \qquad R > D \quad (2-63)$$

Let the sum of carrying cost and order cost for one year be dented by $TVC = C_O \frac{D}{Q} + C_h \bar{I}$;   substituting Q^* from Eq.2-63 in TVC yields:

$$TVC^* = \sqrt{2DC_O C_h (1-\frac{D}{R})} \qquad (2-64-1)$$

$$TVC^* = C_h Q^* \left(1 - \frac{D}{R}\right) \hspace{2cm} (2 - 64 - 2)$$

$$TVC^* = C_h Q_w \sqrt{1 - \frac{D}{R}} \hspace{2cm} (2 - 64 - 3)$$

In this model if **D = R** *or* $\frac{D}{R}$ **= 1,** no inventory is deposited.   If the production rate or the purchase rate is largish $\frac{D}{R} \cong \mathbf{0}$ and the model converts to the classic EOQ model.

2-14-1-1 The reorder point in EPQ model –single item

If the time of consumption in each cycle time is t_D(line HS in Fig.2-19) then the reorder point would be (Hajji, 2012 page 66):

$$ROP = r_h =$$

$$\begin{cases} DT_L - KQ & T_L - KT < t_D \\ T_L(D - R) + (K + 1)\left(\frac{R}{D} - 1\right)Q & T_L - KT > t_D \end{cases} \hspace{1cm} (2 - 65)$$

Where $K = \left[\frac{T_L}{T}\right]$ and T_L is the lead time.

Example 2-16

50 tons of a kind of chemical fertilizer is produced in a workshop . The fertilizer contains 30% urea which is produced in another work-shop which could produce 20 tons urea per year. T_L=2 days and for each setup the workshop shuts down for 2 working days but 10 people have to adjust and fix the machine for producing urea. When The workshop incurs A dollars during the shutdown and pays $10 per hour to each of these 10 people. There are 8 hours in each working day. The cost of producing 1 ton urea is P and annual I=0.1.   Find the

optimum values for Q, production time (t_p) and cycle time(T). Stockout is not permitted.
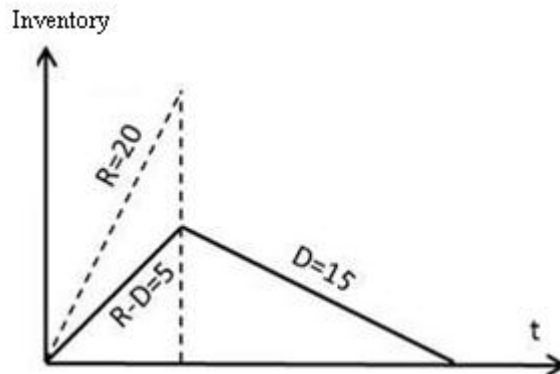
Solution

For the production of urea:

R=20;   D=50 × .30 = 15. yr C_O=(2×8×10×10)+2*A=1600+2A

If A is given the following relationship could be used :

$$Q^* = \sqrt{\frac{2 \times 15 \times C_O}{(0.1)P(1 - \frac{D}{20})}} \qquad\qquad t_P^* = \frac{Q^*}{R} \qquad\qquad T^* = \frac{Q^*}{D}.$$

End of example▲

The following table compares some relationships in EOQ and EPQ models.

	EPQ	EOQ
Order Quantity	$Q^* = \dfrac{Q_W}{\sqrt{1 - \dfrac{D}{R}}}$	Q_W
Maximum inventory on hand	$Q_W \sqrt{1 - \dfrac{D}{R}} = Q^*(1 - \dfrac{D}{R})$	Q_W
Average inventory	$\dfrac{Q_W}{2} \sqrt{1 - \dfrac{D}{R}} = \dfrac{Q^*}{2}(1 - \dfrac{D}{R})$	$\dfrac{Q_W}{2}$
TVC	$\sqrt{2DC_OC_h(1 - \dfrac{D}{R})}$	$\sqrt{2DC_OC_h}$
TVC	$Q_W C_h \sqrt{1 - \dfrac{D}{R}}$	$Q_W C_h$
TC	$\sqrt{2DC_OC_h(1 - \dfrac{D}{R})} + PD$	$\sqrt{2DC_OC_h} + P'D$

There are some variations for EPQ model including discounted EPQ model, EPQ model with stockout.   The description of backordered EPQ model follows.

2-14-2 Single-item EPQ model with backorders

In a Single-item backordered EPQ model, as depicted in Fig. 2-20, when the inventory reaches zero the production phase does not start and the demand continues with rate D. When the shortage reaches the allowable amount b the production phase begins.

Symbols

    b   maximum allowable shortage
    π fixed shortage cost per unit
    $\hat{\pi}$ shortage cost per unit product per year$(\hat{\pi} \neq 0)$

It is assumed that $\hat{\pi} \neq 0$ and when the production starts again and the product arrives, the backorders are fulfilled.

To Find the optimal order quantity(Q) and maximum allowable shortage(b), the total cost of the model has to be written and its partial derivatives be set to zero. The final results are:

$$Q^* = \sqrt{\frac{2DC_O}{C_h(1-\frac{D}{R})} - \frac{\pi^2 D^2}{C_h(C_h+\hat{\pi})}} \sqrt{\frac{\hat{\pi}+C_h}{\hat{\pi}}} \qquad (2-66)$$

$$b^* = \frac{[C_h Q^* - \pi D](1-\frac{D}{R})}{\hat{\pi}+C_h} \qquad (2-67)$$

$$I^*_{max=}Q^*\left(1-\frac{D}{R}\right)-b^* \qquad (2-68)$$
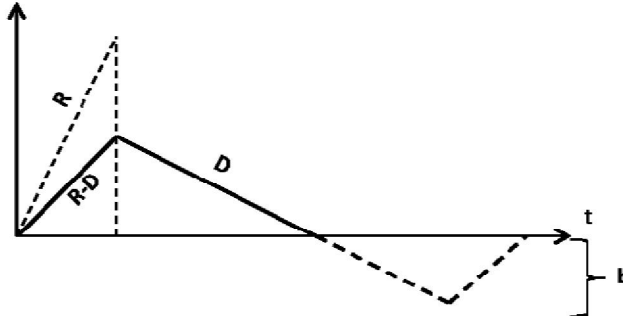
Fig. 2-20 A single-item EPQ inventory model with backorder

2-14-2-1 EPQ model with backorder - $\pi = 0$ & $\widehat{\pi} \neq 0$

Substituting $\boldsymbol{\pi = 0}$ in Eqs. 2-66 &2-67 results in the followings:

$$Q^* = \sqrt{\frac{2DC_O}{C_h\left(1-\frac{D}{R}\right)}} \sqrt{\frac{\hat{\pi}+C_h}{\hat{\pi}}} \qquad (2\text{-}66\text{-}1)$$

$$TVC^* = \sqrt{2DC_O C_h\left(1-\frac{D}{R}\right)} \sqrt{\frac{\hat{\pi}}{\hat{\pi}+C_h}} \qquad (2\text{-}69)$$

$$TVC^* = C_h Q_W \sqrt{\left(1-\frac{D}{R}\right)\frac{\hat{\pi}}{\hat{\pi}+C_h}}$$

$$TVC^* = C_h Q^*\left(1-\frac{D}{R}\right)\sqrt{\frac{\hat{\pi}}{\hat{\pi}+C_h}}$$

It is worth mentioning that optimal b reduces to $\boldsymbol{b^* = \dfrac{1-\frac{D}{R}}{1+\frac{\hat{\pi}}{C_h}}\boldsymbol{Q^*}}$ when

$\boldsymbol{\pi = 0}$.

2-15 Make or Buy Decision

A make-or-buy decision involves an act of using cost-benefit to make a choice between manufacturing a product internally or purchasing it from an external source. To cope with this decision problem , use EOQ model for buying and EPQ model for manufacturing , choose the one with less total cost (TC not TVC).

2-16 Economic Production Quantity:Multiple-item

Symbols

$(C_h)_i$	Annual holding cost for product # i
$(C_O)_i$	Setup cost for product # i
D_i	Annual demand for product # i
$d_i = \frac{D_i}{N}$	daily demand for product # i
$\bar{I}_i$	Average inventory of product # i
$m = \frac{D_i}{Q_i}$	Annual number of cycles (production runs)
N	Number of working days in a year
P_i	Unit production cost of product # i
Q_i	Order Quantity for product # I per cycle
R_i	Annual potential production rate of product # i
S_i	The setup time required for product # i
$(t_p)_i$	The production time for product # i
$t_{P_i}^*$	Optimal $(t_p)_i$
T^*	The time between two successive setups
TC_i	annual total cost of product # i
T_o^*	The time between two successive setups for the case the setup times are negligible

For determining the production quantity of each product in multiple- item EPQ, 2 cases are distinguished: case 1 in which each of our *n* products are produced on *n* separate machines and case 2 in which our *n* products are produced on *only one* machine or station where the number of cycles are the same for all n products.

2-16-1 Multiple-item EPQ model: n machines for n products with no constraints

When we have n products that could be manufactured on n separate machines and there is no constraint, the purpose of is to determine the optimal production lot size of each product in order to minimize the total cost (TC)of system including set up costs, holding costs of raw materials and finished products as well as production costs i.e.

$$TC = \sum_{i=1}^{n} TC_i = \sum_{i=1}^{n} \frac{C_{O_i} D_i}{Q_i} + \sum_{i=1}^{n} \frac{C_{h_i}}{2} Q_i \left(1 - \frac{D_i}{R_i}\right) + \sum_{i=1}^{n} P_i D_i.$$

To find the optimal values of Q_i's, the partial derivatives are set equal to zero:

$$\frac{\eth TC}{\eth Q_i} = 0 \quad \Longrightarrow \quad -\frac{C_{O_i} D_i}{Q_i^2} + \frac{C_{h_i}}{2}\left(1 - \frac{D_i}{R_i}\right) = 0 \quad \Longrightarrow$$

$$Q_i^* = \sqrt{\frac{2 D_i C_{O_i}}{C_{h_i}\left(1 - \frac{D_i}{R_i}\right)}} \qquad i = 1, 2, \dots, n \qquad (2-70)$$

The optimal total cost and cycle times are obtained from:

$$TC^* = \sum_{i=1}^{n} \sqrt{2 D_i C_{O_i} C_{h_i}\left(1 - \frac{D_i}{R_i}\right)} + \sum_{i=1}^{n} P_i D_i \qquad (2-71)$$

$$T_i^* = \frac{Q_i^*}{D_i} \qquad\qquad (2\text{-}72)$$

The required time for producing product # i is derived from $t_{p_i} = \frac{Q_i}{R_i}$.

If we use Fig 2-19,the average inventory of Product No. i is calculated as follows:

$$\overline{I}_i = \frac{\frac{(OS)(HM)}{2}}{R_i - D_i^{OS} Q_i} = \frac{HM}{2} = \frac{(R_i - D_i)t_{p_i}}{2} \implies$$

$$\overline{I}_i = \frac{R_i - D_i}{2}(\frac{Q_i}{R_i}) = \frac{Q_i}{2}(1 - \frac{D_i^2}{R_i}),$$

and the maximum inventory of product # i would be equal to:

$$I_{Max})_i = (R_i - D_i)t_{P_i}$$

In this model the annual number of setups for a product in not necessarily equal to that of the other product.

2-16-1 Multiple-item EPQ model: 1 machine for n products

Suppose would like to apply EPQ model to plan manufacturing of n products on the same machine and each product has to be produced m times a year. The following assumptions are needed in the multiple-item EPQ model

-Each tome, only one product is produced on the machine

- The number of setups and cycles for manufacturing all n products are assumed the same and constant.

-The number denoted by m equals $m = \frac{D_i}{Q_i}$, $i = 12, \dots, n$.

- The reciprocal of m is the time between two consecutive: $T = \frac{1}{m}$.

-$D_i, R_i,$ demand and production rates for product # i,   ares assumed the same during all production cycle times and so is the production rate.

-The setup cost for product # i is assumed independent of the order of producing the items on the machine

To deal with this model, 2 situations are supposed to be discussed:

Either the setup times are negligible($S_i \cong 0$) or they are considerable and cannot be ignored($S_i \neq 0$).

2-16-2-1 Multiple-item EPQ model: 1 machine & $S_i \cong 0$

Here we would like to consider the multiple-item EPQ model having 1 machine with negligible setup times $(S_i \cong 0)$ available for producing n products (Fig. 2-21) where $\frac{D_i}{R_i} < 1$ $i = 1,2, \dots ,n$.
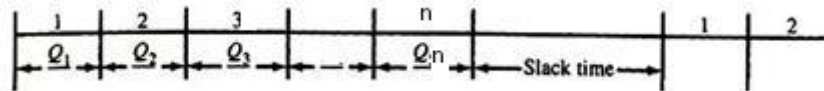
Fig. 2-21 EPQ model- multiple item& $S_i = 0$

To reach reasonable results in this model, as you will notice later, we must have $\sum_{i=1}^{n} \frac{D_i}{R_i} < 1$. The average inventory of product # i could be written as follows:

$$\bar{I}_i = \left(\frac{R_i - D_i}{2}\right)\left(\frac{Q_i}{R_i}\right) = \frac{Q_i}{2}\left(1 - \frac{D_i}{R_i}\right) = \frac{D_i}{2m}\left(1 - \frac{D_i}{R_i}\right),$$

Therefore the total cost of Product # i is:

$$TC_i = C_{O_i}m + C_{h_i} \times \frac{D_i}{2m}\left(1 - \frac{D_i}{R_i}\right) + P_i D_i = C_{O_i}\frac{D_i}{Q_i} + \frac{C_{h_i}}{2}Q_i\left(1 - \frac{D_i}{R_i}\right) + P_i D_i,$$

The total cost of the system:

$$TC = \sum_{i=1}^{n} TC_i = \sum_{i=1}^{n} C_{O_i}m + \sum_{i=1}^{n} C_{h_i}\left(\frac{D_i}{2m}\left(1 - \frac{D_i}{R_i}\right)\right) + \sum_{i=1}^{n} P_i D_i.$$

$$\frac{dTC}{dm} = 0 \Longrightarrow$$

$$m^* = \sqrt{\frac{\sum_{i=1}^{n}(C_h)_i D_i \left(1 - \frac{D_i}{R_i}\right)}{2\sum_{i=1}^{n}(C_O)_i}} \qquad\qquad (2-73)$$

then

$$TC^* = TC(m^*) = 2m^* \sum_{i=1}^{n}(C_O)_i + \sum_{i=1}^{n} P_i D_i \qquad (2-74)$$

$$Q_i^* = \frac{D_i}{m^*} = D_i T_0^* \qquad\qquad (2\text{-}75)$$

Let T_0 dentote the cycle time when the setup times are negligible; then:

$$TC^* = \frac{2\sum_{i=1}^{n}(C_O)_i}{T_0^*} + \sum_{i=1}^{n} P_i D_i \qquad\qquad (2\text{-}76)$$

where $T_0^* = \frac{1}{m^*}$.

This EPQ model cannot be use unless $\sum_{i=1}^{n}\frac{\frac{D_i}{R_i}}{N}$ (Tersine,1994,page 128).;or $\sum_{i=1}^{n}\frac{D_i}{R_i} < 1$. The difference of right hand side from left hand side is dented by α which is a dimentionless ratio:

$$\alpha = 1 - \sum_{i=1}^{n}\frac{D_i}{R_i} \qquad\qquad (2\text{-}77).$$

α is sometimes called the free or idle time of the station or the machine used for production. That is because αN is the number of working days the machine is idle.This time in year is equal to $\frac{\alpha N}{N} = \alpha$. The multiple- item EPQ model has feasible answer if α > 0. In this model

$$T_0^* = \sqrt{\frac{2\sum_{i=1}^{n}(C_O)_i}{\sum_{i=1}^{n}[(C_h)_i D_i\left(1-\frac{D_i}{R_i}\right)]}} \qquad (2\text{-}78).$$

Note that T_0^* is also valid for the case in which the setup times(S_i) are not zero but their sum in a year is less than α;however , if their

sum is greater than α, as you will see later if $\alpha = 1 - \sum_{i=1}^{n} \frac{D_i}{R_i} > 0$ the

cycle time is calculated from $T^* = Max \left\{ T_0^*, \frac{\sum_{i=1}^{n} S_i}{\alpha} \right\}$.

Example 2-18

What do you suggest the production cycle for the group of products in the following table. Assume $S_1 = S_2 = S_3 = S_4 = S_5 \cong 0 \ and \ 250$ working days per year . what is the optimal production run size and total cost(Tersine, 199, page 129).

product	Annual Demand	price	Daily production rate	Annual holding Cost per unit	Setup cost
i	D_i	P_i		C_{hi}	$(C_o)_i$
1	5000	6	100	1.6	30
2	1000	5	400	1.4	25
3	7000	3	350	0.6	30
4	15000	4	200	1.15	27
5	4000	6	100	1.65	80
sum					202

Solution

$\alpha = 1 - \sum_{i=1}^{n} \frac{D_i}{R_i}$

$=1 - (\frac{5000}{250 \times 100} + \frac{10000}{250 \times 400} + \frac{7000}{250 \times 350} + \frac{15000}{250 \times 200} + \frac{4000}{250 \times 100}) = 0.16$

Since $\alpha > 0,$ the problem has answer to the optimal production runs(m^*).

$$m^* = \sqrt{\frac{\sum_{i=1}^{n} (C_h)_i D_i \left(1 - \frac{D_i}{R_i}\right)}{2 \sum_{i=1}^{n} (C_O)_i}} = \sqrt{\frac{40\ 483}{2 \times 202}} \cong 10.$$

This means the there are 10 runs per year for each product to meet the corresponding demands.

When the setup times are negligible ($S_i \cong 0$), the number of production runs are dented by m_0 whose oiptimal value in the example is $m_0^* = 10$.   The production cycle (the time between 2 successive production runs for each of the 5 products is equal to:$T_0^* = \dfrac{1}{m_0^*} = \dfrac{1}{10}$ yr.

The production run size for each product calculated from

$\qquad Q_j^* = D_j T_0^*$ $j = 1,2,3,4,5$ is given in the following table:

product	i	$D_i(1 - \dfrac{D_i}{R_i})$	$C_{h_i} D_i(1 - \dfrac{D_i}{R_i})$	Production run size $Q_i^* = \dfrac{D_i}{m^*} = D_i T^*$	number of days in each cycle machine busy producing Q_i^* $tp_i = \dfrac{Q_i^*}{R_i}$
	1	4000	6400	500	5
	2	9000	12600	1000	2.5
	3	6440	4864	700	2
	4	1500	12074	1500	7.5
	5	3360	5544	400	4
	su		40483		21
Note:R_i is the annual production rate for product # i. e.g.$R_1 = 250 \times 100 = 25000$					

The machine cycle time is $\dfrac{N}{m^*} = \dfrac{250}{10} = 25$ days and according to

the above table $\sum_{i=1}^{5} t_{P_i} = 5 + 2.5 + 2 + 7.5 + 4 = 21$ days. Then in each cycle the machine is idle for 4 days. The optimal total cost is given by Eq.2-74:

$$TC(m^*) = 2m^* \sum_{i=1}^{n}(C_O)_i \ + \sum_{i=1}^{n} D_i P_i$$

$TC(m^*) = 2(10)(202) + (30000 + 50000 + 21000 + 60000 + 24000)$
$\qquad\qquad = 189040$

End of example. ▲

2-16-2-2   Multiple-item EPQ model: 1 machine & $S_i \neq 0$

This section deals with multiple-item EPQ model when the machine setup time for each product is not negligible and the production runs(m) for each product in a year is such that:

$$m = \frac{D_1}{Q_1} = \dots = \frac{D_n}{Q_n};$$

and the cycle time is equal to :

$$T = \frac{Q_1}{D_1} = \dots = \frac{Q_n}{D_n}.$$

The time required by the machine to produce the amount Q_j of product #j is

$$t_{P_j} = \frac{Q_j}{R_j} \qquad j = 1,2,\dots,n.$$

Let T denote the time between two successive setups for product j including the non zero setup time S_j: $T = \frac{Q_j}{D_j}$.

The optimal T $(T^* = \frac{Q_j^*}{R_j})$ is not less than T_0^* (the time between two successive setups for product j when $S_j = 0$): $T^* \geq T^*_0$ (**I**)

In each machine cycle time, each product is produced once and It is obvious that :

$$\sum S_j + \sum t_{P_j} \leq T \qquad\qquad \sum S_j + \sum \frac{Q_j}{R_j} \leq T.$$

The number of production runs for product j to produce amount D_j is equal to $m = \frac{D_j}{Q_j}$ and in the optimal state $m^* = \frac{D_j}{Q_j^*}$.

Since $\sum S_j + \sum \frac{Q_j}{R_j} \leq T$ and $Q_j^* = D_j T^*$ then

$$\sum S_j + \sum \frac{D_j}{R_j} T^* \leq T^* \implies T^* \geq \frac{\sum S_j}{1 - \sum_{j=1}^{n} \frac{D_j}{R_j}}$$

Let $T_{min} = \frac{\sum S_j}{1 - \sum_{j=1}^{n} \frac{D_j}{R_j}}$ therefore $T^* \geq T_{min}$ **(II)**

Considering Eq. **(I)** & **(II)** we could write:

$$T^* = \text{Max}\{T_0^*, T_{min}\} \qquad (2\text{-}79)$$

where

$$T_0^* = \sqrt{\frac{2 \sum_{i=1}^{n}(C_O)_i}{\sum_{i=1}^{n}(C_h)_i D_i \left(1 - \frac{D_i}{R_i}\right)}} \qquad T_{min} = \frac{\sum_{j=1}^{n} S_j}{1 - \sum_{j=1}^{n} \frac{D_j}{R_j}} = \frac{\sum S_j}{\alpha}.$$

$\alpha = 1 - \sum_{j=1}^{n} \frac{D_j}{R_j} > 0$ is a necessary condition for the existence T^*.

Note that:
- S_j 's are not necessarily equal.
-production run size is
$$Q_j^* = D_j T^* \qquad (2\text{-}80)$$
-Eq. (2- 79) is also applicable when setup times are zero.

Example 2-19

Assuming 250 working days in ayear solve example 2-18 again if

$(aS_1 = S_2 = S_3 = S_4 = S_5 = 0.5 = $ half a day

b) $S_j = 1\, day \qquad j = 1,..,5$

Solution

a)

The necessary condition $\alpha = 1 - \sum_{j=1}^{5}\frac{D_j}{R_j} = 0.16 > 0$ holds

therefore Eq. 2-79 could be utilized:

$$\alpha = 1 - \sum_{j=1}^{5}\frac{D_j}{R_j} = 1 - \frac{20}{100} - \frac{40}{400} - \frac{28}{350} - \frac{60}{200} - \frac{16}{100} = \frac{40}{250} = 0.16$$

$$m^* = \frac{1}{T^*} \qquad \sum_{j=1}^{5} S_j = 2.5\, days = \frac{1}{100}\, \text{yr}$$

$$T^* = Max\{T_0^*, T_{min}\}$$

$$T_{min} = \frac{\sum S_j}{1-\sum_{j=1}^{5}\frac{D_j}{R_j}} = \frac{5(0.5)}{1-\frac{20}{100}-\frac{40}{400}-\frac{28}{350}-\frac{60}{200}-\frac{16}{100}} = \frac{2.5}{0.16}\, day = \frac{2.5}{0.16} \times \frac{1}{250} = \frac{1}{16}\text{yr}$$

$$T_0^* = \sqrt{\frac{2\sum_{i=1}^{n}(C_O)_i}{\sum_{i=1}^{n}(C_h)_i D_i\left(1-\frac{D_i}{R_i}\right)}} = \sqrt{\frac{2\times202}{40483}} = \frac{1}{10}\, \text{yr} \Longrightarrow$$

$$T^* = Max\left\{\frac{1}{10}, \frac{1}{16}\right\} = \frac{1}{10}\, \text{yr} \qquad m^* = \frac{1}{T^*} = 10$$

The production quantities in each run are obtained from

$$Q_j = D_j T^* = \frac{D_j}{10} \qquad j = 1,2,3,4,5$$

Therefore

$$Q_1 = \frac{D_1}{10} = 500 \quad Q_2 = 1000 \quad Q_3 = 700 \quad Q_4 = 1500 \quad Q_5 = 400$$

b) $T^* = Max\{T_0^*, T_{min}\} \quad T_0^* = \frac{1}{10}\, \text{yr}, \; S_j = 1\, day$, then:

$$T_{min} = \frac{\sum S_j}{1-\sum_{j=1}^{5}\frac{D_j}{R_j}} = \frac{(5)(1)}{1-\frac{20}{100}-\frac{40}{400}-\frac{28}{350}-\frac{60}{200}-\frac{16}{100}} = \frac{5}{0.16}\quad day$$

$$T_{min} = \frac{5}{0.16}\times\frac{1}{250} = \frac{1}{8}\, \text{yr}, \; T_0^* = \frac{1}{10}\, \text{yr}$$

$$T^* = Max\left\{\frac{1}{10}, \frac{1}{8}\right\} = \frac{1}{8}\text{yr}, \qquad m^* = \frac{1}{T^*} = 8 \qquad\qquad Q_j = D_j/m^*$$

$$Q_1 = D_1 T^* = 5000\times\frac{1}{8} = 625, \quad Q_2 = 1250, Q_3 = 875, Q_4 = 1875,$$

$$Q_5 = 500 \blacktriangle$$

2-17 Multiple-item EOQ model

In this section, EOQ model is extended to simultaneous purchase of several items. Here we either have a constraint such as the having a case where the number of orders for all items must be the same or we may not have any constraint or precondition.

2-17-1 Unconstrianed multiple-item EOQ model

In a multiple –item EOQ model in which there is no constrain or preconditions, and the items could be dealt separately e.g. our n products could be bought from n suppliers, the optimal order quantity for each item is derived as follows:

$$TC_i = C_{O_i} \frac{D_i}{Q_i} + C_{h_i} \frac{Q_i}{2} + P_i D_i$$

$$TC = \sum_{i=1}^{n} TC_i = \sum_{i=1}^{n} \left(C_{O_i} \frac{D_i}{Q_i} + C_{h_i} \frac{Q_i}{2} + P_i D_i \right)$$

$$\frac{\partial TC}{\partial Q_i} = 0 \quad \Rightarrow \quad Q_i^* = \sqrt{\frac{2 D_i C_{O_i}}{C_{h_i}}}$$

Substituting Q_i^* in TC yields:

$$TC^* = \sum_{i=1}^{n} \sqrt{2 D_i C_{O_i} C_{h_i}} + \sum_{i=1}^{n} P_i D_i \qquad (2-81)$$

2-17-2 Multiple-item EOQ model- annual number of orders the same for all

Here every time we place an order, we would like to order n products; therefore the annual number of orders for all products is the same and equals:

$$m = \frac{D_1}{Q_1} = \cdots = \frac{D_n}{Q_n}$$

or equivalently the cycle time(T) is the same for all:

$$T = \frac{Q_1}{D_1} = \cdots = \frac{Q_n}{D_n}.$$

In this regard two cases will be dealt with below; in one case one single order cost is paid to place an order of several items. In the other case each item has its own order cost.

2-17-2-1 Multiple-item EOQ Model : order cost independent of number and quantity of items

In this case we pay the order cost C_O to purchase n items. C_O is independent of the Q_j's and n. The number of orders and the cycle time is the same for all items. The stockout is assumed not to happen. With the symbols:

C_{h_j} The annual holding cost of product j
C_O Order cost
D_j Annual demand of product # j
Q_j Order quantity of product # j
$T = \frac{Q_j}{D_j}$, the cycle time of all items.

We could write the total cost as follows:

$$TC = \frac{C_O}{T} + \sum_{j=1}^{n} C_{h_j}\left(\frac{D_j T}{2}\right) + \sum_{j=1}^{n} P_j D_j \quad (2-81)$$

$$\frac{dTC}{d\,T} = 0 \Longrightarrow$$

$$T^* = \sqrt{\frac{2C_O}{\sum C_{h_j} D_j}} \qquad\qquad (2-82)$$

$$Q_j^* = T^* D_j = \frac{D_j}{m^*} \qquad\qquad (2-83)$$

It is assumed that the number of orders are the same and independent of items.

Example 2-20

Given the annual demand, unit price and annual holding cost of each unit for 5 items in the following table, if we want to have the same number of orders for the all items and the order cost is independent of the items and equals $40.5, find the optimal order quantity for each item.

Item #(j)	1	2	3	4	5
annual D_j	5000	10000	7000	15000	4000
Pj	6	5	3	4	6
annualC_{h_j}	1.6	1.4	0.6	1.15	1.65

Solution

$$T^* = \sqrt{\frac{2C_O}{\sum C_{h_j}D_j}} = \sqrt{\frac{2\times40.5}{5000\times1.6+\cdots+4000\times1.65}} = 0.0402\text{year}$$

$$m^* = \frac{1}{T^*} \cong 24, Q_1^* = \frac{5000}{24}, \qquad Q_2^* = \frac{10000}{24}, \qquad \ldots \qquad Q_5^*=\frac{4000}{24}.$$

End of example▲

Example 2-21

Two products A&B are ordered simultaneously. The annual demand for the products are respectively 500 &1500. If the annual holding cost for each unit is $10 and cost of joint order of these two is $100, find the optimal order quantities.

Solution

$$T^* = \sqrt{\frac{2C_O}{C_{h_1}D_1 + C_{h_2}D_2}} = 0.1 \text{ سال} \Longrightarrow m^* = 10$$

$$Q_1^* = \frac{D_1}{m^*} = \frac{500}{10} = 50, \quad Q_2^* = \frac{D_2}{m^*} = \frac{1500}{10} = 150\blacktriangle$$

2-17-2-1 Multiple-item EOQ Model : separate order cost for items

In this case several items are purchased simultaneously with its own order cost. The number of orders and the cycle time are the same for all items:

$$m = \frac{D_1}{Q_1} = \frac{D_2}{Q_2} = \cdots \quad \Rightarrow \quad T = \frac{Q_1}{D_1} = \frac{Q_2}{D_2} = \cdots$$

Substituting these relationships into

$$TC_j = C_{O_j}\frac{D_j}{Q_j} + C_{h_i}\frac{Q_i}{2} + P_jD_j \text{ yields:}$$

$$TC_j = C_{O_j}\left(\frac{1}{T}\right) + C_{h_j}\frac{D_jT}{2} + P_jD_j \qquad j = 1,2,3,\ldots$$

Since $TC = \sum TC_j$ Then:

$$TC = \sum_{j=1}^{n} C_{O_j}\left(\frac{1}{T}\right) + \sum_{j=1}^{n} C_{h_j}D_j\frac{T}{2} + \sum_{j=1}^{n} P_jD_j \qquad (2\text{-}84)$$

$$T^* = \sqrt{\frac{2\sum C_{O_j}}{\sum C_{h_j}D_j}} \qquad\qquad\qquad (2\text{-}85)$$

$$Q_j^* = D_jT^* \qquad\qquad\qquad\qquad (2\text{-}86)$$

2-18 Deterministic continuous & periodic review Models

In deterministic models sometimes we encounter deterministic FOS and FOI models. They are briefly introduced below.

2-18-1 Deterministic continuous review=deterministic (r,Q) Model= Deterministic (FOS)Model

This model deals with a system where the stock level of the product is calculated each time a product   moves in or moves out the system. The demand rate for the product   is fixed and deterministic; whenever the inventory reaches fixed level *r* an order of fixed quantity   Q is placed. Note that some real world inventory systems, such as the one shown in Fig. 2.22 where the demand is not fixed, could be approximated with this deterministic continuous review model.
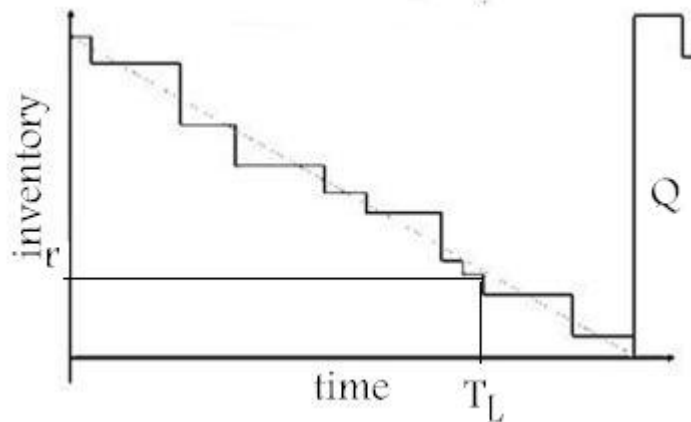
Fig. 2-22 Approximation of a real model with (r,Q) model

The total variable cost in this system equals:

$$TVC = C_o \frac{D}{Q} + C_h \frac{Q}{2}$$
(2-87)

2-18-2 Deterministic periodic review=deterministic (R,T) Model= Deterministic (FOI)Model

The periodic   review   model is   one   of   the   inventory   policies that reviews physical inventory at specific interval of time T and places an order with the quantity equal to the difference between the maximum level of inventory(R) and the current level of inventory(A) *i.e.*

$$Q = \begin{cases} R - A & A < R \\ 0 & A \geq R \end{cases} \qquad \text{(2-88)}$$

TVC in this model equals:

$$TVC = \frac{C_o}{T} + \frac{C_h DT}{2}. \qquad \text{(2-89)}$$

It worth mentioning that, classic EOQ and EPQ models are both FOS and FOI.

2-19 Inventory Models for Deteriorating Items

In the models discussed so far, the products were assumed to have long life and does not deteriorate or the deterioration rate is negligible. The items that incur a gradual loss in quality or quantity over time while in inventory are usually called deteriorating items.

There are many references which deal with deteriorating items. One could refer to references such as Bakkar(2012), Goyal &Giri (2001), Hung(2011) to study these kind of inventory models.

Exercises

1-(Tersine,1994 page 141)A company needs 54000 ball bearing sets each year. Each set costs the company $40. Annual holding cost per unit set is $9 and each order costs$20. Find

  a) The optimal order quantity,
  b) Annual number of orders,
  c) the reorder point , if the lead time is 1 month.

2-(Tersine,1994 page 141)A firm needs 38000 units of a product whose unit price is $4. Each order costs $9. The annual carrying cost is 25% of of the unit price. There are 52 working week in a year.

a) What is the optimal order quantity for this product?
b) How much is the annual total cost of ordering the economic quantity?
c) The maximum number of inventory in the warehouse?
d) What is the average number of inventory
e) What is the interval between 2 successive orders in weeks.

3- A company buys and sells 5 items and the at the time being the places a 5-item order at the end of each month. The order quantity for each item is one twelfths ($\frac{1}{12}$) of the corresponding annual demand. The company intends to shift from the current FOI system to FOS system. The ordering cost per each item is $10. The annual carrying cost of $1 is $0.2(I=20%). Using the table below, calculate the total cost for the FOI and FOS systems. Is shifting to the FOS system economic?

Item (i)	Annual demand D_i	Unit price $)(p_i$	Annual order cost	Average cost of holding inventory in FOI $(\frac{1}{2}I \times p_i \times \frac{D_i}{12}$
1	600	3	12×10	0.2× 75
2	900	10	120	0.2×375
3	2400	5	120	0.2×500
4	12000	5	120	0.2×2500
5	18000	1	120	0.2×750
sum			600	0.2×4200

The solution is in Tersine(1994) page 277.

4-(Tersine,1994 page 142) If a firm overestimates its annual demand by 50%, calculate the ratio of the total variable cost in overestimate case to the total variable cost when the demand is not overestimated.

   5-(Tersine,1994 page 142)The annual demand for an item is 6000 units, the unit price is $15, each order costs $25, annual holding cost per unit=$3, lead time is 3 weeks and there are 50 working weeks in a year. Suppose the customers agree to backordering. Each unit backordered costs $2 /yr.

   What is the

a) size of economic order quantity?
b) maximum inventory level in the optimal case?
c) reorder point?
d) number of backordered units during each order cycle?

6-(Tersine,1994 page 143)An electronics company uses 20000 particle beams each year. The supplier of the beams offers them at the following prices

Quantity	Unit Price($)
1-799	11
800-1199	10
1200-1599	9
≥ 1600	8

the cost of an order is $50.00, and the holding cost is 20% of the unit value per year. Find

a)The optimal order size that minimizes for an all-unit- discount model.

b) The optimal order size in an incremental discount model.

7-If we buy a product from out of the company it costs $5 per unit and the ordering cost is $1 and if we manufacture it in the company it costs $4 per unit and the setup cost is $10. The production rate in the company is 5000 /yr. The annual holding cost of each unit is 10% of its price. If monthly demand is 100 units, what policy do you suggest:

Buy or manufacture why? What is the reorder point and the optimal quantity per order in your suggested policy?

Ans: TC in buy policy is $6034 and in make policy is $4852.

7-If $C_h = 1$, which of the following choices are correct?

a)Q_w and TC_W both have the same quantity regardless of their dimension.

b) the quantity of Q_W is half of that of TC_W.

c) the quantity of TC_W is half of that of Q_W.

9-Which phrase is _not_ correct for completing the phrase

"In classic EOQ model it is assumed that"
a) the unit shortage cost is largish
b) The products is not deteriorating
c) the demand rate probabilistic
d) There is no constraint on, space, capital and the number order runs
Ans: choice (c)

11-Suppose the annual holding cost in classic EOQ model is estimated as much C_h', while the actual value is C_h. With this assumption, Compute the the ratio of total variable cost in terms of C_h' to the optimal total variable cost (in terms of C_h).

12-The annual holding cost of $1 is $0.05, the unit price is $100 and the product is supplied in 100-unit boxes, find the optimal order quantity(ans:200). What would be the answer if there were no constraint on order quantity.

13-The order quantity has to satisfy Q=100k, where k is an integer i.e. k=1,2,3,…if the annual demand is 2400 kilo gram, the annual holding cost per unit product is $5 and the ordering cost is $C_o = \$22$, find the optimal value for k.

Hint: $Q_W = 145$ is not a multiple of 100; use the following relationship:

$$Q^*(Q^* - n) \le Q_W^2 \le Q^*(Q^* + n).$$

14- (Tersine,1994, page143)

The demand in a firm is annually 3000 units. The ordering cost is a fixed cost of $250 and holding costs are computed at 25% of unit value per year. Source A will sell the component for $10 regardless of the order size. Source B will only accept orders of at least 600 units at a unit price of $9,50.

Source C will charge $9.00 per item but requires a minimum order of 800 units, (a) What Quantity should be purchased and from which source? (b) What are the cost savings in comparison with the other two sources?

15-(Tersine,1994, page143 Pr#13)The Supplier for the firm in Problem 2 is offering a special discount and temporarily reducing the unit price of the product by $2.

   a)What lot size should the firm order to take the advantage of the discount?

   b)What cost saving would result from this order?

16-The supplier in Problem 2 has decided to increase the unit price of its component from $4.00 to $4.2 tomorrow. If the reorder point is 1500 units and the current stock position is 2200 units,

   a) What lot size Should be ordered today
   b) What cost savings will be sacrificed if no special order is placed prior to the price increase?
   c) If the current stock position were 1500 instead of 200, what lot size should be ordered today?

17- (Tersine,1994, page144)A tire manufacturing plans to produce 40000 units of a special type of tire next year. The production rate is 200 tires per day, and there are 250 working days available. The set up cost is $200 per run, and the unit production cost is $15. If holding costs are $11.50 per unit per year,
   a) what is the economic production quantity?
   b) how many production runs should be made each year?
   c) If the production lead time is 5 days, what is the reorder point?
   18- The current order quantity in a firm is 1000 units. Suppose customers agree to backordering. If the annual holding costs per unit is $6 and each unit backordered costs $3/yr.

19- (Tersine,1994, page144) A firm produces five products in a work center. The available information is shown in the table:

i	annual demand	p_i	daily production date	annual $(C_h)_i$	$(C_o)_i$
1	6000	6	300	2. 1	80
2	20000	4	500	1.4	40
3	8000	6	160	1.8	100
4	8000	2	200	0.5	50
5	15000	4	200	1.5	50

If there are 250 working days available:

a) What is the best production cycle?
b) What is the optimum production run size for each product?
c) What is the annual demand time?

20-(Tersine,1994, page146) A firm orders eight items from the same vendor, as shown in the table. The ordering costs are $10 per purchase order and $0.25 per item. Carrying costs are 15% per year.

a) What is the economic order interval?
b) If the lead time is one month, what is the maximum inventory level for each item?

\

item (i)	Annual Demand (Di)	Unit Cost (pi)	Order Cost (Co)
1	175	1	$175
2	425	0.6	225
3	115	2.1	241
4	90	3	270
5	810	0.75	607
6	70	4	280
7	190	5	950
8	210	2	420
sum			3199

21-What is the effect of the error in C_h on TVC and also the effect of error in all parameters related to TVC on it?

22- In Classic EOQ, let $\dfrac{Q}{Q_w} = \beta$ & $\alpha = \dfrac{TVC(Q)}{TC_w}$. Show $\beta = \alpha \pm \sqrt{\alpha^2 - 1}$.

23- What happens in backordered EOQ model if $\pi D > TC_W$ & $\hat{\pi} \neq 0$?

24-(Tersine,1994, page142) Jane wants to determine the optimum amount of money to withdraw from an automatic teller machine (ATM) per transaction. The bank charges $.30 per ATM withdrawal transaction and a flat service charge of $5.00 per month. Jane spends an average of $10.00 per day. She figures there is a 10% chance that she will lose her wallet or be robbed in any given year. The bank pays 6% per year on checking account balances.

  a) What is her optimal withdrawal amount per transaction?

  b) How might the amount of Jane's withdrawals be altered if she moved to a high crime area?

Solution

On Sat, 6/23/18, Tersine, Richard J. wrote:
Subject: Re: THe solution of a problem
To: "Hamid Bazargan"
Date: Saturday, June 23, 2018, 10:54 AM
Hamid,
The problem solution is as follows:
(a) the unit price is $1.00; ordering cost is $.30/transaction; annual demand is 365(10)= 3650; the annual
holding cost fraction is the opportunity cost fraction plus the probability of loss or .16 (.06+.10); the fixed
service charge of $5.00 is irrelevant in lot size determination.
optimum Q = sq. root {2(.30)3650/1(.16)} = $117.00
(b) If Jane moves to a high crime area, she may need to increase her holding cost fraction. This would
effectively lower the optimal withdrawal amount per transaction.
Since the text materials were completed about 25 years ago, understandably they are no longer available.
Best wishes,
Richard J. Tersine

From: HamidBazargan<bazarganh@yahoo.com>
Sent: Friday,June22,2018 9:49AM
To: Tersine,RichardJ.;Tersine,MicheleG.
Subject: The solution of a problem
DearProfessor
I hope this email shall find you in the best of health and spirits.
I teach Inventory control to BS students; my mail reference is:
Prof.Tersine,RichardJ.1994
    Principles of Inventory and Materials Management -
    Prentice Hall

Could please tell me where I can find the solution of the following problem of the book:
Page142 of 4th edition1994.

You can never satisfy people by your property. So, you can attract their satisfaction by your behavior

Chapter3

Constrained Inventory Control Problems

Chapter 3

Constrained Inventory Control Problems

Aims of the chapter

This chapter deal with the problems of inventory control in which some constraints on budgets, cycle time, ware house space, number of replenishments, the holding costs, etc…   are considered. The chapter briefly describes the Lagrange multiplies technique and Karush-Kuhn-Tacker conditions , widely used in solving nonlinear programming problems which arises in various fields including constrained inventory control.

3-1 Lagrange multiplies technique and Karush-Kuhn-Tacker conditions

Lagrange multiplies technique is used for finding the extrima of a nonlinear optimization problem with *equality* constrains. Karush-Kuhn-Tacker conditions generalize the Lagrange method. Below some cases of the nonlinear problems are distinguished and the above techniques are described briefly. Before discussing the cases and the methods, note the following definition.

Definition of Lagrange's function

In constrained optimization if you multiply the function of each constraint by a multiplier and add the product to the objective function, you obtain a new function which is called Lagrange function or Lagrangian.

3-1-1 Nonlinear optimization problems with equality constraints

Consider a constrained optimization problem, where the constraints are in equality form and their functions are continuous and differentiable. Equality constraints restrict the feasible region to points lying on some surface inside R. To solve this equality-constrained problem, Lagrange suggest to assign a variable(known as Lagrange multiplier) to each constraint.   Then write the Lagrangian function. Deriving the gradient of the Lagrangian and setting it to zero and solving the simultaneous equations usually gives the answer of the equality-constrained   problem. A mathematical description   of is provided below.

Consider the following minimization problem, and assign a Lagragrange multiplier to each constraint:

$min \ Z = f(x_1, \dots, x_n)$

   s.t.

$h_1(x_1 \dots x_n) = b_1$ λ_1:Lagraqnge Multiplier

$h_2(x_1 \dots x_n) = b_2$ λ_2: Lagraqnge Multiplier

   $\vdots$ $\vdots$

$h_m(x_1 \dots x_n) = b_m$ λ_m: Lagraqnge Multiplier

   The Lagrangian is as follows:

$L = f(x_1 \cdots x_n) + \lambda_1 [h_1(x_1 \cdots x_n) - b_1] + \cdots + \lambda_m [h_m(x_1 \cdots x_n) - b_m]$

   Set the gradient of L (partial derivates of L with respect to $x_j's$ and $\lambda_i's$) equal to zero:

$$\begin{cases} \dfrac{\partial L}{\partial x_j} = 0 & j = 1, \dots, n \\[2mm] \dfrac{\partial L}{\partial \lambda_i} = h_i - b_i = 0 & i = 1, \dots, \text{m} \end{cases}$$

The feasible points where the partial derivatives of L are simultaneously zero are the optimal point of function L, and usually

provide   the   solution   for   the   above   equality-constrained problem(Winston,1994 page 684).

In fact the above simultaneous equations which could written as follows:

$$
\begin{cases}
\nabla_X f(x) + \sum_{i=1}^{m} \lambda_i \nabla_X h_i(x) = 0 & (3\text{-}1) \\[2mm]
b_i - h_i(x) = 0 & i = 1,\dots,m \quad (3\text{-}2)
\end{cases}
$$

are the necessary conditions for optimality and under proper convexity assumptions they are also sufficient. In Eq. (3-1) $\nabla_X f$ denotes the gradient of function f i.e. partial derivatives of f with respect to variables x_j $'s$.

In the above problem, if all functions are differentiable and continuous, f is convex, h_i's are convex[e.g. linear] then the solution to Eq. (3-1) & (3-2) is always the solution to the above optimization problem(extracted from Winston, 1994 page 685). Therefore for solving such a problem, set the derivatives of the lagrangian with respect to x_j و λ_i equal to zero; then find the solution to the simultaneous equations. If The answers to $\lambda_i's$ are specific numbers, then the answers to $x_j's$ constitute the optimal solution of the optimization problem under consideration.

It is worth knowing that Eq.(3-1)&(3-2) are some times called Karush-Kuhn-Tucker{KKT) conditions for the aforementioned equality-constrained problem.

Example 3-1
Write the Lagrangian and KKT conditions for the following problem:
$$Min \ \ Z = 2x_1 + 2x_2^2$$
   s.t.
$$4x_1 - x_2 = 6$$
$$x_1, x_2 \quad \text{unrestricted in sign}$$

Solution

The Lagrangian is

$L=2x_1 + 2x_2^2 + \lambda(4x_1 - x_1 - 6)$

KKT conditions :

$$\begin{cases} \nabla_X L(x) = \nabla_X f + \sum_{i=1}^{m} \lambda_i \nabla_X h_i(x) = 0 \\[2mm] b_i - h_i(x) = 0 \qquad i = 1,...,m \end{cases}$$

$$\begin{cases} \begin{pmatrix} 2+4\lambda \\ 4x_2 - \lambda \end{pmatrix} = 0 \\ 4x_1 - x_2 = 6 \end{cases} \Longrightarrow x_1 = 1.4688 \ , x_2 = -0.125 \ , \lambda = -0.5,$$

$x_1 = 1.4688 \ , x_2 = -0.125$ could be the optimal point . Since all functions of the problem are continuous and differentiable; furthermore f is convex and the function in the constraint is linear , therefore

$x_1 = 1.4688 \ , x_2 = -0.125$ is the optimal solution to the problem.

Solution with Lingo Software:

```
min=2*x1+2*(x2)^2;
4*x1-x2=6;
@free(x1);@free(x2);
end
Local optimal solution found at iteration:   11
  Objective value: 2.968750
```

Variable	Value	Reduced Cost
X1	1.468750	0.000000
X2	-0.125000	0.000000

End of example▲

3-1-2   optimization of nonlinear problems with in-equality constraints

In minimization problems with constraints of type inequality, assign a variable known as Lagrange multiplier to each constraint and write the Lagrangian function and the KKT conditions as will be shown. If the answer to the KKT conditions is a feasible solution for the problem, it might also be an optimal solution to the problem. To illustrate this case consider a problem with following form:

$minZ = f(x_1, ..., x_n)$

s.t.

$$g_1(x_1 \ldots x_n) \leq b'_1 \qquad\qquad \theta_1. Lagrange\ Multiplier$$

$$g_2(x_1 \ldots x_n) \leq b'_2 \qquad\qquad \theta_2: Lagrange\ Multiplier$$

$$\vdots \qquad \vdots$$

$$g_m(x_1 \ldots x_n) \leq b'_m \qquad\qquad \theta_m: \ Lagrange\ Multiplier$$

Suppose   all the functions are continuous and differentiable; the constraints are of the type $g_i \leq b'_i$ any other form has to be converted to this form even though the right had side becomes negative.

The Lgrangian is as follows:

$$L = f(x_1 \cdots x_n) + \theta_1[g_1(x_1 \cdots x_n) - b'_1] + \cdots + \theta_m[g_m(x_1 \cdots x_n) - b'_m]$$

The optimal solution of L satisfies the following conditions known as the Karush-Kuhn-Tucker{KKT) conditions:

$$\nabla_X L = \cdot \ \ or \ \nabla f + \theta_1 \nabla g_1 + \theta_2 \nabla g_2 + \ldots = \cdot \ or \ \ \frac{\partial L}{\partial x_j} = \cdot \ \ j = 1, .., n$$

$$or \qquad\qquad\qquad\qquad\qquad (3\text{-}3)$$

$$\frac{\partial f}{\partial x_j} + \theta_1 \frac{\partial g_1}{\partial x_j} + \ldots + \theta_m \frac{\partial g_m}{\partial x_j} = \cdot \qquad j = 1, .., n$$

$$\theta_i [b'_i - g_i(x_1, \ldots, x_n)] = \cdot \qquad i = 1, .., m \qquad\qquad (3\text{-}4)$$

$$\theta_i \geq \cdot \qquad i = 1, .., m \qquad\qquad\qquad (3\text{-}5)$$

In many cases[1] any point $(x_1*\ldots x_n*, \theta_1*, \ldots, \theta_n*)$ which satisfies the above conditions as well as the constraints, is the optimal solution to the aforementioned optimization problem. Note that since $g_i(x_1 \ldots x_n) \leq b'_i$ is equivalent to $g_i(x_1 \ldots x_n) + S_i = b'_i, S_i \geq 0$; then $\theta_i[b'_i - g_i(x_1, \ldots, x_n)] = 0$ and $\theta_i S_i = 0$ are equivalent.

---

[1] Winston(1994) page 684

3-1-3   Nonlinear optimization problems with equality and in-equality constraints

When a nonlinear optimization problem has inequality constraints of type $\leq$ and equality constraints:

$min \; Z = f(x_1, \dots , x_n)$

    s.t.

$g_i(x_1 \dots x_n) \leq \; b'_i \qquad i = 1, \dots, \qquad \theta_i \; : Lagrange \; Multiplier$

$h_j(x_1 \dots x_n) = b_j \qquad j = 1, \dots, \qquad \lambda_j \; : Lagrange \; Multiplier$

The Lgrangian :

$\quad L = f(x_1 \cdots x_n) + \theta_1[g_1(x_1 \cdots x_n) - b'_1] + \cdots + \lambda_1[h_1(x_1 \cdots x_n) - b_1] + \dots .$

If all the functions are continuous and differentiable, the necessary optimality conditions, according to Karush, Kahn and Tacker would be (Bazaraa,et al 2006 page205):

$$\begin{cases} \nabla f(x) + \theta_1 \nabla g_1(x) + \theta_2 \nabla g_2(x) + \dots + \lambda_1 \nabla h_1(x) + \lambda_2 \nabla h_2(x) + \dots = 0 \quad (3\text{-}6) \\ \theta_i[b'_i - g_i(x)] = 0 \qquad\qquad i = 1, \dots m \qquad\qquad\qquad (3-7) \\ \theta_i \geq 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad (3-8) \end{cases}$$

If a point wants to be optimal for the above-mentioned nonlinear optimization problem, it has to satisfy the KKT conditions as well the constraints (whether equality or non-equality).

In a problem is of the above form(minimization with both equality and non-equality ($\leq$)constraints), the optimal values obtained for the Lagrange multipliers of equality constraint could be <u>negative, zero or positive</u> numbers; however for   the   constraint   of $\leq$ type,   the corresponding Lagrange multipliers must be non-negative. In other words if the optimal value is negative the KKT conditions are not satisfied.   For more details on KKT conditions refer nonlinear programming text books.

Example 3-2

Solve the following problem:

$$\min f(\mathbf{x}) = x_1(x_1 - 30) + x_2(2x_2 - 50) + 3x_1 + 5x_2 + 10x_3$$

$s.t.$

$$x_1 + x_2 \le x_3 \quad or \quad g_1(x_1, x_2, x_3) = x_1 + x_2 - x_3 \le 0$$
$$x_3 \le 17.25 \quad or \quad g_2 = x_3 - 17.25 \le 0$$

Solution

With Lingo:

```
model:
min=x1*(x1-30)+x2*(2*x2-50)+3*x1+5*x2+10*x3;
x1+x2<=x3;
x3<=17.25;
x1>=0;
x2>=0;
x3>=0;
end
```

Solve Menu:

Rows= 6   Vars=3   No. integer vars=0

Nonlinear rows=   1 Nonlinear vars=   2 Nonlinear constraints=   0

Nonzeros=   11 Constraint nonz=   7 Density=0.458

Optimal solution found at step: 6

Objective value: -225.3750

Variable	Value	Reduced Cost
X1	8.500000	0.0000000
X2	8.750000	0.0000000
X3	17.25000	0.0000000

The second way to solve the problem is to write the KKT conditions:

Let $\mathbf{x} = x_1, ..., x_n$;The KKT conditions are:

$$\begin{cases} \nabla f(\mathbf{x}) + u_1 \nabla g_1(\mathbf{x}) + u_2 \nabla g_2(\mathbf{x}) = 0 \\ u_i[b_i' - g_i(\mathbf{x})] = 0 \qquad\qquad i = 1,2 \\ u_i \ge 0 \end{cases}$$

Or

$$
\begin{cases}
\begin{pmatrix} \overset{\nabla f}{} \\ 2x_1 - 30 + 3 \\ 4x_2 - 50 + 5 \\ 10 \end{pmatrix} + u_1 \begin{pmatrix} \overset{\nabla g_1}{1} \\ 1 \\ -1 \end{pmatrix} + u_2 \begin{pmatrix} \overset{\nabla g_2}{0} \\ 0 \\ 1 \end{pmatrix} = 0 \\
u_1(-x_1 - x_2 + x_3) = 0 \\
u_2(17/25 - x_3) = 0 \\
u_1 \geq 0 \\
u_2 \geq 0
\end{cases}
$$

or
$$
\begin{cases}
2x_1 - 30 + 3 + u_1 = 0 \\
4x_2 - 50 + 5 + u_1 = 0 \\
10 - u_1 - u_2 = 0 \\
u_1(-x_1 - x_2 + x_3) = 0 \\
u_2(17.25 - x_3) = 0 \\
u_1 \geq 0 \\
u_2 \geq 0
\end{cases}
$$

To try to solve the above simultaneous equations, notice that u_1 is either 0 or 1 and therefore 2^m possible cases are identified for (u_1, ...,u_m) where m is the number of constraints; in this case $m=2$ and the four possible cases are: $(u_1 = 0, u_2 = 0)$, $(u_1 = 0, u_2 > 0)$، $(u_1 > 0, u_2 = 0)$ and $(u_1 > 0, u_2 > 0)$.

Now let consider start with case $(u_1 = 0, u_2 = 0)$

I) $u_1 = 0$ $u_2 = 0$ $Eq.(3) \Rightarrow 10 - 0 + 0 = 0$ impossible

II) $u_1 = 0$ $u_2 > 0$ $(3) \Rightarrow 10 - 0 + u_2 = 0 \Rightarrow u_2 = -10$ unacceptable

III) $u_1 > 0 \quad u_2 = 0$

$$(3) \Rightarrow 10 - u_1 = 0 \Rightarrow \qquad u_1 = 10$$
$$(1) \Rightarrow x_1 = 8.5$$
$$(2) \Rightarrow x_2 = 8.75$$
$$(4) \Rightarrow 10(-8.5 - 8.75 + 1.3) = 0 \Rightarrow x_3 = 17.25$$

This point satisfies the constraints. Therefore $\overline{x} = (8.5, \ 8.75, \ 17.25)$ is a feasible point with acceptable Lagrange multipliers . Therefore is a KKT point.

There is no need to investigate the case $(u_1 > 0 \quad u_2 > 0)$, because we have come up with the solution to the problem. ▲

3-1-4 Nonlinear optimization problems inequality constraints and nonnegative $x_j's$

The Karush-kahn Tacker conditions for the case where we have non-negative variables as well as inequality constraints are given in references such as Wiston(1994) page 694. Needless to say if one finds the KKT point of the problem, ignoring the nonnegativity, and the point is nonnegative, the point is a KKT point for the problem having non-negative variables.

3-1-5 Interpretation of Lagrange multiplies

. In the subject of inventory control, positive Lagrange multiplier could be interpreted as shadow price of the resources(invested capital, warehouse space, number of orders, etc). In minimization problems, the shadow price is the amount of reduction in the objective function, when the right-hand side value of the corresponding constraint increases by one unit. Of course If the objective function is TVC,this is valid until the TVC reaches TC_W

3-2 Constraint in inventory systems

In this section, We have several products and there are some constraints on the budget, warehouse space, number of orders or machine setups, maximum inventory and the cycle time, etc.

$minZ = f(Q)$
$s.t.$
$g_1(Q) \leq 0$
$...$
$g_m(Q) \leq 0$

$Q = (Q_1, Q_2, ...) \geq 0$

A solution method

A method for solving these kind of problem is as follows:
Solve the problem as if is there is no constraint. If the calculated Q_{jw}'s satisfy the constraints, you have come up with the Solution to the constrained problem; otherwise the constraint which is not satisfied is called active and KKT conditions is used for finding the solution to the problem.

The Lagrangian function (i.e. the objective function together with the constraint's function times the Lagrange multiplier) is as follows:

$$L = f(Q_1 ... Q_n) + \sum_{i=1}^{m} \theta_i \, g_i(Q_1 ... Q_n)$$

The point(s)that minimize L,satisfy the following conditions kown as KKT conditions:

$$\begin{cases} \nabla_Q L = 0 \\ \theta_i[g_i] = 0 \quad i = 1, ..., m \\ \theta_i \geq 0 \end{cases}$$

Note that in writing L, the non-negativity of the variables ($Q_j \geq 0$) was not included, instead the Q_j's obtained from the KKT conditions have to be checked for their non negativity and feasibility and the obtained θ_i's have to be nonnegative ($\theta_i \geq 0$). A few cases will follow to illustrate solving constrained inventory problems.

3-2-1 Constraint on the space or surface of the warehouse

Suppose we have *n* products. The order quantity of Product # j is Q_j and

each unit of the product occupy f_j of the space or the surface area of our warehouse. If the maximum available space or surface area is F, then $\sum_{j=1}^{n} f_j Q_j \leq F$.   We want to determine the order quantity Q_j in such a way that total cost is minimized and the constraint is satisfied.

$$Min \sum_{j=1}^{n} \left(\frac{Co_j D_j}{Q_j} + \frac{Ch_j Q_j}{2} \right)$$

$$s.t. \sum_{j=1}^{n} f_j Q_j - F \leq 0 \quad , \quad Q_j \geq 0$$

If the order quantities calculated from Wilson formula (Q_{jw}'s) satisfy the constraint, they are the optimal solution to the constrained problem; otherwise , using, the Lagrange multipliers technique, Lagrangian function is formed:

$$L = \sum_{j=1}^{n} \left(\frac{Co_j D_j}{Q_j} + \frac{Ch_j Q_j}{2} \right) + \theta(\sum_{j=1}^{n} f_j Q_j - F)$$

where $\theta \geq 0$ is the multiplier assigned to the constraint.

$Q_j's$, as well as feasibility, must satisfy the following KKT conditions:

$$
\begin{cases}
\dfrac{\partial L}{\partial Q_j} = 0 \quad\quad \Longrightarrow Q_j{}^* = \sqrt{\dfrac{2Co_jD_j}{Ch_j + 2\theta f_j}} \quad j = 1, \dots, n \\[2em]
\theta\left(F - \displaystyle\sum_{j=1}^{n} f_jQ_j\right) = 0 \quad j = 1,2,..,n \\[2em]
\theta \geq 0
\end{cases}
$$

After finding the optimal θ, $Q_j{}^* \; j = 1, \dots, n,$ are obtained.

If the model is of the following form:

$$
Min \sum_{j=1}^{n} \left(\frac{Co_jD_j}{Q_j} + \frac{Ch_jQ_j}{2}\right)
$$
s.t.
$$
\sum_{j=1}^{n} f_jQ_j - F = 0 \quad,
$$
$$
Q_j \geq 0 \; j=1,2,\dots
$$

To find the optimal values of Q_j, set the gradient of L equal to zero i.e. differentiate L with respect to $Q_j, j = 1,2, \dots$ and θ; set the results equal to zero

$$\nabla L = 0 \equiv$$

$$
\begin{cases}
\dfrac{\partial L}{\partial Q_j} = 0 \rightarrow \\[1em]
\dfrac{\partial L}{\partial \theta} = 0 \quad \rightarrow
\end{cases}
$$

$$
\begin{cases}
\dfrac{\partial L}{\partial Q_j} = 0 \rightarrow -\dfrac{Co_jD_j}{Q_j{}^2} + \dfrac{Ch_j}{2} + \theta f_j = 0 \Longrightarrow = \dfrac{Ch_j + 2\theta f_j}{2} \Longrightarrow Q_j = \sqrt{\dfrac{2Co_jD_j}{Ch_j + 2\theta f_j}} \quad j = 1,2,\dots \\[2em]
\dfrac{\partial L}{\partial \theta} = 0 \quad \rightarrow \displaystyle\sum_{j=1}^{n} f_jQ_j - F = 0
\end{cases}
$$

Solve the resultant equations for the Q_j; insert $Q_j{}'s$ in

$\sum_{j=1}^{n} f_j Q_j - F = 0$ to find the optimal value of θ.

$$\Rightarrow \begin{cases} Q_j = \sqrt{\dfrac{2Co_j D_j}{Ch_j + 2\theta f_j}} & j = 1,2,..,n \\[4mm] \displaystyle\sum_{j=1}^{n} f_j Q_j = F \end{cases}$$

This value of θ easily gives the numerical value of $Q_j{}^*$.

Example 3-3

The maximum available space for keeping five products in a warehouse is $2000\, m^3$. Using the information in the following table, calculate the optimum order quantity for each product.   The annual holding cost of 1 dollar is approximately \$ 0.2.

Product No.(j)	Annual demand (D_j)	price (P_j)	Unit space requirement in m^3 (f_j)	Co_j
1	600	3	1	10
2	900	10	1.5	10
3	2400	5	0.5	10
4	1200	5	2	10
5	1800	1	1	10

Solution

The model is as follows:

$$Min\ Z = \sum_{j=1}^{5} \left(\frac{Co_j D_j}{Q_j} + \frac{I\, P_j Q_j}{2} \right)$$

s.t.

$$\sum f_j Q_j \le 2000$$

$$Q_j \ge 0$$

If we calculate Qw_j, j=1,…,5 from Wilson formula, we will niotice that these order quantities do not satisfy the constraint; then we

proceed with Lagrange multiplies. Assigning u as a multiplier to the constraint, we have:

$$L = \sum_{j=1}^{5} \left(\frac{Co_j D_j}{Q_j} + \frac{IP_j Q_j}{2} \right) + u(\sum_{j=1}^{5} f_j Q_j - 2000)$$

The KKT conditions are as follows:

$$\begin{cases} \frac{\partial L}{\partial Q_j} = 0 \rightarrow Q_j = \sqrt{\frac{2Co_j D_j}{Ch_j + 2uf_j}} \\ u\left(2000 - \sum f_j Q_j\right) = 0 \\ u \geq 0 \end{cases}$$

$$Q_1 = \sqrt{\frac{2(10)(600)}{(0.2 * 3) + 2u * 1}}, \cdots, Q_5 = \sqrt{\frac{2(10)(18000)}{(0.2 * 1) + 2u * 1}}$$

Since $u(2000 - \sum f_j Q_j) = 0$, either both are zero or one of them is zero;

and since $u \geq 0$ n it is possible that

1) $u = 0$ and $\left(2000 - \sum f_j Q_j\right)$ is nonzero,

2) $u > 0$ and $\sum f_j Q_j - 2000 = 0$,

3) both are zero.

In cases 1 & 3, where $u = 0$, $Q_j = \sqrt{\frac{2Co_j D_j}{Ch_j + 2uf_j}}$ converts to Wilson formula, and hence unacceptable. Then necessarily:

$u > 0$ and $\sum f_j Q_j - 2000 = 0$. Substituting $Q_1, \ldots, Q_5$ yields an equation whose variable is u. The equation could be solved by trial and error or fzero command in MATLAB:

fzero(@(u)   2000-(1*sqrt(2*10*600)/(.2*3+2*u*1)+

1.5*sqrt(2*10*900)/(.2*10+2*u*1.5)+0.5*sqrt(2*10*2400)/(.2*5+2*u*0.5)

+2*sqrt(2*10*12000)/(.2*5+2*u*2) +1*sqrt(2*10*18000)/(.2*1+2*u*1)),0.1)

gives u=0.1674.

Trial and error:

```
clc;d=0:.0001:.21;D=1000000000;i=1;
while abs(D)>= d(i);
   for u=0:0.0001 :0.2;
      D=2000-(1*sqrt(2*10*600)/(.2*3+2*u*1)+
1.5*sqrt(2*10*900)/(.2*10+2*u*1.5)+0.5*sqrt(2*10*2400)/(.2*5+2*u*0.5)
+2*sqrt(2*10*12000)/(.2*5+2*u*2)+1*sqrt(2*10*18000)/(.2*1+2*u*1));
      if (abs(D)<=d(i));
        break;
      end;
   end;
   i=i+1;
end;
disp(sprintf(' u=  %6.4f D=   %5.4f', u ,  d(i-1) ));
```

gives u=0.1674 $\cong$ 0.17.

$Q_1,\ldots,$ Q_5 would be117, 53, 188,,293, 1122 approximately for this value of u, which satisfy the constraint.

Interpretation of u=0.17: If one unit is added to the right hand side of the constraint(in this case the space of the warehouse), the objective function of the minimization problem (in this case the total cost) will decrease as much as $0.1674 \cong 0.17$. Of course this will be true until the function reaches its potential minimum. ▲

3-2-2 Constraint on the budget

This section deals with 2 constraints related to the budget i.e.

$$\sum_{j=1}^{n} p_j Q_j = C \quad \text{and} \quad \sum_{j=1}^{n} p_j Q_j \leq C .$$

3-2-2-1 The budget for ordering is exactly C dollars

If we have C dollars budget and want to order n products with unit price $p_j, j = 1, .., n,$ in such a way to minimize the total cost of the inventory system then the model would be

$$Min \ \ TVC = \sum_{j=1}^{n} \left(\frac{Co_j D_j}{Q_j} + \frac{Ch_j Q_j}{2} \right)$$

s.t.

$$\sum_{j=1}^{n} P_j Q_j = C$$

$$Q_j \geq 0$$

Assigning Lagrange multiplier λ to the constraint, the Lagrangian would be :

$$L = \sum_{j=1}^{n} \left(\frac{Co_j D_j}{Q_j} + \frac{Ch_j Q_j}{2} \right) + \lambda (\sum_{j=1}^{n} P_j Q_j - C)$$

Since we have only equality constraint, to solve the model it is enough to solve $\Delta L = 0$ or equivalently the following:

$$\begin{cases} \frac{\partial L}{\partial Q_j} = 0 \ \rightarrow \ Q_j \ = \sqrt{\frac{2Co_j D_j}{Ch_j + 2\lambda P_j}} = \sqrt{\frac{2Co_j D_j}{P_j(I + 2\lambda)}} & j = 1, \dots, n \\ \frac{\partial L}{\partial \lambda} = 0 \ \rightarrow \ \sum_{j=1}^{n} P_j Q_j = C \end{cases}$$

To find the optimal λ, substitute $Q_j, j = 1, 2, \dots, n$ from the first equations in $\sum_{j=1}^{n} P_j Q_j = C$. After finding λ, it is easy to find Q_j's. It is worth mentioning that in models of this kind which have equality constraints the optimal value of the LaGrange multiplier could be negative, zero or positive.

3-2-2-2 The budget for ordering is less than or equal to C

Suppose the model is as follows:

$$Min \; TVC = \sum_{j=1}^{n} \left(\frac{Co_j D_j}{Q_j} + \frac{Ch_j Q_j}{2} \right)$$

s.t.

$$\sum_{j=1}^{n} P_j Q_j \leq C$$

$$Q_j \geq 0$$

The optimal values of Q_j must satisfy the following KKT conditions:

$$\begin{cases} \frac{\partial L}{\partial Q_j} = 0 \; j = 1,2, \dots, n \rightarrow \quad Q_j = \sqrt{\frac{2Co_j D_j}{P_j(I+2\theta)}} \\ \theta(C - \sum_{j=1}^{n} P_j Q_j) = 0 \\ \theta \geq 0 \end{cases}$$

Example 3-4

Three products are to be ordered simultaneously. The maximum budget available is \$14000 to order the 3 products each time. No shortage is permitted and the annual holding cost of 1 dollar is approximately \$ 0.2 (I=20%). Using the data in the table calculate the optimal value of the ordering quantities.

Dj	Pj($)	Coj($)
1000	20	50
500	100	75
2000	50	100

Solution
The model of the problem is:

$$Min\ Z = \sum_{j=1}^{3} \left(\frac{Co_j D_j}{Q_j} + \frac{Ch_j Q_j}{2} \right)$$

s.t.

$$\sum_{j=1}^{3} P_j Q_j \leq 14000$$
$$Q_j \geq 0$$

At the outset we solve the problem, ignoring the constraint:

$$Q_{w1} = \sqrt{\frac{2(50)(1000)}{(0.2)(20)}} \cong 158 \qquad Q_{w2} \cong 61 \qquad Q_{w3} \cong 200$$

$$\sum_{j=1}^{3} P_j Q_j = 19260 > 14000$$

Since these values do not satisfy the constraint, the KKT conditions are utilized:

$$\frac{\partial L}{\partial Q_j} = 0 \ j = 1,2,3 \rightarrow Q_j^* = \sqrt{\frac{2Co_j D_j}{p_j(I + 2\theta)}} \ \ j = 1,2,3$$

$$\theta(14000 - \sum P_j Q_j) = 0$$

$$\theta \geq 0$$

$$\begin{cases} Q_1 = \sqrt{\frac{2Co_1 D_1}{p_1(I+2\theta)}} = \sqrt{\frac{2(50)(1000)}{20(0.2+2\theta)}} = \sqrt{\frac{10^5}{40(0.1+\theta)}} = \frac{50}{\sqrt{0.1+\theta}} \\[3mm] Q_2 = \sqrt{\frac{2(75)(500)}{100(0.2+2\theta)}} = \sqrt{\frac{75000}{200(0.1+\theta)}} = \frac{19.3649}{\sqrt{0.1+\theta}} \\[3mm] Q_3 = \sqrt{\frac{2(100)(2000)}{50(0.2+2\theta)}} = \sqrt{\frac{4*10^5}{100(0.1+\theta)}} = \frac{63.2456}{\sqrt{0.1+\theta}} \end{cases}$$

Since the product of θ and $(\sum P_j Q_j - 14000)$ is zero,
Either both are zero or only one of them is zero.

θ cannot be zero therefore $\sum P_j Q_j - 14000 = 0$

$$\sum P_j Q_j = 14000 \Rightarrow$$

$$20 \times \frac{50}{\sqrt{0.1 + \theta}} + 100 \times \frac{19.3649}{\sqrt{0.1 + \theta}} + 50 \times \frac{63.2456}{\sqrt{0.1 + \theta}} = 14000$$

$$x = \frac{1}{\sqrt{0.1 + \theta}} \quad \Rightarrow \quad x \cong 2.2955$$

$$\Rightarrow \quad \theta = 0.09 \rightarrow Q_1 = 114 \, , Q_2 = 44 \, , Q_3 = 145$$

Since these $Q_j's$ satisfy the constraint and the Lagrange multiplier θ in not negative, they form the optimal solution to the problem:

$$Q_1^* = 114 \qquad , Q_2^* = 44 \qquad , Q_3^* = 145.$$

The optimal value of the total variable cost is
TVC*=\$4064

Interpretation of $\theta = 0.09$:
If one unit is added to the right hand side of the constraint(in this case the space of the warehouse), the objective function of the minimization problem (in this case the total cost) will decrease as much as 0.09. Of course this will occur until the function reaches its potential minimum. ▲

Note that in the above 2 examples, if instead of maximum budget, the average inventory or the average budget involved with inventory were given, we would substitute Q_j in the constraint with $\frac{Q_j}{2}$.

3-2-3 Constraint on the number of orders of multiple items

Sometimes there is a constraint on the number of orders that can be placed per unit time say per year i.e.$\sum_{j=1}^{n} m_j = \sum \frac{D_j}{Q_j} \leq \ell$. To deal with this case we suppose either the ordering cost C_o is negligible or not negligible.

3-2-3-1 Constraint on annual number of orders-C_o negligible

If there is a constraint on annual number of orders of multiple-item case and the ordering costs are negligible, then the model of the problem would be:

$$\text{Min } TVC = \sum C_{h_j} \frac{Q_j}{2} + 0$$

s.t.

$$\sum \frac{D_j}{Q_j} \leq \ell$$

To determine the $Q'_j s$, the Lagrange function and the KKT conditions are

Written:

$$L = \sum C_{h_j} \frac{Q_j}{2} + \theta\left(\sum \frac{D_j}{Q_j} - \ell\right)$$

$$\begin{cases} \frac{\partial L}{\partial Q_j} = 0 \ , j = 1,2,\dots \\ \theta\left(\ell - \sum \frac{D_j}{Q_j}\right) = 0 \\ \theta \geq 0 \end{cases} \Rightarrow \begin{cases} \frac{C_{h_j}}{2} - \frac{\theta D_j}{Q_j^2} = 0 \\ \theta\left(\ell - \sum \frac{D_j}{Q_j}\right) = 0 \\ \theta \geq 0 \end{cases}$$

The equality of $\theta\left(\ell - \sum \frac{D_j}{Q_j}\right)$ with zero imply that either both are zero or one of them is zero;

and since $\theta \geq 0$ n it is possible that

1) $\theta = 0$ and $\left(\ell - \sum \frac{D_j}{Q_j}\right)$ is nonzero,

2) $\theta > 0$ and $\sum \frac{D_j}{Q_j} - \ell = 0$

3) both are zero.

Cases 1 and 3 cannot be valid, because with $\theta = 0$, the first equation i.e. $\frac{C_{h_j}}{2} - \frac{\theta D_j}{Q_j^2} = 0$ does not hold. Therefore $\theta > 0$, $m \sum \frac{D_j}{Q_j} - \ell = 0$ and the KKT conditions reduces to :

$$\begin{cases} Q_j = \sqrt{\frac{2D_j\theta}{C_{h_j}}} \\ \ell - \sum \frac{D_j}{Q_j} = 0 \\ \theta > 0 \end{cases}$$

Substituting Q_j in the second equation, we have:

$$\sum D_j \sqrt{\frac{C_{h_j}}{2D_j\theta}} = \ell \implies \theta^* = \frac{1}{2\ell^2}\left(\sum_{j=1}^{n}\sqrt{D_j \times C_{h_j}}\right)^2.$$

θ^* is derived from the above relationship and Q_j from $Q_j^* = \sqrt{\frac{2D_j\theta^*}{C_{h_j}}}$,

whose feasibility have to be verified.

3-2-3-2 Constraint on annual number of orders-C_o significant

If there is a constraint on annual number of orders of multiple-item case and the ordering cost is not negligible, then the model of the problem would be:

$$\text{Min TVC} = \sum_{j=1}^{n}\left(\frac{C_{o_j}D_j}{Q_j} + \frac{C_{h_j}Q_j}{2}\right)$$

s.t.

$$\sum_{j=1}^{n}\frac{D_j}{Q_j} \leq \ell$$

$$Q_j \geq 0$$

Again here the order quantities calculated from Wilson formula would be answers to the problem if they satisfy the constraint. Otherwise the lagrangian function and KKT conditions has to be written as follows:

$$\begin{cases} \dfrac{\partial L}{\partial Q_j} = 0, j = 1,2,\dots \\ \theta\left(\ell - \sum\dfrac{D_j}{Q_j}\right) = 0 \\ \theta \geq 0 \end{cases}$$

$$L = \sum C_{h_j} \frac{Q_j}{2} + \theta \left(\sum \frac{D_j}{Q_j} - \ell \right)$$

$$\begin{cases} \dfrac{\partial L}{\partial Q_j} = 0 \Rightarrow -\dfrac{Co_j D_j}{Q_j^2} + \dfrac{C_{h_j}}{2} - \dfrac{\theta D_j}{Q_j^2} = 0 \Rightarrow \dfrac{D_j Co_j + \theta D_j}{Q_j^2} = \dfrac{C_{h_j}}{2} \\[2ex] \theta \left(\ell - \sum \dfrac{D_j}{Q_j} \right) = 0 \\[2ex] \theta \geq 0 \end{cases}$$

$$\Rightarrow \begin{cases} Q_j = \sqrt{\dfrac{2D_j(Co_j + \theta)}{C_{h_j}}} \\[3ex] \theta \left(\ell - \sum \dfrac{D_j}{Q_j} \right) = 0 \\[2ex] \theta \geq 0 \end{cases}$$

$\theta \left(\ell - \sum \frac{D_j}{Q_j} \right) = 0$ implies either both θ and $\left(\ell - \sum \frac{D_j}{Q_j} \right)$ are equal zero or one of the 2 is zero. For $\theta = 0$, the value of Q_j will convert into Wilson formula. If this Q_j is feasible it is the answer; otherwise $\sum \frac{D_j}{Q_j} - \ell = 0$. To compute the optimal values of θ substitute $Q_j = \sqrt{\frac{2D_j(Co_j + \theta)}{C_{h_j}}}$, j=1,2,… into

$\sum \frac{D_j}{Q_j} - \ell = 0$. After finding the value of θ, if it is positive, calculate $Q_j, j = 1,2,…,n$ by substituting the optimal value of $.\theta$; then check the feasibility of them, if feasible they are optimal since they satisfy KKT conditions.

Example 3-5

Three products are to ordered by a firm. There is no stock-out cost and the annual carrying cost of $1 is $ 0.20(I=0.20). Considering

the data in the table and either of the following constraint, calculate the optimal order quantities,

a) $\sum \frac{D_j}{Q_j} \leq 25$ b) $\sum \frac{D_j}{Q_j} \leq 15.$

Dj	Pj($)	Coj($)
1000	20	50
500	100	75
2000	50	200

Solution
a)
The model of the first part of the problem is as follows:

$$Min\ Z = \sum_{j=1}^{3} \left(\frac{Co_j D_j}{Q_j} + \frac{Ch_j Q_j}{2} \right)$$

s.t.

$$\sum \frac{D_j}{Q_j} \leq 25$$

$$Q_j \geq 0$$

Ignoring the constraint, would yield

$$Q_{w1} = \sqrt{\frac{2(50)(1000)}{(0.2)(20)}} \cong 158, \qquad Q_{w2} \cong 61 \quad \text{and} \quad Q_{w3} \cong 282$$

$$\sum \frac{D_j}{Q_j} = \frac{1000}{158} + \frac{500}{61} + \frac{2000}{282} = 21 \leq 25$$

The above quintiles satisfy the constraint; therefore they are the answers to the first part.

b)The model for this part is

$$Min\ Z = \sum_{j=1}^{3} \left(\frac{Co_j D_j}{Q_j} + \frac{Ch_j Q_j}{2} \right)$$

s.t.

$$\sum \frac{D_j}{Q_j} \leq 15$$

$$Q_j \geq 0$$

If the Lagrangian function and KKT conditions are written as done above at the beginning of this section to come up with the solution of the model, We have: $Q_j = \sqrt{\frac{2D_j(Co_j+u)}{c_{h_j}}}$, j=1,2,3. Since $\sum (D_j \times \frac{1}{Q_j}) \leq$ 15; therefore we have to find u in such away that

$$1000 \sqrt{\frac{4}{2000(50+u)}} + 500 \sqrt{\frac{20}{1000(75+u)}} + 2000 \sqrt{\frac{10}{4000(200+u)}} = 15.$$

Using MATLAB command fzero:

fzero(@(u)   15-
(1000*sqrt(4/(2000*(50+u)))+500*sqrt(20/(1000*(75+u)))+2000*sqrt(10/(4000*(200+u)))), 200)

yields u=93.975 which is nonnegative.

The optimal value of $Q'_j s$ are obtained by substituting u in

$$Q_j = \sqrt{\frac{2D_j(Co_j+u)}{c_{h_j}}}$$ which yields : $Q_1 = 269, Q_2 = 92, Q_3 = 343.$

It is evident that these quantities satisfy the constraint: $\sum \frac{D_j}{Q_j} = \frac{1000}{269} + \frac{500}{92} + \frac{2000}{343} = 14.98 < 15,$ and u is nonnegative; therefore the optimal answer is $Q_1^* = 269, Q_2^* = 92$ and $Q_3^* = 343.$

Note if u were negative or the quantities did not satisfy the constraint, we would conclude the problem in this case does not have optimal answer. ▲

3-2-4 Constraint on the number of orders of multiple items having the same number of orders

Suppose a firm places order for several items which have the same number of orders per year. Also suppose there is a constraint on the number. In other words the goods have the same cycle time T ($\frac{Q_1}{D_1} = \cdots = \frac{Q_n}{D_n} = T$) and the there is a constraint on T. This case is illustrated below.

Example 3-6

The annual demand of 2 items, which ordered simultaneously, are 1000 and 2000 respectively. The holding cost is \$ 2 per year. The ordering cost is \$100. The annual number of orders must not exceed 5 times. Find the optimal order quantity of each item.

Solution

The items have the same cycle time T and the model of the problem is as follows:

$$Min\ TVC = \sum_{j=1}^{2} \left(\frac{Co_j D_j}{Q_j} + \frac{C_{h_j} Q_j}{2} \right) = \sum_{j=1}^{2} \left(\frac{Co_j}{T} + \frac{C_{h_j} TD_j}{2} \right)$$

$$\text{s.t.}$$
$$\frac{1}{T} \le 5,$$
$$T > 0$$

Let us find the solution of the model ignoring the constraint:

$$\frac{dTVC}{dT} = 0 \Rightarrow$$

$$T = \sqrt{\frac{2 \sum Co_j}{C_{h_1} D_1 + C_{h_2} D_2}} = \sqrt{\frac{200}{2(1000 + 1200)}} = 0.2132$$

This value of T satisfies the constraint and is optimal i.e. $T^* = 0.2132$. Therefore:

$$Q_1{}^* = T^* D_1 = 2132 \ , Q_2{}^* = T^* D_2 \cong 2559 \ \blacktriangle$$

Example 3-7

The annual demand of 2 items, ordered simultaneously, are 8000 and 16000 respectively. The per unit holding cost is $ 5 per year. The ordering cost is $1000. The annual number of orders must not exceed 4 times. Find the optimal order quantity of each item.

Solution

The items have the same cycle time T and the model of the problem is as follows:

$$Min\ TVC = \sum_{j=1}^{2} \left(\frac{Co_j}{T} + \frac{C_{hj}TD_j}{2} \right)$$

s.t.

$$\frac{1}{T} \leq 4, T > 0$$

Let us find the solution of the model ignoring the constraint:

$$\frac{dTVC}{dT} = 0 \Rightarrow$$

$m = \frac{1}{T} = 5.47 > 4 \quad \Rightarrow$ The constraint is active; and we write the Lagrangian   and   KKT   conditions:

$$L = \sum_{j=1}^{2} \left(\frac{Co_j}{T} + \frac{IP_jD_jT}{2} \right) + \theta \left(\frac{1}{T} - 4 \right)$$

$$\begin{cases} \dfrac{\partial L}{\partial T} = 0 \\[2mm] \theta\left(4 - \dfrac{1}{T}\right) = 0 \\[2mm] \theta \geq 0 \end{cases}$$

$$\frac{\partial L}{\partial T} = 0 \implies \quad T = \sqrt{\frac{2 \sum Co_j + 2\theta}{\sum C_{h_j} D_j}} \quad .$$

Since $\theta \geq 0$ and the second equations implies that :

$\theta = 0$ & $4 - \frac{1}{T} \neq 0$ or

$\theta = 0$ & $4 = \frac{1}{T}$ or

$\theta > 0$ & $4 - \frac{1}{T} = 0$;

θ cannot be zero because $T = \sqrt{\frac{2 \sum Co_j}{\sum C_{h_j} D_j}} = 0.18$ does not satisfy the

constraint. Therefore $\theta > 0$ and $4 - \frac{1}{T} = 0$.

$$T = \frac{1}{4} \implies \begin{cases} Q_1 = D_1 T = 8000 \times \dfrac{1}{4} = 2000 \\ \\ Q_2 = D_2 T = 16000 \times \dfrac{1}{4} = 4000 \end{cases}$$

$$4 - \frac{1}{T} = 0 \implies T = \sqrt{\frac{2 \sum Co_j + 2\theta}{\sum C_{h_j} D_j}} = \frac{1}{4} \implies \sqrt{\frac{2 \times 2000 + 2\theta}{(5 \times 8000) + (5 \times 16000)}} = \frac{1}{4} \implies$$

$\theta = 1750 > 0$.

Since the Lagrange multiplier θ is not negative and $Q_1 \,_\& \, Q_2$ are feasible, they could be the optimal solution. ▲

3-2-5 constraint on the cycle time of classic EOQ model-single item

In this section we would like to consider a classic EOQ model whose cycle time is constrained i.e. $\frac{Q}{D} = T \leq T'$ or $\frac{Q}{D} - T' \leq 0$. The model is therefore:

$$Min \; TVC = \frac{CoD}{Q} + C_h \frac{Q}{2} = \frac{Co}{T} + C_h \frac{DT}{2}$$
s.t.

$T - T' \leq 0 \, \& \, \text{T} > 0$

To find the optimal value of the cycle time, Q_w and   T   is calculated, if $T^* = \frac{Q_w}{D} = \sqrt{\frac{2Co}{ChD}} < T'$, T^* is the optimal cycle time and $Q^* = T^*D$; otherwise L and KKT conditions are utilized:

$$L = \frac{Co}{T} + Ch\frac{DT}{2} + \theta(T - T')$$

KKT conditions:

$$
\begin{cases}
\dfrac{\partial L}{\partial T} = 0 \implies -\dfrac{Co}{T^2} + \dfrac{ChD}{2} + \theta = 0 \\[2mm]
\theta(T' - T) = 0 \\[2mm]
\theta \geq 0
\end{cases}
$$

After getting the answer of θ and T related to the above conditions, if T is feasible and $\theta \geq 0$ then $Q^* = TD$

3-2-6 Constraint on the capital associated with inventory maximum in EPQ and EOQ models

The following examples show how to deal with the EOQ and EPQ model in which the monetary value of the maximum of the inventory in warehouse is constrained.

Example 3-8

In an EOQ model, the capital devoted to the maximum inventory is restricted to $10,000, = $200 , I = 20% yearly , the unit price is $50 and annual demand is 4000. Find the economic order quantity and the corresponding annual total cost.

Solution

The problem model is as follows:

$$TC = \frac{C_oD}{Q} + C_h\frac{Q}{2} + PD$$

s.t.

$P \times I_{max} \leq 10000$ or $PQ \leq 10000$

Ignoring the constraint would yield:

$$Q_w = \sqrt{\frac{2DC_o}{C_h}} = 400 \ \& \ \ I^*_{max} = Q_w = 400.$$

$P \ I_{max} = 50 \times 400 > 10000 \Rightarrow$ The constraint is active and 400 cannot be optimal.   We use Lagrangian function and KKT conditions:

$$L(Q,\theta) = \frac{C_oD}{Q} + C_h\frac{Q}{2} + +PD + \theta[PQ - 10000]$$

Karush_Kahn-Tacker conditions:

$$\begin{cases} \dfrac{\partial L}{\partial Q} = 0 \\ \theta[(10000 - PQ)] = 0 \\ \theta \geq 0 \end{cases}$$

$$\frac{\partial L}{\partial Q} = 0 \quad \Rightarrow \quad Q = \sqrt{\frac{2DC_o}{(C_h + 2P\theta)}}$$

θ cannot be zero because $Q = \sqrt{\frac{2DC_o}{(C_h+0)}}$ doesn't satisfy the constraint; therefore $\theta > 0$ and $10000 - PQ$=0 and KKT conditions reduces to :

$$\begin{cases} Q = \sqrt{\dfrac{2DC_o}{(C_h + 2P\theta)}} \\ 1000 - PQ = 0 \\ \theta > 0 \end{cases}$$

$50Q - 10000 = 0 \Rightarrow Q = 200 \Rightarrow \sqrt{\frac{2(4000)(200+\theta)}{0.2(50)+100P}} = 200 \Rightarrow \theta = 0.3.$

θ is positive and Q satisfies the constraint therefore $Q^* = 200$ is the answer.

The total cost for this amount of order quantity is

TC(Q=200)= $\frac{200 \times 4000}{(200)} + \frac{0.2 \times 50 \times 200}{2} + 50 \times 4000 = \$205000.$

Interpretation of u=0.3: If one unit is added to the right hand side of the constraint(in, the objective function of the minimization problem (in this case the total cost) will decrease as much as 0.3. Of course this will be true until the function reaches its minimum. ▲

Example 3-9

The capital associated with the maximum inventory of a product in a warehouse is restricted to $20000. The annual demand is 4000. The production capability rate is R=8000, the setup cost (C_o) is 2500 dollars and the carrying cost per unit per year is $200. Find the economic production quantity.

Solution
The model of the problem:
$$Min \ TVC = \frac{CoD}{Q} + C_h \frac{Q}{2}\left(1 - \frac{D}{R}\right)$$
s.t.
$$PI_{max} = P \times Q \times \left(1 - \frac{D}{R}\right) \leq 20000$$

Ignoring the constraint yields $Q^* = \sqrt{\frac{2DCo}{C_h(1-\frac{D}{R})}} \cong 633$ based on EPQ model. This answer does not satisfy the constraint; therefore we apply Lagrange multiplier technique to obtain the optimal solution of the model.

$$L = \frac{CoD}{Q} + C_h \frac{Q}{2}\left(1 - \frac{D}{R}\right) + \theta\left[PQ\left(1 - \frac{D}{R}\right) - 20000\right]$$

Karush_Kahn-Tacker conditions:

$$\begin{cases} \frac{\partial L}{\partial Q} = 0 \\ \theta\left[20000 - PQ\left(1 - \frac{D}{R}\right)\right] = 0 \\ \quad \geq 0 \end{cases}$$

$$\frac{\partial L}{\partial Q} = 0 \implies$$

$$-\frac{CoD}{Q^2} + \frac{C_h}{2}\left(1 - \frac{D}{R}\right) + \theta P \left(1 - \frac{D}{R}\right) = 0 \implies Q$$
$$= \sqrt{\frac{2DCo}{(C_h + 2P\theta)(1 - \frac{D}{R})}}$$

θ cannot be zero because $Q = \sqrt{\frac{2DCo}{(C_h+0)(1-\frac{D}{R})}}$ doesn't satisfy the constraint; therefore $\theta > 0$ and $20000 - PQ\left(1 - \frac{D}{R}\right) = 0$. KKT conditions reduces to:

$$\begin{cases} Q = \sqrt{\dfrac{2DCo}{(C_h + 2P\theta)(1 - \frac{D}{R})}} = \sqrt{\dfrac{2*2500*4000}{(100 + 400 \times \theta)(1 - \frac{4000}{8000})}} = \sqrt{\dfrac{400000}{1 + 4\theta}} \\ PQ\left(1 - \dfrac{D}{R}\right) = 20000 \\ \theta \geq 0 \end{cases}$$

$$PQ\left(1 - \frac{D}{R}\right) - 20000 = 0 \implies$$

$$200\sqrt{\frac{400000}{1 + 4\theta}}\left(1 - \frac{1}{2}\right) = 20000 \implies \sqrt{1 + 4\theta} = \sqrt{2} \implies \theta^* = \frac{9}{4} \implies$$

$$Q^* = \sqrt{\frac{400000}{1 + 4\left(\frac{9}{4}\right)}} = 200.$$

Since the multiplier θ is not negative and Q^* is feasible, therefore Q^* could be accepted as the optimal solution to the problem. ▲

3-2-7 Multiple-constraint inventory models

Sometimes several restrictions may constrain the operation of an inventory system. In this case, at first solve the problem without considering the constraint; if the solution satisfy the constraints it is the optimal solution. Otherwise find the optimal solution move the constraints to the objective function to obtain the Lagrangian form, and writing the KKT conditions.

As an illustration suppose the monetary value of the average inventory in the warehouse and also the space available for a product are restricted; then the model and the lagrange multipliers would be:

$$Min \ \ TVC = \sum_{j=1}^{n} (\frac{Co_j D_j}{Q_j} + \frac{IP_j Q_j}{2})$$

$s.t.$	Lagrange Multiplier
$g_1 = \sum P_j \dfrac{Q_j}{2} \leq M$	θ_1
$g_2 = \sum f_j Q_j \leq F$	θ_2

If the constraints are not satisfied with the optimal solution of the unrestricted problem, KKT conditions will be written:

$$L = \sum_{j=1}^{n} (\frac{Co_j D_j}{Q_j} + \frac{IP_j Q_j}{2}) + \theta_1 (\sum P_j \frac{Q_j}{2} - M) + \theta_2 (\sum f_j Q_j - F)$$

$$\begin{cases} \nabla_Q L = 0 \ \ or \ \ \dfrac{\partial L}{\partial Q_j} = 0 \quad j = 1,2,\dots,n \\[2mm] \theta_1 \left(M - \sum P_j \dfrac{Q_j}{2} \right) = 0 \\[2mm] \theta_2 \left(F - \sum f_j Q_j \right) = 0 \\[1mm] \theta_1 \geq 0 \\ \theta_2 \geq 0 \end{cases}$$

After calculating Q_j $'s$ from $\frac{\partial L}{\partial Q_j} = 0$ in terms of θ_1 & θ_2 and substituting in the other 2 equations, θ_1 & θ_2 and then the values for $Q_j's$ are obtained. Note that if θ_1 & θ_2 are non negatives and the calculated values for Q_j's are feasible, they are usually optimal.

It is worth mentioning that several computer softwares easily solve constrained problems.

Exercises

1-(Extracted from: Example 3, Tersine,1994,page 284)

A firm buys and sell 5 items.   The ordering cost of each item is $10 per order. , The holding cost is %20 per year ie. the annual holding cost of 1 dollar is $0.2. The unit price and annual demand for each item is as follows:

item no.(i)	annual demand(D_i)	unit price(p_i)
1	600	3
2	900	10
3	2400	5
4	12000	5
5	18000	1

With continuous review system, the mean investment calculated in its optimum state(i.e. $\sum p_i \frac{Q_i^*}{2}$) with the above data is obtained equal to $3130. Suppose the budget for this purpose is restricted to $2000. What is the economic order if

a)The monetary value of the average inventory is $2000

i.e. $\sum_{i=1}^{5} \frac{p_i Q_i}{2} = 2000$.

Answer in Tersine(1994) page287.

b)The monetary value of the average inventory for all items is totally $2000.

2-(Example 5, Tersine,1994,page 290)

The maximum space in Problem 1 is 1500cubic feet and each unit of the 5 item occupies respectively 1,1.5, 0.5,2,1cubic feet and also the capital for all items is restricted to $2000 maximum.   Find the economic order quantity for each item.   Answer on page 291 of Tersine(1994).

3-(Asadzadeh et al. 2006)

A firm buys 3 kind of electrical circuits. The management cannot pay more than $ 15000 on each order run.   Annual holding cost fraction is 20%; and Stockout is not permitted. Annual demand, unit price and the ordering cost for each item is given in the Table:

Item no.	1	2	3
Annual Demand	1000	1000	2000
Unit price	50	20	80
Ordering cost	50	50	50

Find the economic order quantity of each item

Let us take the advantage of the present time which is a divine present, and not live either in the past or in the future

.

Chapter4 Dynamic Lot Sizing Techniques

Written with cooperation of

Engineers

Mr. Milad MirNajafi, Mr Mohsen Esfahani, Mr. Ali Soltanpour,

Mr. Mostafa Hasankhani, Mr. Mostafa Tahami

Mrs. Behnaz Sarmat and Mrs. Najmeh Kafashian

Chapter 4

Dynamic Lot sing Techniques

Aims of the chapter

The present chapter addresses lot sizing problem in inventory control where the demand changes considerably from 1 period to another.   Several algorithms are presented for finding the best orders sizes which cover the periods in a time horizon .

4-1 Introduction

In the deterministic models presented in Chapter 2, such as EOQ model for purchase and EPQ for production , implicitly it was assumed that the demand is continuous and the demand rate is constant. This chapter deals with one-item cases where the demand is discrete and changes from one period to another due to factors such as changes in season, social, economical , political issues or machine maintenance. The following table is an example which shows the demand varies from one period to another but for each period is deterministic and fixed.

period(t)	1	2	3	4	5	6	...	T
demand(D_t)	10	0	15	24	0	1	...	4

One such case is when we would like to produce a certain amount of a product over a T-period time horizon and the production capacity in the periods are different (Zenon et al, 2003). To do this it is required to determine   the number of orders and the order sizes in such a way that minimizes the ordering and holding costs, the two significant factors that are considered while determining the economic

order quantity for any business. For this purpose we have to determine how much to purchase or produce at the beginning   of the T periods and determine the number of periods that this amount covers. Needles to say that for some periods the scheduled order size would be zero and, instead, for some the size would be more than its own requirement.

4-2 Dynamic Lot Sizing Problem

When in an inventory planning problem, there is period-varying determi- nestic demand for a single storable product over some finite periods and the order size changes with period, it is called dynamic lot sizing problem. This problem deals with the determination of the production or purchase plan that minimizes the total costs incurred over the planning horizon. In other words , dynamic lot sizing problem is a planning task for a multi-period time horizon to minimize the total cost of the inventory system.

The rest of this chapter describes some of the algorithms that have been proposed for these problems. Before introducing the algorithms some assumptions are needed to be explained.

4-2-1 Assumptions of Dynamic Lot Sizing Algorisms

The algorithms of dynamic lot sizing described in this chapter have been developed under the following assumptions(based on Chang,2001 and Tesine, 1994 page 179):

1. The time horizon is finite and the periods of the horizon are of the same time length.

2. The demand is known but varies from one period to another period.

3. The replenishment always occurs at the beginning of a period.

4. No orders are scheduled to be received at the beginning of a period in which demand is zero.

5. Orders placed at the beginning of a period are assumed to be available in time to meet the requirements .

6. The entire order quantity is delivered outright at the beginning of a period.

7. No shortages is permitted.

8. The holding(carrying) cost is applied to the inventory available at the end of periods and only to inventory held from one period to the next.

9. All variables except demand and except specified ones are assumed to be constant,

10. The manufacturer or vendor pays for the delivery cost.

11. The replenishment of raw material to the manufacturer is assumed instantly, and the quantity is the same as the production quaintity of a production period.

12. No inventory is held after the last period.

The first models for Lot sizing problem was developed in 1950s and still is being improved. For the history and more information on this model, the reader could refer to the books by Bramel and Simchi-Levi (1997), Johnson and Montgomery (1974) and Silver et al. (1996).

This model see Johnson &Montgomeri(1974), Bramel&Simchi, Silver et al(a996)

4-2-2 Dynamic Lot Sizing Classic Model

A mathematical model for dynamic lot sizing problem in its simplest form i.e. deterministic uncapacitated single item, zero lead-time, without backlogging, is as follows:

Symbols

T Number of periods in the planning horizon

t Period index;$t \in \{1,.., T\}$

D_t The quantity of demand for tth period

C_{o_t} The ordering cost for period t

C_{h_t} The cost for holding one unit at the end of Period t; not necessarily the same for all periods

Q_t The planned quantity purchased or produced for the beginning of period t

I_t The on-hand inventory at the end of period t

Zt $\begin{cases} 1 & \text{if} & Q_t > 0 \\ 0 & \text{if} & Q_t = 0 \end{cases}$

M A largish number e.g. the sum of the demands for all periods

Total variable cost: $\text{TVC} = \sum_{i=1}^{T} C_{h_t} I_t + \sum_{i=1}^{T} C_{o_t}$

TVC   If C_o & C_h are the same for the periods, then

$\text{TVC} = C_h \times \sum_{i=1}^{T} I_t + (T) \times C_o$

$$\text{Min} \sum_{t=1}^{T} (C_{o_t} \times z_t + C_{h_t} \times I_t) \qquad\qquad (1)$$

s.t.

$$I_{t-1} + Q_t = I_t + D_t \qquad t = 1,2,\dots T \qquad (2)$$
$$Q_t \leq M \times z_t \qquad\qquad t = 1,2,\dots T \qquad (3)$$
$$I_t, Q_t \geq 0, \quad t = 1,2,\dots T \qquad z_t \in \{0,1\} \qquad (4)$$

In this model

Line (1)

Represents minimization of the objective function (sum of setup/order cost and holing cost). Note that when an order is placed, there will be an incurred ordering cost.

Line (2)

The inventory – balance constraints

Line (3)

States that the quantity of each order cannot exceed a level.

Line (4):

Denotes the nonnegativity of the models variables. Note that (Simchi & Bramel, 1997 page 106):

the inventory can be rewritten as $I_t = \sum_{i=1}^{t}(Q_i - D_i)$ and therefore I_t variables can be eliminated from the model.

In fact, the above model does a trade-off between the holding and the order/setup cost. The answer of the model is the solution to the classic dynamic problem. In this model, assuming the holding and setup/order costs do not depend on t, the total variable cost (TVC) is obtained from the following relationship:

$\text{TVC} = C_h \sum_{i=1}^{T} I_t + (T) \times C_o \qquad (4\text{-}1)$

Example 4-1

Write the mathematical model for the following dynamic problem. The ordering cost is $100 /order and the unit holding per period id $2. The inventory at the beginning and the end of the 8- period time horizon is zero (i0=0; i8=0). There is no backlogging and the lead time is ignorable. Solve the model with a software:

Period(t)	1	2	3	4	5	6	7	8	sum
demand(D_t)	10	25	15	40	30	0	5	10	135

Solution:

$$\text{Min } 100 \sum_{t=1}^{8} z_t + 2 \sum_{t=1}^{8} I_t$$

  s.t.
  I0+Q1=I1+10;
  I1+Q2=I2+25;
  I2+Q3=I3+15;
  I3+Q4=I4+40;
  I4+Q5=I5+30;
  I5+Q6=I6+0;
  I6+Q7=I7+5;
  I7+Q8=I8+10;
  I0 = 0; I8 = 0;

Big M is set equal to the sum of the demands.

  Q1<=135*z1;
  Q2<=135*z2;
  Q3<=135*z3;
  Q4<=135*z4;
  Q5<=135*z5;
  Q6<=135*z6;
  Q7<=135*z7;
  Q8<=135*z8;

$z_t \in \{0,1\}$ $t = 1,2,\dots 8$
$Q_t \geq 0,$ $t = 1,2,\dots 8$
$I_t \geq 0,$ $t = 1,2,\dots 8$

  Solution from Lingo Software:

  We typed the following phrases in LINGO environment. Note that since Lingo does not accept i(0), i(1) denotes initial inventory at the beginning of the 8-period time horizon and i(9) denote the inventory at the end of the horizon.

  sets:
    index1/1..8/:z;
  index2/1..9/:i;

```
   end sets
   min = 100*(@sum(index1:z))+2*(@sum(index2:i));
   i(1)+q1=i(2)+10;
i(2)+q2=i(3)+25;i(3)+q3=i(4)+15;i(4)+q4=i(5)+40;i(5)+q5=i(6)+30;
   i(6)+q6=i(7)+0;i(7)+q7=i(8)+5;i(8)+q8=i(9)+10;q1<=135*z(1);
   q2<=135*z(2);q3<=135*z(3);q4<=135*z(4);
   q5<=135*z(5);q6<=135*z(6);q7<=135*z(7);q8<=135*z(8);
@FOR(index1:@BIN(z));
   !@BIN( z(1));
   !@BIN( z(8));


   i(1)=0;i(2)>=0;i(3)>=0;i(4)>=0;i(5)>=0;i(6)>=0;i(7)>=0;
i(8)>=0;i(9)=0;
   q1>=0;q2>=0;q3>=0;q4>=0;q5>=0;q6>=0;q7>=0;q8>=0;
   end
```

Of course there is no need to write the non- negativity of the variables; because it is the default in Lingo.

Lingo gives the following answer with Solve command:

```
Global optimal solution found at iteration:      39
  Objective value:   480
               Variable     Value      Reduced Cost
                  Z1      1.000000       100.0000
                  Z2      0.000000      -170.0000
                  Z3      0.000000      -440.0000
                  Z4      1.000000       100.0000
                  Z5      0.000000      -170.0000
                  Z6      0.000000      -440.0000
                  Z7      0.000000      -710.0000
                  Z8      0.000000      -980.0000

                  I1     40.00000       0.000000
                  I2     15.00000       0.000000
                  I3      0.000000       6.000000
                  I4     45.00000       0.000000
                  I5     15.00000       0.000000
                  I6     15.00000       0.000000
                  I7     10.00000       0.000000
                  I8      0.000000       0.000000
                  I0      0.000000       0.000000
                  Q1     50.00000       0.000000
                  Q2      0.000000       0.000000
                  Q3      0.000000       0.000000
                  Q4     85.00000       0.000000
```

Q5   0.000000   0.000000
Q6   0.000000   0.000000
Q7   0.000000   0.000000
Q8   0.000000   0.000000

Assuming zero lead time, the result is as follows:

Period (t)	1	2	3	4	5	6	7	8	Sum
Demand(D_t)	10	25	15	40	30	0	5	10	135
Quantity ordered (Q_t)	50	-	-	85	-	-	-	-	135
Inventory available at the end of period (I_t)	40	15	0	45	15	15	10	0	

Costs:

The software gives the optimum cost : $\sum_{t=1}^{8} 100z_t + \sum_{t=1}^{8} 2I_t = 480$ calculated as follows:

>> $i_1=40$; $i_2=15$; $i_3=0$; $i_4=45$; $i_5=15$; $i_6=15$; $i_7=10$; $i_8=0$;

>> $z_1=1$; $z_2=0$; $z_3=0$; $z_4=1$; $z_5=0$; $z_6=0$; $z_7=0$; $z_8=0$;

>>TVC=   100 $(z_1+z_2+z_3+z_4+z_5+z_6+z_7+z_8)+2(i_1+i_2+i_3+i_4+i_5+i_6+i_7+i_8)$

TVC = 100 (1+0+0+1+0+0+0+0)+2(140)=200+280 = 480

Or according to Eq.(4-1):

TVC= $C_h \sum_{i=1}^{T} I_t$ + (T) $\times C_o$

$C_o = 200$, $C_h = 2$

$2\sum_{t=1}^{8} I_t = 2(40 + 15 + 0 + 45 + \cdots + 0) = 2 \times 140 = 280$

TVC =280+(2)(100)=   480.End of example ▲

4-3   Model Solution Techniques

Many exact, heuristic and meta-heuristic have been proposed for solving dynamic lot sizing problems in the last decades. The answer given by Lingo software for Example 4-1 is considered an exact solution. Among   other   exact   solution   techniques   is   dynamic programming(DP). One of the DP algorithms is Wagner-Within algorithm which will be discussed at the end of this chapter.

Several meta-heuristic algorithms such as Genetic Algorithm, Ant Colony, Variable Neighborhood Search have been applied to solve the dynamic lot sizing. These kind of algorithms have been discussed in references such as Zenon(2003&2006).

Heuristc Algorithms

This part discusses a number of heuristic approaches for finding the answer to the dynamic lot sizing model. Although the heuristic algorithms are approximate and do not give an optimal solution but some of them give good solution. It is very common in practice to use an approximate method. One reason is that the approximate methods are easy to understand. It is also easy to check the computations manually(Axsater,2015,page60).

Orlicky (1975) divides lot sizing into static and dynamic defined as follows(Yilmaz, dated-nil, page44)

Static order quantity is defined as the one that once compute, continues unchanged in the planned order schedule. A dynamic order quantity, on the other hand, is subject to continuous re-computation. According to Orlicky (1975), only The so-called Fixed order Quantity technique is always static, while others can be used for dynamic repaining at the users option

 (End of quote).

Among the heuristic techniques used for solving dynamic lot sizing are the following ones:

1- Lot for Lot L4L(LFL)

2 - Fixed order quantity (FOQ)

   2-1 Economic order Quaintly (EOQ)

3- Fixed Period Requirement (FPR) or Fixed Order Period(FOP) or Periods Of Supply (POS)algorithm

   3-1 Economic Order Interval (EOI) or Period order Quantity (POQ)

        or Fixed order Interval(FOI) algorithm

4- Least Unit Cost (LUC) algorithm

5- Total cost (LTC) Algorithm = Part Period Algorithm (PPA)

6- Part Period Balancing (PPB))

7- Incremental Part Period Algorithm (IPPA)

8-Silver-Meal (SM) Algorithm

As well as the above algorithms described below, there are other heuristic techniques such as Least Period Cost (LPC)method, Uniform Order Quantity (UOQ) lot sizing technique, Foris Webster, Fix - Relax method and Groff's method which have applied to solve dynamic lot sizing problems.

Axsater(2015),Tersine(1994), Peterson& Silver(1991), Winston (1994) are among references which deal with dynamic lot sizing techniques.

4-3-1 Lot –for –Lot (LFL=L4L)method

In lot-for-lot rule or method, an order is placed for each period in which there is a non-zero demand in the exact quantity required for that period. If the lead time is zero, the quantity planned for the beginning of the period (Q_t) is equal to $Q_t = D_t, t = 1, 2, .., T$. Therefore the number of orders is large and is generally used for the products that have storage restrictions such as deteriorating products. LFL method is also suitable for high-volume continuous production (assembly lines). The lead time should be small. This ordering rule is the simplest among the dynamic ordering techniques. Although the method does not use costs for determining the amount of orders, but it is suitable for goods with high holding cost or in other words(Yilmaz, dated-nil) for goods that have a high unit price and a slight ordering cost. This technique (Yilmaz, dated-nil) minimizes the inventory holding cost.

Example 4-2

Determine the lot sizes by LFL rule from the data below; also calculate TVC if $C_o = 100$ and $C_h \cong 0$. $T_{L \cong 0}$.

t	1	2	3	4	5	6	7	8
D_t	-	43	19	35	58	-	-	12

Solution

The second row of the following table shows the demand of each period and the third row gives the lot sizes to be placed by lot-for lot rule, assuming the lead time is zero. Rows 5 and 6 show the costs

t	1	2	3	4	5	6	7	8	sum
Dt	-	43	19	35	58	-	-	12	
Q_t	-	43	19	35	58	-	-	12	
C_O		100	100	100	100			100	500
Holding cost	0	0	0	0	0	0	0	0	0
TVC	0	100	100	100	100	0	0	100	500

The total variable cost for this example is TVC=500.

Note that if the lead time is not zero all orders are placed before the beginning of the periods; e.g. with T_L=1 all orders would be placed one period ahead.

4-3-2 Fixed order Quantity (FOQ)method

In fixed order quantity rule, there is a constraint: a fixed amount or an integer multiple of it must be ordered, every time an order is placed for a particular item to be purchased or produced. The fixed quantity(Q) depends upon the restrictions on transportation capacity, packaging, storage capacity, quantity discounts and production capacity. It is required to order the smallest multiple that is immediately greater the required demand to satisfy the demand and prevent shortage. Yilmaz points out that "this technique would be applicable to items with an ordering cost sufficiently high to rule out ordering in net requirement quantities, period by period". In this technique the order quantity is fixed but the time interval between the orders is not usually the same.

Example 4-3

A workshop produces an item in batches of size 100. The table shows the equipments of a 10-period horizon. Prepare a production plan for the time horizon using FOQ rule and calculate the costs if $C_h = \$2$ and the setup cost per run is \$1000.

t	1	2	3	4	5	6	7	8	9	10
Dt	20	50	10	50	50	10	20	40	20	30

Solution

Costs:

t	1	2	3	4	5	6	7	8	9	10	su
Dt	20	50	10	50	50	10	20	40	20	30	300
Qt	100	0	0	100	0	0	100	0	0	0	300
tI	80	3	20	70	20	1	90	50	3	0	

ordering cost:$3 \times C_o = 3000$,

jolding cost: $C_h \sum_{t=1}^{10} I_t = 2(80 + 30 + \cdots + 30 + 0) = 800$,

Total variable cost : TVC=3000+800=3800.

Example 4-4

The following table shows the requirements schedule for the nine periods. Determine the order sizes by FOQ policy. Use lot sizes of multiples 15. $T_L \cong 0$.

t	1	2	3	4	5	6	7	8	9
Dt	0	40	10	25	35	0	10	10	35

Solution

Third row of the following table gives the solution. The 4th row is the inventory at the end of period t.

t	1	2	3	4	5	6	7	8	9	sum
Dt	0	40	10	25	35	0	10	10	35	165
Qt	-	45	15	15	45	-	-	15	30	165
It	0	5	10	0	10	10	0	5	0	

Note:

The demands of some periods are greater than 15; that is why lot sizes of more than 15 were ordered.

Costs: Assuming the cost per order is C_O and the unit holding cost per period is C_h, then

ordering cost: $6 \times C_O$

holding cost : $C_h * (5 + 10 + 0 + 10 + 10 + 0 + 5 + 0) = 40C_h$

$$TVC = 6C_o + 40C_h \blacktriangle$$

4-3-2-1 Economic order Quantity (EOQ) lot sizing policy

EOQ policy is a special case of FOQ policy in which the average of the demands of the periods$(\bar{D})$ is used to calculate EOQ according to Wilson Formula for purchase or production lot, if the range of demand changes is not too much. The calculated EOQ is rounded to the immediate greater integer. EOQ may not be necessarily suitable for lot size. If the EOQ does not satisfy the demand of any period, use the smallest multiple of it(2×EOQ, 3×EOQ,…) that will avoid shortage (Winston 1994, page 947). The more the discontinuous and non-uniform the demand, the less effective the EOQ will prove to be(Yilmaz, dated-nil).

Example 4-5

The demand for all coming 10 months is the same and equal to 25. $T_L=0$ and the setup cost Co=$80. The unit holding cost per period is

$C_h = 1.5$. Determine the order sizes by EOQ policy. What are the costs?

Solution

Since $\textbf{EOQ} = \sqrt{\dfrac{2DC_O}{C_h}} = \textbf{sqrt}(2 * 25 * 80/1.5) = 51.64$ therfore

Q is set equal to 52 and we could have the plan given in the following table.

Q_w Error factor	0.1	0.2	0.3	0.4	0.5	...	1	1.2	1.4	...	2
Relative increase in TVC(%)	405	160	81	45	25		0	1.7	5.7	...	25

Costs

Ordering cost: $5C_o$

Holding cost $= C_h(27 + 2 + \cdots + 35 + 10) = 185\ C_h$

$C_h = 1.5, Co = 80$

TVC $= 5C_o + 185\ C_h = 677.5$.

If the lot size were chosen Q=50 then

t	1	2	3	4	5	6	7	8	9	10
D_t	25	25	25	25	25	25	25	25	25	25
Q_t	50	-	50	-	50	-	50	-	50	-
I_t	25	0	25	0	25	0	25	0	25	0

Ordering cost: $5C_o$

Holding cost$=\ C_h(25 + 25 + 25 + 25 + 25) = 125\ C_h$

$C_h = 1.5, ‹Co = 80$

TVC $= 5C_o + 125\ C_h = 587.5$.

4-3-3 Fixed Order Period (FOP) or Periods of Supply (POS) policy

In Fixed Order Period method of lot sizing, the item is ordered every T time i.e. the time interval between successive orders is a fixed time such as T, due to some restrictions. This method is also called Periods of Supply (POS) policy; and it is not necessarily economical.   In this method the order size changes but the interval between successive orders is constant. In a special form of FOP called Fixed Period Requirement(FPR),the fixed T is set equal to m periods and

$$Q_t = \sum_{i=t}^{t+m-1} D_i$$

where

m　　The time interval between two successive orders
　　　　(in number or periods)
Q_t　　The order to be received at the beginning of Period t
D_t　　The demand of period t

Example 4-6

The following table shows the future monthly demands for a product. The lead time is 3 months and orders are set to exactly match the requirements of 2 months. The unit holding cost per period for all periods is equal to C_h. Determine the lot sizes and the costs for the time horizon by FPR rule.

period	Sep	Oct	Nov	Dec	Jan	Feb	Mar	Apr	May	Jun
demand	-	-	-	5	10	15	20	35	5	25

Solution

With FPR rule:

t	1	2	3	4	5	6	7	8	9	10
D_t	0	0	0	5	10	15	20	35	5	25
Q_t	15		35	-	40	-	25	-	-	-
t	0		0	10	0	20	0	5	0	0

ordering cost $= 4C_o$,

holding cost $= \sum_{t=1}^{10} C_h \times I_t = C_h * (10 + 20 + 5) = 35C_h$

$$TVC = 4C_o + 35C_h \blacktriangle$$

Example 4-7[1]

Apply POS method for the data given below. Order for 3 weeks ahead. The lead time is 2 weeks and the safety stock is 80. The initial inventory is 370. The unit holding cost per period is $C_h = 1.5$.

t(week)	1	2	3	4	5	6	7	8
D_t	130	160	120	260	130	120	185	115

Solution

From the initial inventory, 80 units are left after period 2. As the following table shows 2 more orders are needed:

t (TL = 2 weeks	Initial inventory	1 2	3	4	5	6	7	8
Gross Requirement(Dt)		130 160	120	260	130	120	185	115
Planned Receipts			510			420		
Planned Order Releases(Qt)		510		420				
Projected Available (I_t)	370	240 80	470	210	80	380	195	80

Costs:

Ordering cost: $C_o \times 2$

Holding cost :

$$C_h \sum_{t=1}^{8} I_t = C_h * (240 + 80 + 470 + 210 + 80 + 380 + 195 + 80)$$

$$= 1735 C_h$$

$$TVC = 2C_o + 1735C_h \blacktriangle$$

[1] Extracted from: http://www.slideshare.net/anandsubramaniam/lot-sizing-techniques

4-3-3-1 Economic Order Interval ([1]EOI) method or Period Order Quantity (POQ) or Fixed Order Interval(FOI)

In this heuristic method which is a kind of Fixed Order Period method and sometimes called Fixed Order Interval method, a fixed number of periods is used for ordering. This fixed number (T) is derived from :

$$T = \frac{EOQ}{\bar{D}} = \sqrt{\frac{2C_o}{\bar{D} \times I \times P}}$$
(4-2)

Where

$\bar{D}$ The average of the period requirements

If the calculated $T = \frac{EOQ}{\bar{D}}$ is not integer, round it.. If it is possible to calculate the average inventory cost per period, from the integers (less than or the greater than T) choose the one with less cost. The consumption during time T is sometimes dented by POQ :

Consumption during time T=POQ.

It is worth knowing that together with a fixed number of periods, some- times another number is given as the maximum inventory in this method. If so the amount for placing an order is calculated from the difference between the maximum and the inventory available at the time of placing an order.

For more details see Tersine (1994) page 134, Peterson &Silver(1991) page 327

Example 4-8

Apply   FOI rule to the following data in order to determine the order quantities which cover the 9-period horizon. $C_o = \$100, T_L \cong 0$, $P = \$50$, $I = 2\%$.   The unit holding cost is the same for all periods.

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[1] Economic Order Interval

t(month)	1	2	3	4	5	6	7	8	9
D_t	10	3	30	100	7	15	80	50	15

Solution

$$\bar{D} = \frac{\sum\limits_{t=1}^{9} D_t}{9} = 34.5 \quad \text{و} T^* = \sqrt{\frac{2C_o}{\bar{D}IP}} = 2.41 \quad \rightarrow T^* = 3$$

t	1	2	3	4	5	6	7	8	9	sum
D_t	10	3	30	100	7	15	80	50	15	310
Q_t	43			122			145			310
I_t	33	30	0	22	15	0	65	15	0	180

If the planning for the receipt of the orders were such that the demand after the receipt of the order was zero, schedule the order for the next period with positive demand. For example if the demand of Period 4 were zero instead of 122, the order would be scheduled for the beginning of Period 5 that has a positive demand.

Cost:

Ordering cost: $3C_o$

Holding cost:

$$C_h \sum_{t=1}^{10} I_t = C_h(33 + 30 + 22 + 15 + 65 + 15 = 180C_h$$

Total Variable cost

$$C_h = I \times P = .02 \times 50 = 1$$

$$TVC = 3 * C_o + C_h \sum_{t=1}^{10} I_t = 3 \times 100 + (1)(180) = 480 \quad \blacktriangle$$

Example 4-9[1]

The demands of the next 8 periods for a product are given in the following table. The unit price is \$1.5, the setup cost is $C_o = 100$ and annual I=%25 for all periods. T_L =2 weeks. The

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[1] From Subramaniam(2009)

initial invent- tory is 370. Apply POQ method to determine the lot sizes. Calculate the costs, assuming the unit holding cost per period is C_h

T(week)	1	2	3	4	5	6	7	8
D_t	130	160	120	260	130	120	185	115

Solution

$$C_o = 10, \qquad C_h = \frac{0.25 \times 1.50}{52} \text{ per week}$$

$$\overline{D} = \frac{130 + 160 + 120 + 260 + 130 + 120 + 185 + 115}{8} = 152.5 \text{ per week}$$

$$EOQ = \sqrt{\frac{2\overline{D}C_o}{C_h}} = \sqrt{\frac{2(152.5)(10)}{\frac{0.25 \times 1.5}{52}}} = 650.33 \to 650$$

$$T = \frac{EOQ}{\overline{D}} = \frac{650}{152.5} = 4.262 \cong 4$$

As observed from the table, the initial inventory suffices period 1 and 2. A lot of size 630 is placed for Periods 3 to 6 at the start of Period 1(note that we have a lead time of 2 weeks). To cover the Periods 7 & 8 a lot of size 300 is placed at the start of Period 5. Row 4 of the table shows the remaining inventory at the end of the periods; e.g. the on-hand inventory at the end of periods 3,7 & 8 are:

t	t
t =3	I_3=80+ 630 -120=590
t =7	I_7 =80+300-185=195
T=8	I_9 =195-115=80

TL=2 SS= 80	Initial inventory	week							
t		1	2	3	4	5	6	7	8
Dt		130	160	120	260	130	120	185	115
Planned to be received				630			300		
Qt		630			300				
t	370	240	80	590	330	200	80	195	80

costs: $2C_0$

Holding cost:

$$C_h \sum_{t=1}^{8} I_t = C_h * (240 + 80 + 590 + 330 + 200 + 80 + 195 + 80) = 1795C_h$$

Total Variable cost:

$$TVC = 2C_0 + 1795C_h \blacktriangle$$

Example 4-10

The demands for a product during the next 8 periods and the unit holding cost per period for various periods are given below:

t	1	2	3	4	5	6	7	8
D_t	45	60	35	50	70	50	60	80
C_{h_t}	10	12	14	15	18	20	20	20

the lead time is negligible and every 2 periods an order is placed (2-period FOI rule). The maximum on-hand inventory is set to be 140 units and no safety stock is necessary. Find the order lot sizes in order to plan for the time horizon, Also calculate the TVC.

Solution

Since the lead time is zero and we have a ceiling for inventory, the order quantities(Q_t's) are obtained from the difference between the maximum i.e.140 and the on hand inventory at the beginning of the period as shown in the table below:

period(t)	1	2	3	4	5	6	...	T
demand(D_t)	10	0	15	24	0	1	...	4

t	1	2	3	4	5	6	7	8	sum
D_t	45	60	35	50	70	50	60	80	450
Q_t	140	-	$140 - 35 = 105$	-	$140 - 55 = 85$		120	-	450
I_t	95	35	105	55	70	20	80	0	

Ordering cost: $4C_o$

Holding cost : $\sum_{t=1}^{8}(C_h)_t \times I_t =$

$10(95) + 12 \times 35 + 14(105) + 15(55) + 18(70) + 20(20) + 80(20) + 0$

$\qquad = 6925$

$\text{TVC} = 4 \times C_o + 6925 \blacktriangle$

4-3-4 Least Unit Cost (LUC)[1] Algorithm

Suppose we would like to place an order which covers the next i periods and would like to know how many periods the order should cover. Least unit cost (LUC) method is based on minimization of ordering and holding cost per unit product. This cost denoted by $UC(i)$ $i = 1,2, \dots$ is defined as follows:

$$UC(i) = \frac{\text{ordering cost} + \text{holding cost}}{\text{The sum of demand up to ith perios}} = \frac{C_O + C_h \times \sum_{t=1}^{i}(t-1)D_t}{\sum_{t=1}^{i} D_t}$$

$$UC(i) = \frac{C_O + C_h \times \sum_{t=1}^{i-1}(t)D_{t+1}}{\sum_{t=1}^{i} D_t}$$

$$= \frac{C_O + C_h(1D_2 + 2D_3 + \cdots + (i-1)D_i)}{\sum_{t=1}^{i} D_t} \qquad (4-3)$$

where

i   The period through the end of which the order covers

C_O Ordering cost

C_h Holding cost per unit hold at the end of period

D_t The requirement of period t

The algorithm of LUC may take several iterations to complete the planning horizon. During the process, the periods for which the planning has been performed are put away and new iterations are performed until all periods are planned.

In the first iteration the starting period is Period 1. UC(i) is consecutively calculated for the starting period and the next periods (i=1,2,…) until UC(i) for a particular i satisfy the following two conditions:

$$UC(i) \leq UC(i - 1)$$

and

$$UC(i) < UC(i + 1)$$

$$(4\text{-}4)$$

Denote this i by i_1. Place an order to cover Periods 1 through i_1.

For the second iteration take $i_1 + 1$ as the starting period and calculate   UC(i)   for $i = i_1 + 1, i_1 + 2, …$ from: $UC(i) = \frac{C_O + C_h \times \sum_{t \geq i_1 + 1}(t - (i_1 + 1))D_t}{\sum_{t \geq i_1 + 1} D_t}$.

The stopping criterion here is the same as that of the previous itera- tion. Denote the period satisfying Eq. 4-4 by i_2. Perform new iterations until the entire time horizon is covered.

If when increasing i=1,2,… in any iteration, you reach the end of the time horizon and the stopping criterion namely Eq. 4-4 is not satisfied, then stop and place the last order in such a way it cover the remaining periods of the iteration (the unplanned periods).

Example 4-11

Find the order lot sizes for the time horizon given in the table below using LUC heuristic method. If the order cost is \$100 and the unit holding cost per period is \$2, calculate the costs.

t	1	2	3	4	5	6	7	8
D_t	1	2	1	4	3	0	5	1

Solution

The problem is solved by LUC method through 3 iterations; in each iteration UC(i) is consecutively computed, when UC(i) starts to increase the iteration stops and an order is placed for the sum of the requirements of the first period in the iteration and all its successive periods before the period in which the increase occurs.

1st Iteration: the stating period is 1

$$UC(i) = \frac{C_O + C_h \times \sum_{t=1}^{i}(t-1)D_t}{\sum_{t=1}^{i} D_t}$$

$$UC(1) = \frac{C_O + C_h \times \sum_{t=1}^{1}(t-1)D_t}{\sum_{t=1}^{1} D_t} = \frac{C_O + 0}{D_1} = \frac{100}{10} = 10$$

$$UC(2) = \frac{C_O + C_h \times D_2}{D_1 + D_2} = \frac{100 + 2 \times 25}{10 + 25} = 4.28$$

$$UC(3) = \frac{C_O + C_h \times (1D_2 + 2D_3)}{D_1 + D_2 + D_3} = \frac{100 + 2 \times 25 + 2 \times 2 \times 15}{10 + 25 + 15}$$
$$= 4.2$$

$$UC(4) = \frac{C_O + C_h \times (1D_2 + 2D_3 + 3D_4)}{D_1 + D_2 + D_3 + D_4}$$
$$= \frac{100 + 2 \times 25 + 2 \times 2 \times 15 + 2 \times 3 \times 40}{10 + 25 + 15 + 40} = 5$$

The stopping criterion i.e. Ineq. 4-4 is satisfied for i=3:

$$UC(3) \le UC(3-1) \&$$
$$UC(3) < UC(3+1)$$

Now the first order is placed such that it covers period 1,2 and 3 with quantity 10+25+15=50.

2nd Iteration:Although the starting period in this iteration is 4 but we set i equal to 1 for the calculation.

$$UC(1) = \frac{C_O + 0D_4}{D_4} = \frac{100}{40} = 2.5;$$

$$UC(2) = \frac{C_O + C_h \times D_5}{D_4 + D_5} = \frac{100 + 2 \times 30}{40 + 30} = 2.2857$$

$$UC(3) = \frac{C_O + C_h \times (1D_5 + 2D_6)}{D_4 + D_5 + D_6} = \frac{100 + 2 \times 30 + 2 \times 2 \times 0}{40 + 30 + 0}$$
$$= 2.2857$$

$$UC(4) = \frac{C_O + C_h \times (D_5 + 2D_6 + 3D_7)}{D_4 + D_5 + D_6 + D_7}$$
$$= \frac{100 + 2 \times 30 + 2 \times 2 \times 0 + 2 \times 3 \times 5}{40 + 30 + 0 + 5} = 2.53$$

The stopping criterion i.e. Ineq. 4-4 is satisfied for i=3:

$$UC(3) \leq UC(3-1),\ \ UC(3) < UC(3+1)$$

Then the second order is placed such that it covers 3 more periods i.e 4,5 and 6 with quantity 40+30+0=70.

3rd Iteration:Although the starting period is 7 but for the calculation we set i equal to 1.

$$UC(1) = \frac{C_O + 0}{D_7} = \frac{100}{5} = 20$$

$$UC(2) = \frac{C_O + C_h \times D_8}{D_7 + D_8} = \frac{100 + 2 \times 10}{5 + 10} = 8$$

Then the third and final order is placed for the remaining periods 7 and 8 with size of 5+10=15.   The results are summarized in the following table:

period(t)	1	2	3	4	5	6	7	8
requirement(D_t)	10	25	15	40	30	0	5	10
order(Q_t)	50			70			15	
Inventory at the end of period (I_t)	40	15	0	30	0	0	10	0

Costs:

Ordering cost : 3×100

Holding cost   : $C_h \sum_{t=1}^{8} I_t = 2(40 + 15 + 30 + 10) = 2(95) = 190$

  $\text{TVC} = 3 \times 100 + 2 \times 95 = 490$▲

4-3-5 Least total Cost (LTC) method or Part Period Algorithm(PPA)

   Part Period algorithm was first introduced by DeMatteis(1968) . This researcher points out that it works well for all environments especially for the cases having a limited number of periods. The algorithm tries to find a number of periods whose holding costs equals the ordering cost. The logic of this procedure is the same as that of classic EOQ model in which the inventory cost is minimized at the point where the holding cost equals the ordering cost. It is worth mentioning that when the demand is discrete the holding cost and the ordering cost do not become equal. Then the aim is to minimize their difference.

Symbols

C_o	Cost per order
C_h	Unit holding cost per period
D_i	Requirement of ith period
$pp = (i-1)D_i$	Part Period(PP) related to i^{th} period
$App = \sum_{i=1}^{n}(i-1)D_i$	Accumulated Part-Period for n periods

Definition of Part-Period and Accumulated Part-Period

   One of the measurement units used in inventory subject is part-period(pp)[1]. By 1 pp it is meant that 1 unit of a product is held for 1

---

[1] In industry we have other such measurement units as man-hour or machine-hour.

period. If one unit of a kind of a product is held for ten periods or 2 units are held for 5 periods or 10 units are held for 1 period we say that the part-period(pp) value of all these 3 cases are the same and equal to 10pp.

Suppose we place an order for the requirements of n periods to receive a lot of size $Q = D_1 + D_2 + ... + D_n$ at the beginning of Period 1. From the amount Q, as much as D_1 is consumed during Period 1. The pp measurement unit for this amount is $0 \times D_1$. From the amount Q, as much as D_2 is consumed during Period 2. Noting that D_2 was held for one period before being consumed in Period 2, the pp measurement unit for this amount is $1 \times D_2$.

From the amount Q, as much as D_3 is consumed during Period 3. Noting that D_3 was held for 2 periods before being consumed in Period 3, the pp measurement unit for this amount is $2 \times D_3$ … the sum of these products i.e. $0D_1 + 1D_2 + ... + (n-1)D_n = \sum_{i=1}^{n}(i-1)D_i$ is called accumulated part-period for n periods and is denoted by APP_n:

$$APP_n = \sum_{i=1}^{n}(i-1)D_i$$

Determination of order lot sizes

To determine the lot sizes or the orders for a time horizon with PPA algorithm you may require several iterations. In each iteration try to find the that number of periods(n=1,2,….) for which $C_h \times APP_n = C_o$ or find that n which makes $|C_h \times APP_n - C_o|$ minimum. Therefore in iteration 1   when an increase happened after several consecutive decrease in $|C_h \times APP_n - C_o|$,   stop the iteration and place an order that cover the *n-1* periods.

In the next Iteration take Period n+1 as the starting period and act similar to iteration 1.   Do this procedure until all periods in the horizon are covered. If in an iteration the stopping criterion is not satisfied place an order which cover the unplanned periods.

Example 4-12

Find the order lot sizes for the time horizon given in the table below using LTC heuristic method. If the order cost is $300 and the unit holding cost from one period to the next immediate period is $2, calculate the costs.

t	1	2	3	4	5	6	7	8	9	10
D_t	30	40	0	50	10	20	30	0	55	0

Solution

$C_o = 300, C_h = 2$

Iteration (k)	Covered periods	Quantity	$\|APP_n - C_o\| = \|C_h \sum_{i=1}^{n}(i-1)D_i - C_o\|$
1st	1	30	$\|0 - 300\| = 300$
	1,2	70=30+40	$\|2(40 \times 1) - 300\| = 220$
	1.2.3	0+70=70	220
	1.2.3.4	50+70=120	$\|2(40 \times 1 + 50 \times 3) - 300\| = 80$
	1.2.3.4.5	10+120= 130	$\|2(40 \times 1 + 50 \times 3 + 10 \times 4) - 300\| = 160$

In Period 4 the difference $\|APP_n - C_o\|$ has reached its minimum and an order is placed to cover periods 1 through 4

	5	10	$\|0 - 300\| = 300$
	5.6	30	$\|2(20 \times 1) - 300\| = 260$
2nd	5.6.7	60	$\|2(20 \times 1 + 30 \times 2) - 300\| = 140$
	5.6.7.8	60	140
	5.6.7.8.9	115	$\|2(20 \times 1 + 30 \times 2 + 55 \times 4) - 300\| = 300$

In period 8 the difference $\|APP_n - C_o\|$ has reached its minimum and an order is placed to cover periods 5 through 8 and for the 3rd time for Period 9(and 10)

3rd	9	55	300

The summary of results are shown in the following Table

Results of Example 4-12 with LTC or PPA method									
t	1	2	3	4	5	6	7	8	9	10
D_t	30	40	0	50	10	20	30	0	55	0
Q_t	120	-	-	-	60	-	-	-	55	-
I_t	90	50	50	0	50	30	0	0	0	

$$TVC = 3C_O + 3C_h \sum_{t=1}^{8} I_t = 3 \times 300 + 2(90 + 50 + 50 + 50 + 30) = 1440$$

End of example▲

4-3-6 Part Period Balancing(PPB) algorithm

The part period balancing algorithm determines the lot sizes by a procedure similar to LTC algorithm. It tries to balance the holding costs and ordering costs. Let

$$APP_n = \sum_{i=1}^{n} (i - 1)D_i \qquad (4\text{-}4)$$

$$EPP = \frac{C_o}{C_h} = \frac{C_o}{I \times p} \qquad (4\text{-}5)$$

If the C_h of periods are not equal use their average in the denominator.

In each iteration the aim is to find the n which APP and EPP equal. Practically stop the iteration when you reach the smallest n which satisfy the following(Yilmaz, dated-nil)

$$APP_n \geq EPP \qquad (4\text{-}6)$$

To determine the suitable n in the first iteration, calculate

$APP_n = \sum_{i=1}^{n} (i - 1)D_i$ for n=1,2,… consecutively, When for the first

time *APP* exceeds EPP stop and place an order for the periods up to the period for which the increase happen. Denote the last period before the increase stats by n.

In the second iteration take n+1 as the starting period(i=1) and act similar to iteration 1. Continue the procedure until the horizon is

completed. PPB algorithm usually gives results similar to those of PPA.

   "Refinements   to this algorithm have been developed. These refinement called look-ahead and look-backward can improve performance" see Tersine(1994) page 191.

Example 4-13

   Find the order lot sizes for the time horizon given in the table below using PPB heuristic method. If the order cost is $120 and the unit holding cost from one period to the next immediate period is $2, calculate the costs.

t	1	2	3	4	5	6	7	8	9
D_t	40	15	0	35	0	20	5	15	30

Solution

Iteration 1		Iteration 2		
n	$App_n = \sum_{i=1}^{n}(i-1)D_i$	period	n	$App_n = \sum_{i=1}^{n}(i-1)D_i$
1	$(1-1)(D1) = 0 < EPP = \dfrac{120}{2}$	4	1	$(1-1)(35)=0< EPP=60$
2	$0+(2-1)(15)=15< EPP = 60$	5	2	$0+(2-1)(0)=0< EPP$
3	$15+(3-1)(0)=15< EPP$	6	3	$0+(3-1)(20)=40< EPP$
4	$15+(4-1)(35)=120> EPP$	7	4	$40+(4-1)(5)=55< EPP$
		8	5	$55+(5-1)(15)=115> EPP$
Since APP exceeds $\dfrac{C_o}{C_h}$ = EPP = 60 a lot of size 55 is placed for the periods before period 4				Since APP exceeds EPP a lot of size 60 is placed for periods 4 through 7

The third ordering quantity is Q_3 =30+15 for periods 8 and 9. The summary of calculations is given in the table below.

period	1	2	3	4	5	6	7	8	9	sum
requirement	40	15		35		20	5	15	30	160
inventory carrying period (i)	0	1	2	3						
$pp=(i-1)D_i$	0	15	0	105						
$APP_n=\sum_{i=1}^{n}(i-1)D_i$	0	15	15	120						
i				0	1	2	3	4		
$pp=(i-1)D_i$				0	0	40	15	60		
APP				0	0	40	55	115		
Q_t	55			60				45		160
It	15			25	25	5		30		

Ordering cost: $3 \times 120 = 360$

Holding cost:

$$C_h \sum_{t=1}^{8} I_t = 2(15+0+0+25+25+5+0+30+0) = 200$$

$$TVC = 360 + 200 = 560 .$$

End of example▲

4-3-7 Incremental Part- Period Algorithm(IPPA)

Increment Part-Period algorithm which was presented in Patterson and Forge (1985), is similar to PPB algorithm, but tries to balance incremental holding costs to ordering cost.   In this algorithm, the lot size is continually increased as long as the incremental holding costs is less than or equal to the ordering cost(Tersine, 1994 ,p 193). La Forge(1982) showed through simulation technique that IPPA is preferable to PPB(Shih& Fu,1995). The objective in IPPA algorithm id to determine lot sizes that include an integer number of period requirements so that(Tersine 1994,page193)

$$C_h(n\text{-}1)D_n=C_o \quad or \quad IPP_n=(n\text{-}1)D_n=\frac{C_o}{C_h} \qquad (4\text{-}7)$$

where

C_o	The ordering cost
$C_h = IP$	Unit holding cost
D_n	The requirement of nth period
PP_n	Incremental part-period=(n-1)D_n
EPP	Economical Part-Period $= \dfrac{C_o}{C_h}$

This algorithm may require several iterations. In iteration 1, calculate $IPP_n = (n-1)D_n$ for n=1,2,…   Stop whenever IPP_n exceeds EPP; record the last value of n and denote the value of (last n-1) by n*. Place an order for the periods 1 through n*. Some references ignore the equality of IPPn with EPP ;however the author of this bookbelieves that the actual objective is to find an integer that satisfy the   equality $C_h(n-1)D_n = C_o$. Therefore if for a particular n the equality happened, stop and set n* equal to this n. If the horizon is not ended perform another similar iteration with n*+1 as the starting period.

If in an iteration the stopping criterion is not satisfied place an order which cover the unplanned periods in the horizon.

IIPA has been extended to discount case (see Fu and Shih,1995). This method has easy understanding and has less calculations with respect to Silver-Meal and PPA methods. The following Flowchart helps to understand each iteration of the algorithm.
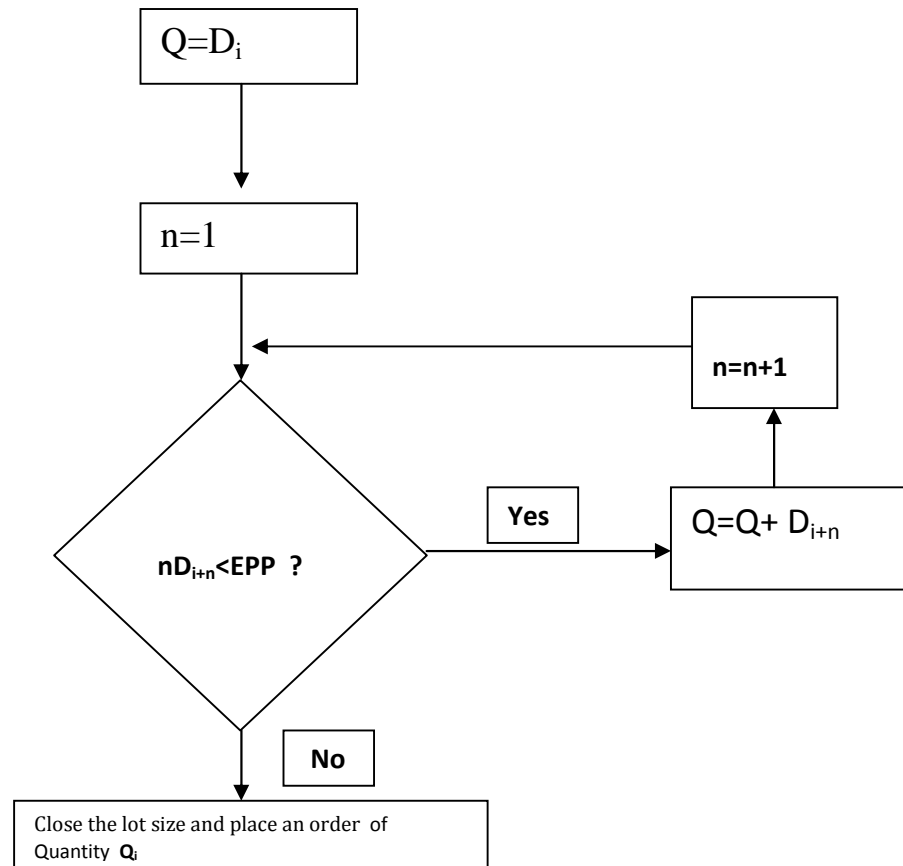
Fig 4-1 The algorithm for determining each lot size in IPPA
assuming zero inventory for i^{th} period (Vera&Laforge,1985)

Example 4-14

Find the order lot sizes for the time horizon given in the table
below using IPPA heuristic method. If the order cost is $100 and
the fraction of unit holding cost from one period to the next
immediate period is 2%, and the unit price is $50 calculate the
costs.

Solution

$$EPP = \frac{C_o}{C_h} = \frac{100}{0.02 \times 50} = 100 \, .$$

The calculations for IPPA method are as follows:

Period(t)	n	D_n	$IPP_n = (n-1)D_n$
1	1	75	$0 \times 75 = 0 < 100$
2	2	0	$1 \times 0 = 0 < 100$
3	3	33	$2 \times 33 = 66 < 100$
4	4	28	$3 \times 28 = 84 < 100$
5	5	0	$4 \times 0 = 0 < 100$
6	6	10	$5 \times 10 = 50 < 100$

Since IPP does not exceed EPP=100 in any period only one order is enough to be placed with size $75 + 0 + 33 + 28 + 0 + 10 = 146$ for all the horizon.

.

Costs:
Order cost $1 \times 100 = 100$
Holding Cost
$C_h \sum_{t=1}^{6} I_t =$

$0.02 \times 50(71 * 1 + 71 * 1 + 38 * 1 + 10 * 1 + 10 * 1) = 200$
TVC=100+200=300
The summary of the results are given in the table below:

t	1	2	3	4	5	6	su
D_t	75	0	33	28	0	10	14
Q_t	146	-	-	-	-	-	14
I_t	71	71	38	10	10	0	20
Accumulated variable cost	100 + (146 - 75) × (0.02 × 50) = 171	171 + (146 - 75 - 0) × (.02 × 50) = 242	280	290	300	300	

End of example ▲

Another example is given at the end of this chapter.

4-3-8 Silver –Meal algorithm

Edward Silver and Harlen Meal in 1973 proposed an algorithm for dynamic lot sizing. They did not want to minimize unit cost or total cost, but tried to minimize average cost per period(Yilmaz, dated-nil).

This method has less calculations compared to that of Wagner-Wittin and gives near optimal answer(Based on Winston,1994 Page1050). In this algorithm. starting from a period, we are in search of that number of periods to place an order whose   cost per period is minimum. The costs consists of the ordering cost plus the carrying costs related to the requirements of the periods being considered. Defining [1]AC(j) as

$$AC(j) = \frac{\text{Ordering cost} + \text{carrying cost}}{j}$$

$$AC(j) = \frac{C_O + \sum_{t=1}^{j} (C_h)_t \times (t-1)D_t}{j} \quad \text{ﻟ}$$

$$AC(j) = \frac{C_O + \sum_{t=2}^{j} (C_h)_t \times (t-1)D_t}{j} \qquad (4\text{-}8\text{-}1)$$

If the $(C_h)_t$'s are the same and equal to C_h, then we have

$$AC(j) = \frac{C_O + C_h \times \sum_{t=1}^{j}(t-1)D_t}{j}$$

$$= \frac{C_O + C_h(0D_1 + 1D_2 + 2D_3 + .. + (j-1)D_j)}{j}$$

$$AC(1) = \frac{C_O + C_h(0)D_1}{1} = C_O \quad (4\text{-}8\text{-}2)$$

where

$AC(j)$	Average cost per period
j	Number of periods
C_O	Ordering cost
$(C_h)_t$	Unit holding cost related to period t
C_h	Unit holding cost for all periods
D_t	Requirement for period t

This is an iterative method. In each iteration the aim is to find say j periods whose AC(j), when starting from a particular period, is minimum.

To perform Silver =Meal algorithm, at the outset in **iteration 1** set j=1. It is assumed that all units assigned to Period 1(D_1) is consumed

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[1] Average Cost

and none is transferred to the next period; therefore the holding cost for it is supposed to be zero and

$$AC(1) = \frac{ordering\ cost + 0}{1}$$

Then increase j one by one and calculate AC(j) consecutively until for a particular j, as the value of j is increased, AC(j) exceeds AC(j − 1) for the first time. Denote this value of j by j_1. Therefore the iteration is stopped whenever the following inequality is satisfied(Axater,2015 Chap 4):

$$AC(j) \le AC(j_1 - 1) \qquad 2 \le j \le j_1$$

and

$$AC(j_1 + 1) > AC(j_1)$$

The first lot is place to cover periods 1 through j_1: $Q = \sum_{t=1}^{j_1} D_t$

Go to next iteration 2 , set j= $j_1 + 1$,consecutively calculate AC(j) and perform similar iteration until the time horizon is covered.

This approach has performed extremely well in numerous test examples and is recommended for significantly variable demand pattern (Person & Siver, 1991 page 317); however does not give optimal solution. It is worth mentioning that 2 situations where the heuristic does not perform well are (Tersine, 1994 page 187):

1.when the demand rate decreases rapidly with time over several periods,

2. where there are a large number of periods with zero demand.

Example 4-15

Find the order lot sizes for the time horizon given in the table below using Silver-Meal heuristic method. If the order cost is $100 and the unit holding cost from one period to the next immediate iod is $2, Also calculate the costs.

t	1	2	3	4	5	6	7	8
D_t	10	25	15	40	30	0	5	10

Solution

$$AC(j) = \frac{C_O + C_h \sum_{t=2}^{j}(t-1)D_t}{j} \qquad AC(1) = C_O$$

Iteration 1, starting period: 1

Calculation of AC(j) for j=1,2,…:

$$AC(1) = \frac{C_O + 0}{1} = \frac{100}{1} = 100$$

$$AC(2) = \frac{C_O + C_h \times D_2}{2} = \frac{100 + 2 \times 1 \times 25}{2} = 75$$

$$AC(3) = \frac{C_O + C_h \times (D_2 + 2D_3)}{3} = \frac{100 + 2 \times 25 + 2 \times 2 \times 15}{3} = 70$$

$$AC(4) = \frac{C_O + C_h \times (D_2 + 2D_3 + 3D_4)}{4}$$

$$= \frac{100 + 2(25 + 2 \times 15 + 3 \times 40)}{4} = 112.5$$

For the first time when j=4 AC increased; therefore we stop and

plan an order of size Q = 10+15+25 =50 for the requirements of periods 1,2 and 3.

Iteration 2, starting period: 4

$$AC(1) = \frac{C_O + C_h(0)D_1}{1} = \frac{100}{1} = 100$$

$$AC(2) = \frac{C_O + C_h(0D_4 + 1D_5)}{2} = \frac{100 + 2 \times 30}{2} = 80$$

$$AC(3) = \frac{C_O + C_h \times (0D_4 + 1D_5 + 2D_6)}{3}$$

$$= \frac{100 + 2 \times 30 + 2 \times 2 \times 0}{3} = 53.3$$

$$AC(4) = \frac{C_O + C_h \times (0D_4 + 1D_5 + 2D_6 + 3D_7)}{4}$$

$$= \frac{100 + 2(30 + 2 \times 0 + 3 \times 5)}{4} = 47.51$$

$$AC(5) = \frac{C_O + C_h \times (D_5 + 2D_6 + 3D_7 + 4D_8)}{5}$$

$$= \frac{100 + 2(30 + 2 \times 0 + 3 \times 5 + 4 \times 10)}{5} = 54$$

For the first time when j=5 AC increased; therefore we stop and

plan the second order of size Q = 40+30+0+5 =75 for the requirements of periods 4,5,6 and 7.

Iteration 3

No calculations is needed and the third order of size 10 is planed for the last period. The summary of results are given in the following table:

t	1	2	3	4	5	6	7	8	sum
Dt	10	25	15	40	30	0	5	10	135
Qt	50	-	-	75	-	-	-	10	135
CO	100			100				100	300
It	40	15	0	35	5	5	0	0	
$C_h \times I_t$	80	30	0	70	10	10	0	0	200

Costs:

Ordering cost=100$\times$ 3=300

Holding cost: $C_h \sum_{t=1}^{8} I_t$ =2$\times$(I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8)=

2$\times$ (40+15+0+35+5+5+0+0)=200.

TVC=300+200=500▲

4-4 Wagner and Whitin's Exact Algorithm

Wagner and Whitin(1958) presented an algorithm which gives an exact solution for discrete-demand dynamic lot sizing problems of finite time horizon . Their solution causes no shortage. The algorithm assumes the periods of the horizon are of the same time length and the planned orders arrive at the beginning of the periods (not in the middle).   The calculations of the algorithm are based on some theorems. The theorems are mentioned in some references including Winston(1994) page 1047. The algorithm minimizes the inventory costs of the problem.

It is worth mentioning that although the algorithms of Silver& Meal and Wagner_&Whitin cause less inventory costs compared to other dynamic lot sizing rules, but many companies which utilize MRP[1] production planning technique use simple heuristic rules of POQ, PPB and LFL(extracted from Winton,1994,page 946).

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[1] Materials Requirement planning

4-4-1 The steps of Wagner-Whitin Algorithm

This algorithm uses a dynamic programming approach. The steps of this algorithm are mentioned in many references. What follow is based on page 182 Winston (1994).

Step 1

For all possible ordering alternatives related to the given time horizon calculate total variable cost denoted by Z as described below.

Suppose for the beginning of Period t, an order is planned with a size equal to the total requirements of period t through say period e. The cost of this order is calculated from:

$$Z_{te} = Co_t + \sum_{i=t}^{e} Ch_i(Q_{te} - Q_{ti}) + p_t Q_{te} \qquad (4-9)$$

Where

Z_{te}	total variable cost of the order planned for periods t to e
Co_t	ordering cost per order
$Ch_t = IP_t$	unit holding cost for period t
P_t	unit price of period t
Q_{te}	sum of requirements of Period t to Period e: $\sum_{i=t}^{e} D_i = Q_{te}$
N	number of periods available in the time horizon

If for t=1,2,.., N   P $=P_t$, C_o C_{o_t} and C_h C_{h_t} then(Tersine,1994 p182):

$$Z_{te} = C_o + C_h \sum_{i=t}^{e} (Q_{te} - Q_{ti}) \qquad 1 \le t \le e \le N \qquad (4-10)$$

Step 2

Assuming the inventory at the end of Period e is zero, calculate $f_{1,\dots,}f_N$ from:

$$f_e = \underbrace{Min}_{for \quad t=1,\dots,e} (Z_{te} + f_{t-1}) \qquad e = 1,2,\dots,N \qquad f_0 = 0$$

Or

$$f_e = Min(Z_{1e} + f_0 \,, \quad Z_{2e} + f_{1 \,,\dots,} \quad , \quad Z_{ee} + f_{e-1}) \quad e = 1,2,\dots,N$$

Or for $e = 1,2,\dots,N$

$$f_e = Min(f_{1e},\ f_{2e}, \dots, f_{ee}),\ e = 1,2,\dots,N \tag{4-11}$$

Where

f_{1e} The cost of Q_{1e} , the order assigned to period 1 through e: $f_{1e} = Z_{1,e} + f_0,\ ,\ \ f_0 = 0$

f_{2e} The cost of Q_{2e} ,the order assigned to period 2through e: $f_{2e} = Z_{2,e} + f_1,$

…

$f_{e-1,e}$ The cost related to the order assigned to period e-1through e: $f_{e-1,e} = Z_{e-1,e} + f_{e-1},$

$f_{e,e}$ The cost of Q_{ee},the order assigned to period e: $f_{e,e} = Z_{e,e} + f_{e-1},$

Therefore   in   this   step,   for   each   period$(e = 1,2,\dots,N)$ all combinations of ordering alternatives as well as f_e *strategy*   are compared and the combination with lowest   cost is recorded as f_e strategy .   It is proved that the value obtained for f_N is the optimal ordering cost i.e. the cost of the optimal order schedule(Tersine, 1994 page182).

I	$f_N = Z_{w,N} + f_{w-1}$	The last order happens in Period w to meet the requirements of periods w to N
II	$f_{w-1} = Z_{u,w-1} + f_{u-1}$	The order just before the last order is made in Period u to meet the requirements of periods u to w-1 (Z_{uw-1}),
III	$f_{u-1} = Z_{1,u-1} + f_0$	The 1st order is planned for Period 1 to cover the requirements of periods 1 through u-1(Z_{1u-1}).

Step 3

To   convert f_N strategy obtained above into optimal order quantities, act by observing the orders backward .

Example 4-16

´From the data given in the following table determine the order quantities by The Wagner Whitin algorithm ;also calculate the costs assuming C_h =\$ 1 $C_O = \$40.$

t	1	2	3	4	5	6	7	8	9	10	11	12
D_t	2	12	4	8	15	25	20	5	10	20	5	20

Solution

Step 1 : Calculation of $Z_{te} = C_o + C_h \sum_{i=t}^{e}(Q_{te} - Q_{ti})$:

Period 1

$$Z_{11} = C_{o_1} + \sum_{i=1}^{1} C_{h_i}(Q_{11} - Q_{11}) = 40 + 1(2 - 2) = 40$$

$$Z_{12} = C_{o_1} + C_{h_1}(Q_{12} - Q_{11}) + C_{h_2}(Q_{12} - Q_{12})$$
$$= 40 + 1(14 - 2) + 1(14 - 14) = 52$$

$$Z_{13} = 40 + 16 + 4 = 60, Z_{14} = 40 + 24 + 12 + 8 = 84, Z_{15}$$
$$= 144$$

$$Z_{16} = 269 \quad Z_{17} = 389$$

$$Z_{18} = 424 \quad Z_{19} = 504 \quad Z_{1-10} = 684 \quad Z_{1-11} = 734 \quad Z_{1-12}$$
$$= 954$$

Period 2

$$Z_{22} = C_{o_2} + \sum_{i=2}^{2} C_{h_i}(Q_{22} - Q_{22}) = 40 + 1(12 - 12) = 40$$

$$Z_{23} = C_{o_2} + C_{h_2}(Q_{23} - Q_{22}) + C_{h_2}(Q_{23} - Q_{23}) = 40 + 4 = 44$$

$$Z_{24} = 40 + 12 + 8 = 60 \quad Z_{25} = 40 + 27 + 23 + 15 = 105 \quad Z_{26} = 205$$

$$Z_{27} = 305 \quad Z_{28} = 335$$

$$Z_{29} = 405 \quad Z_{2-10} = 565 \quad Z_{2-11} = 610$$

$$Z_{2-12} = 810$$

Period 3

$$Z_{33} = C_{o_3} + C_{h_3}(Q_{33} - Q_{33}) = 40$$

$$Z_{34} = C_{o_3} + C_{h_3}(Q_{34} - Q_{33}) + C_{h_3}(Q_{34} - Q_{34}) = 40 + 8 = 48$$

$$Z_{35} = 40 + 23 + 15 = 78 \quad Z_{36} = 40 + 48 + 40 + 25 = 153$$

$$Z_{37} = 233$$

$$Z_{38} = 258 \quad Z_{39} = 318 \quad Z_{3-10} = 458 \quad Z_{3-11} = 498 \quad Z_{3-12}$$
$$= 678$$

Period 4

$$Z_{44} = C_{o_4} + C_{h_4}(Q_{44} - Q_{44}) = 40$$

$$Z_{45} = C_{o_4} + C_{h_4}(Q_{45} - Q_{44}) + C_{h_4}(Q_{45} - Q_{45}) = 55$$

$$Z_{46} = 40 + 40 + 25 = 105 \quad Z_{47} = 40 + 60 + 45 + 20 = 165$$

$$Z_{48} = 185$$

$Z_{49} = 235$ $Z_{4-10} = 355$ $Z_{4-11} = 390$ $Z_{4-12} = 550$
Period 5

$Z_{55} = C_{o_5} + C_{h_5}(Q_{55} - Q_{55}) = 40$

$Z_{56} = C_{o_5} + C_{h_5}(Q_{56} - Q_{55}) + C_{h_5}(Q_{56} - Q_{56}) = 40 + 25 = 65$

$Z_{57} = 40 + 45 + 20 = 105$ $Z_{58} = 40 + 50 + 25 + 5 = 120$

$Z_{59} = 160$ $Z_{5-10} = 260$ $Z_{5-11} = 290$ $Z_{5-12} = 430$
Period 6

$Z_{66} = C_{o_6} + C_{h_6}(Q_{66} - Q_{66}) = 40$

$Z_{67} = C_{o_6} + C_{h_6}(Q_{67} - Q_{66}) + C_{h_6}(Q_{67} - Q_{67}) = 40 + 20 = 60$

$Z_{68} = 40 + 25 + 5 = 70$ $Z_{69} = 40 + 35 + 15 + 10 = 100$

$Z_{6-10} = 140$

$Z_{6-11} = 205$ $Z_{6-12} = 325$
Period 7

$Z_{77} = C_{o_7} + C_{h_7}(Q_{77} - Q_{77}) = 40$

$Z_{78} = C_{o_7} + C_{h_7}(Q_{78} - Q_{77}) + C_{h_7}(Q_{78} - Q_{78}) = 40 + 5 = 45$

$Z_{79} = 40 + 15 + 10 = 65$ $Z_{7-10} = 40 + 35 + 30 + 20$
                        $= 125$

$Z_{7-11} = 145$ $Z_{7-12} = 245$
Period 8

$Z_{88} = C_{o_8} + C_{h_8}(Q_{88} - Q_{88}) = 40$

$Z_{89} = C_{o_8} + C_{h_8}(Q_{89} - Q_{88}) + C_{h_8}(Q_{89} - Q_{89}) = 40 + 10 = 50$

$Z_{8-10} = 40 + 30 + 20 = 90$ $Z_{8-11} = 40 + 35 + 25 + 5 = 105$

$Z_{8-12} = 185$
Period 9

$Z_{99} = C_{o_9} + C_{h_9}(Q_{99} - Q_{99}) = 40$

$Z_{9-10} = C_{o_9} + C_{h_9}(Q_{9-10} - Q_{99}) + C_{h_9}(Q_{9-10} - Q_{9-10}) = 40 + 20$
            $= 60$

$Z_{9-11} = 40 + 25 + 5 = 70$ $Z_{9-12} = 40 + 45 + 25 + 20 = 130$
Period 10

$Z_{10-10} = C_{o_{10}} + C_{h_{10}}(Q_{10-10} - Q_{10-10}) = 40$

$Z_{10-11} = C_{o_{10}} + C_{h_{10}}(Q_{10-11} - Q_{10-10}) + C_{h_{10}}(Q_{10-11} - Q_{10-11})$
            $= 40 + 5 = 45$

$Z_{10-12} = 40 + 25 + 20 = 85$
Period 11

$Z_{11-11} = C_{o_{11}} + C_{h_{11}}(Q_{11-11} - Q_{11-11}) = 40$

$$Z_{11-12} = C_{o_{11}} + C_{h_{11}}(Q_{11-12} - Q_{11-11}) + C_{h_{11}}(Q_{11-12} - Q_{11-12})$$
$$= 40 + 20 = 60$$

   Period 12

$$Z_{12-12} = 40$$

Step 2 Calculation of $\quad f_e = \underset{\underset{t=1,\dots,e}{for}}{Min} \; (Z_{te} + f_{t-1})$ for (e =

$1, \dots, 12)$

$f_0 = 0$

$f_1 = \min \; (Z_{11} + f_0) = \min(40 + 0) = 40$

$f_2 = \min \; (Z_{12} + f_0, Z_{22} + f_1) = \min(52 + 0 \, , 40 + 40) = 52$

$f_3 = \min \; (Z_{13} + f_0, Z_{23} + f_1, Z_{33} + f_2) = \min(60 + 0 \, , 44 + 40 \, , 40 + 52) = 60$

$\quad f_4 = \min \; (Z_{14} + f_0, Z_{24} + f_1, Z_{34} + f_2, Z_{44} + f_3) = \min(84 +$
$0 \, , 60 + 40 \, , 48 + 52 \, , 40 + 60) \quad = 84$

$f_5 = \min \; (Z_{15} + f_0, Z_{25} + f_1, Z_{35} + f_2, Z_{45} + f_3, Z_{55} + f_4)$
$= \min(144 + 0 \, , 105 + 40 \, , 78 + 52 \, , 55 + 60 \, , 40 + 84) = 115$

$f_6 = \min \; (Z_{16} + f_0, Z_{26} + f_1, Z_{36} + f_2, Z_{46} + f_3, Z_{56} + f_4, Z_{66} + f_5)$
$= \min(269 + 0 \, , 205 + 40 \, , 153 + 52 \, , 105 + 60 \, , 65 + 84 \, , 40 + 115) = 149$

$f_7 = \min \; (Z_{17} + f_0, Z_{27} + f_1, Z_{37} + f_2, Z_{47} + f_3, Z_{57} + f_4, Z_{67} + f_5, Z_{77}$
$\qquad\qquad + f_6)$
$= \min(389 + 0 \, , 305 + 40 \, , 233 + 52 \, , 165 + 60 \, , 105 + 84 \, , 60$
$\qquad\qquad + 115 \, , 40 + 149) = 175$

$f_8 = \min \; (Z_{18} + f_0, Z_{28} + f_1, Z_{38} + f_2, Z_{48} + f_3, Z_{58} + f_4, Z_{68} + f_5, Z_{78}$
$\qquad\qquad + f_6, Z_{88} + f_7)$
$= \min(424 + 0 \, , 335 + 40 \, , 258 + 52 \, , 185 + 60 \, , 120 + 84 \, , 70$
$\qquad\qquad + 115 , 45 + 149 \, , 40 + 175) = 185$

$f_9 = \min \; (Z_{19} + f_0, Z_{29} + f_1, Z_{39} + f_2, Z_{49} + f_3, Z_{59} + f_4, Z_{69} + f_5, Z_{79}$
$\qquad\qquad + f_6, Z_{89} + f_7, Z_{99} + f_8)$
$= \min \; (504 + 0 \, , 405 + 40 \, , 318 + 52 \, , 235 + 60 \, , 160 + 84 \, , 100$
$\qquad\qquad + 115 , 65 + 149 \, , 50 + 175 \, , 40 + 185) = 214$

$f_{10} = \min \; (Z_{1-10} + f_0, Z_{2-10} + f_1, Z_{3-10} + f_2, Z_{4-10} + f_3, Z_{5-10}$
$\qquad\qquad + f_4, Z_{6-10} + f_5, Z_{7-10} + f_6, Z_{8-10} + f_7, Z_{9-10} + f_8, Z_{10-10}$
$\qquad\qquad + f_9)$

$= \min \; (684 + 0 \, , 565 + 40 \, , 458 + 52 \, , 355 + 60 \, ,$

$260 + 84 , 140 + 115, 125 + 149 , 90 + 175 ,60 + 185 ,40 + 214) = 245$

$f_{11} = \min\ (Z_{1-11} + f_0, Z_{2-11} + f_1 , Z_{3-11} + f_2, Z_{4-11} + f_3, Z_{5-11} + f_4, Z_{6-11}$
$\qquad\qquad + f_5, Z_{7-11} + f_6, Z_{8-11} + f_7, Z_{9-11} + f_8, Z_{10-11} + f_9, Z_{11-11} + f_{10})$
$\qquad\qquad =$
$\min\ (734 + 0 , 610 + 40 , 498 + 52 ,$
$\ 390 + 60 , 290 + 84 , 205 + 115, 145 + 149 , 105 + 175 ,70 + 185 ,45$
$\qquad\qquad + 214 ,40 + 245) = 255$

$f_{12} = \min\ (Z_{1-12} + f_0, Z_{2-12} + f_1 , Z_{3-12} + f_2, Z_{4-12} + f_3, Z_{5-12} + f_4, Z_{6-12} + f_5, Z_{7-12} + f_6,$
$Z_{8-12} + f_7, Z_{9-12} + f_8, Z_{10-12} + f_9, Z_{11-12} + f_{10}, Z_{12-12} + f_{11}) =$
$\min(954 + 0 , 810 + 40 , 178 + 52 , 550 + 60 , 430 + 84 , 325 + 115, 245 + 149 , 185$
$\qquad\qquad + 175 ,130 + 185 ,85 + 214 ,60 + 245 , 40 + 255) = 230$
$\qquad\qquad = \text{Optimal Total Cost}$

With MATLAB
   TC= min([954+0   810+40   178+52   550+60   430+84   325+115
245+149 185+175 130+185 85+214 60+245 40+255])   TC =
230

Step 3   Finding optimal combinations and converting the optimal
solution $f_N = \ f_{12} = 295$ into an optimal ordering plan

   The optimal among the costs are
   $f_N = Z_{w,N} + f_{w-1}$
        $f_{12} = \ Z_{12,12} + f_{11} = 295$
   The final order which occurs at Period $w = 12$ covers the demand
of Perio2 12 with size 20

   To determine the order prior to the last order :
   $f_{w-1} = Z_{u,w-1} + f_{u-1}$ w=12
$f_{12-1} = Z_{u,11} + f_{u-1}$
   $f_{11} = 255$ corresponds to f_8 & $Z_{9,11}$ then $Z_{u,11} + f_{u-1} = Z_{9,11} + f_8$
and u =9
   The order prior to the last order is made at period $u = 9$ and covers
the requirements of periods 9 through 11 $w - 1 = 11(Z_{9,11})$ with size
10+20+5=35.
   For the order prior to the final order we considered f_{11}.
   the 3rd order from the end
   For the 3rd order from the end let us consider f_8
$Z_{u,w-1} + f_{u-1} = f_8 = Z_{6,8} + f_5 = 185$

The 3rd order from the end is made for the periods 6 through 8 with size 25+20+5=50.

For the 4th order from the end ,consider f_5

$$Z_{u,w-1} + f_{u-1} = f_5 = Z_{4,5} + f_3 = 175$$

The 4th order from the end is made for the periods 4 and 5 with size 23.

For the 5th order from the end ,consider f_3

$$Z_{u,w-1} + f_{u-1} = f_3 = Z_{1,3} + f_0 = 65$$

The 5th order from the end is made for the periods 1,2,3 with size 18. This is the first order. The horizon is covered.

The orders could be determined from Z's:

$$Z_{1,3}, \quad Z_{4,5}, , Z_{6,8}, \quad , Z_{9,11} \quad , Z_{12,12}$$

Therefore the algorithm give the following plan which is optimal:

The 1st order of size 18 for periods 1 through 3

The second order of size 23 for periods 4 & 5

The third order of size 50 for periods 6 through 8

The 4th order of size 35 for periods 9 through 11

The last order of size 20 for Period 12.

The results summary is mentioned in the following table:

Wagner-Whitin Algorithm											
t	1	2	3	4	5	6	7	8	9	10	11	12
Dt	2	12	4	8	15	25	20	5	10	20	5	20
Q_t	18	-	-	23	-	50	-	-	35	-	-	20
I_t	16	4	0	15	0	25	5	0	25	5	0	0

Cost:

$$TVC = 5C_o + C_h \sum_{t=1}^{12} I_t = 200 + 1(16 + 4 + \cdots + 5 + 0 + 0) = 200 + 95 = 295 \blacktriangle$$

Example 4-17[1]

Find the order lot sizes for the time horizon given in the table below using Wagner-Whitin   method. If the unit price is \$50, the ordering cost is \$100 and the unit holding cost from one period to the next immediate period is \$0.02, Also calculate the costs.

t	1	2	3	4	5	6
D_t	75	0	33	28	0	10

Solution

Step 1   calculations of Z's from $Z_{te} = C_o + C_h \sum_{i=t}^{e}(Q_{te} - Q_{ti})$:

  a)

  calculation of $Z_{1e}, e = 1,2,.., N = 6$**:**

  $C_o = \$100 \qquad C_h = 0.02 \times 50 = 1$ dollar $\qquad\qquad Q_{te} = \sum_{i=t}^{e} D_i$

$$Z_{11} = C_o + C_h \sum_{i=1}^{e=1}(Q_{1e} - Q_{1i}) = 100 + 1(Q_{11} - Q_{11}) = 100$$

$$Z_{12} = C_o + C_h \sum_{i=1}^{e=2}(Q_{1e} - Q_{1i}) =$$

$$100 + C_h(Q_{12} - Q_{11}) + C_h(Q_{12} - Q_{12})$$
$$= 100 + 1(75 + 0 - 75) + 1(75 - 75) = 100$$

$$Z_{13} = C_o + C_h \sum_{i=1}^{e=3}(Q_{1e} - Q_{1i})$$
$$= C_o + C_h(Q_{13} - Q_{11}) + C_h(Q_{13} - Q_{12})$$
$$+ C_h(Q_{13} - Q_{13})$$

$$100 + 1\big((75 + 0 + 33 - 75) + (108 - 75) + (108 - 108)\big) = 166$$

$$Z_{14} = 100 + 1(Q_{14} - Q_{11}) + 1(Q_{14} - Q_{12}) + 1(Q_{14} - Q_{13})$$
$$+ 1(Q_{14} - Q_{14})$$

$$100 + 1\big((75 + 0 + 33 + 28 - 75) + (136 - 75) + (136 - 108)\big) + 1 \times 0 = 250$$

$$Z_{15} = 100 + 1(Q_{15} - Q_{11}) + 1(Q_{15} - Q_{12}) + 1(Q_{15} - Q_{13})$$
$$+ 1(Q_{14} - Q_{14}) + 0$$

---

[1] Extracted from Tersine(1994) page 182

$$= 100 + 1\big((136 - 75) + (136 - 75) + (136 - 108) + 0\big) = 250$$

$$Z_{16} =$$
$$100 + 1(Q_{16} - Q_{11}) + 1(Q_{16} - Q_{12}) + 1(Q_{16} - Q_{13})$$
$$+ 1(Q_{16} - Q_{14}) + 1(Q_{16} - Q_{15})$$
$$= (146 - 75) + (146 - 75) + (146 - 108)$$
$$+ (146 - 136) + (146 - 136) + 0 = 300$$

b)calculation of $Z_{2e}, e = 2,..,6$

$$Z_{22} = C_o + C_h \sum_{i=2}^{e=2}(Q_{2e} - Q_{2i}) = 100 + 0 = 100$$
$$Z_{23} = 100 + 1\big((33 - 0) + (33 - 33)\big) = 133$$
$$Z_{24} = 100 + 1\big((33 + 28 - 0) + (61 - 33) + (61 - 61)\big) = 189$$
$$Z_{25} = C_o + C_h \sum_{i=2}^{e=5}(Q_{2e} - Q_{2i})$$
$$C_o + 1(Q_{25} - C_o Q_{22}) + 1(Q_{25} - Q_{23}) + 1(Q_{25} - Q_{24})$$
$$+ 1(Q_{25} - Q_{25})$$
$$C_o + 1(61 - 0) + 1(61 - 33) + 1(61 - 61) + 1(61 - 61) = 189$$
$$Z_{26} = C_o + C_h \sum_{i=2}^{e=6}(Q_{2e} - Q_{2i}) = 229$$

c)calculation of $Z_{3e}, e = 3,..,6$

$$Z_{33} = 100 + 1[(33 - 33)] = 100,$$
$$Z_{34} = 100 + 1[(61 - 33) + (61 - 61)] = 128,$$
$$Z_{35} = 100 + 1[(61 - 33) + (61 - 61) + (61 - 61)] = 128,$$
$$Z_{36} = 100 + 1[(71 - 33) + (71 - 61) + (71 - 61) + (71 - 71)] = 158,$$

d)calculation of $Z_{4e}, e = 4,5,6$

$$Z_{44} = 100 + 1[(28 - 28)] = 100,$$
$$Z_{45} = 100 + 1[(28 - 28) + (28 - 28)] = 100,$$
$$Z_{46} = 100 + 1[(38 - 28) + (38 - 28) + (38 - 38)] = 120,$$

e)calculation of $Z_{5e}, e = 5,6$ (Z_{55} و Z_{56}

$$Z_{55} = C_o + C_h \sum_{i=5}^{e=5}(Q_{5e} - Q_{5i}) = 100$$

$$Z_{56} = C_o + C_h \sum_{i=5}^{e=6}(Q_{5e} - Q_{5i}) =$$

$$100 + 1(Q_{56} - Q_{55}) + 1(Q_{56} - Q_{56})$$
$$= 100 + 100 + 1(10 - 0) + 1(10 - 10) = 110$$

f)calculation of Z_{66}

$$Z_{66} = C_o + C_h \sum_{i=6}^{e=6}(Q_{6e} - Q_{6i}) = 100 + 1(Q_{66} - Q_{66}) = 100$$

The following table shows the result of calculating Z_{te} 's:

values oftotal variable costs: $Z_{te}, 1 \le t \le e \le N$					(Tersine,1994 page 183)	
e t	1	2	3	4	5	6
1	10	100	166	25	250	300
2		100	133	18	189	229
3			100	12	128	158
4				10	100	120
5					100	110
6						100

Step 2

calculation of minimum of possible cost in periods 1 through e(f_e):
To obtain the minimum of possible cost in periods 1 through e we need to calculate for $e = 1, \dots, N = 6$ the following value:

$$f_e = \underbrace{Min}_{for \quad t=1,\dots,e} (Z_{te} + f_{t-1}) \text{ or}$$

$$f_e = Min(Z_{1e} + f_0 , \quad Z_{2e} + f_{1_{,\dots,}} , \quad Z_{ee} + f_{e-1}) \qquad e = 1, \dots, N$$

$$f_e = Min(Z_{te} + f_{t-1} \quad) \quad \text{for} \quad t = 1, \dots, 6 \quad f_0 = 0$$

$$f_1 = Min(Z_{11} + f_0) = (100 + 0)$$
$$= 100 \quad for \quad Z_{11} + f_0,$$
$$f_2 = Min(Z_{12} + f_0, Z_{22} + f_1) = Min(100 + 0, 100 + 100)$$
$$= 100 \quad for \quad Z_{12} + f_0,$$
$$f_3 = Min(Z_{13} + f_0, Z_{23} + f_1, Z_{33} + f_2) = (166 + 0, 133 + 100, 100 + 100)$$
$$= 166 \quad for \quad Z_{13} + f_0,$$
$$f_4 = Min(Z_{14} + f_0, Z_{24} + f_1, Z_{34} + f_2, Z_{44} + f_3)$$
$$= (250 + 0, 189 + 100, 128 + 100, 100 + 166)$$
$$= 228 \quad for \quad Z_{34} + f_2,$$
$$f_5 = Min(Z_{15} + f_0, Z_{25} + f_1, Z_{35} + f_2, Z_{45} + f_3, Z_{55} + f_4)$$
$$= (250 + 0, 189 + 100, 128 + 100, 100 + 166, 100 + 228)$$
$$= 228 \quad for \quad Z_{35} + f_2,$$
$$f_6 = Min(Z_{16} + f_0, Z_{26} + f_1, Z_{36} + f_2, Z_{46} + f_3, Z_{56} + f_4, Z_{66} + f_5)$$
$$= (300 + 0, 229 + 100, 158 + 100, 120 + 166, 110 + 228, 100 + 228)$$
$$= 258 \quad for \quad Z_{36} + f_2.$$

The table below shows the alternatives of variable costs($Z_{te} + f_{t-1}$ and f_e values:

	Values of variable costs ($Z_{te} + f_{t-1}$) and f_e					
e t	1	2	3	4	5	6
1	10	10	166	250	250	300
2		20	233	289	289	329
3			200	228	228	258
4				266	266	286
5					328	338
6						328
f_e	10	10	166	228	228	258

Step 3   Finding optimal combinations and converting the optimal solution f_N into an optimal ordering plan

Determine the last order by applying Criterion I of step 3 mentioned in the algorithm :

In this example $f_6 = f_N$ corresponds to the combination of "f_2 and Z_{36} i.e. according to Criterion I

$$f_N = Z_{w,N} + f_{w-1} = Z_{36} + f_2$$

According to this criterion the final order is planned for Period w=3 for the requirement of periods 3 through 6 with lot size of $33 + 28 + 0 + 10 = 71$

Determining the order prior to last order by applying Criterion II of step 3 mentioned in the algorithm :

$$f_{w-1} = Z_{u,w-1} + f_{u-1} \qquad w=3$$

$$f_{w-1} = Z_{u,w-1} + f_{u-1} \quad w = 3 \quad f_2 = Z_{u,2} + f_{u-1}$$

f_2 was obtained from the combination of f_0 and Z_{12} therefore $u = 1$. The order is placed at Period 1. This order covers demands in periods u through w-1 i.e. periods 1 and 2 with size $75 + 0 = 75$. These 2 orders suffice to cover the time horizon. The algorithm ends.

Therefore the method gives the following results

t	1	2	3	4	5	6	sum
D_t	75	0	33	28	0	10	146
Q_t	75	-	71	-	-	-	146

Costs:

Ordering cost $= 2 \times 100 =10$

Holding cost:

$C_h \sum_{t=1}^{6} I_t = 1(0 * 1 + 0 * 1 + 38 * 1 + 10 * 1 + 10 * 1) = 58$

$TVC = 200 + 58 = 258$▲

Example 4-18[1]

Using the data given in the following table, Find the solution to this dynamic lot sizing problem by several methods and compare their costs if

C_h per period= \$1   The setup or order cost=$C_O = \$40$

t	1	2	3	4	5	6	7	8	9	10	11	12
D_t	2	12	4	8	15	25	20	5	10	20	5	20

[1] Based on https://www.isye.gatech.edu/~spyros/courses/IE3104/Summer-06/Hw4-Solution.doc

Solution
(i)Silver-Meal
Iteration 1
Starting period:1

$$AC(j) = \frac{C_O + C_h \times \sum_{t=2}^{j}(t-1)D_t}{j}$$

  AC(1) = 40
  AC (2) = (40 + 12)/2 = 26
  AC (3) = [40 + 12 + (2)(4)]/3 = 20
  AC (4) = [40 + 12 + (2)(4) + (3)(8)]/4 = 21   Stop
Iteration 2
Starting period:4
  AC (1) = 40
  AC (2) = (40 + 15)/2 = 27.5
  AC (3) = [40 + 15 + (2)(25)]/3 = 35   stop
Iteration 3
Starting period:6
  AC (1) = 40
  AC (2) = (40 + 20)/2 = 30
  AC (3) = [40 + 20 + (2)(5)]/3 = 23.3333
  AC (4) = [40 + 20 + (2)(5) + (3)(10)]/4 = 25   stop
Iteration 4
Starting period:9
  AC (1) = 40
  AC (2) = $\frac{40 + 20}{2} = 30$
  AC (3) = $\frac{[40 + 20 + (2)(5)]}{3} = 23.3333$
  AC (4) = $\frac{[40 + 20 + (2)(5) + (3)(20)]}{4} = 32.50$ stop.

   Then according to Silver Meal method 5 orders have to be placed with sizes (2+12+4)‹(8+15))‹(25+205)‹(10+20+5 and (20) for Periods 1,4,6&9
   = (2, 12, 4, | 8,15, | 25, 20, 5, | 10, 20, 5, | 20)
   Cost: $C_h = 1$ $C_O = 40$

t	1	2	3	4	5	6	7	8	9	10	11	12
D_t	2	12	4	8	15	25	20	5	10	20	5	20
Q_t	18	-	-	23	-	50	-	-	35	-	-	20

	t	16	4	0	15	0	25	5	0	25	5	0	0

$$C_h \sum_{t=1}^{12} I_t = 1(16 + 4 + 15 + 25 + 5 + 25 + 5) = 95$$

TVC=$5C_o + C_h \sum_{t=1}^{12} I_t = (5)(40)+95=295$.

ii)LUC
Iteration 1
Starting period :1

UC(1) = 40/2 = 20
UC (2) = (40 + 12)/(2 + 12) = 3.71
UC (3) = (40 + 12 + 8) /(2 + 12 + 4) = 3.33
UC (4) = (40 + 12 + 8 + 24) /(2 + 12 + 4 + 8) = 3.23
UC (5) = (40 + 12 + 8 + 24 + 60) /(2 + 12 + 4 + 8 + 15) = 3.51
     Stop.
Iteration 2
Starting period :5
UC (1) = 40/15 = 2.67
UC (2) = (40 + 25)/(15 + 25) = 1.625
UC (3) = (40 + 25 + 40)/(15 + 25 + 20) = 1.75   Stop
Iteration 3
Starting period :7
UC (1) = 40/20 = 2
UC (2) = (40 + 5)/(20 + 5) = 1.8
 UC (3) = (40 + 5 + 20)/(20 + 5 + 10) = 1.86   stop
Iteration 4
Starting period :9
UC (1) = 40/10 = 4
UC (2) = (40 + 20)/(10 + 20) = 2
UC (3) = (40 + 20 + 10)/(10 + 20 + 5) = 2
UC (4) = (40 + 20 + 10 + 60)/(10 + 20 + 5 + 20) = 2.3636
Ṡolution of LUC:
= (2, 12, 4, 8, | 15, 25, | 20, 5, | 10, 20, 5, | 20)
$C_h = 1$ $C_O = 40$

t	1	2	3	4	5	6	7	8	9	10	11	
Dt	2	12	4	8	15	25	20	5	10	20	5	20

Q_t	26	-	-		40	-	25	-	35	-	-	20
t	24	12	8	0	25	0	5	0	25	5	0	0

Cost

$C_h \sum_{t=1}^{12} I_t = 1(24 + 12 + \cdots + 5) = 104$

TVC$=5C_o + C_h \sum_{t=1}^{12} I_t = 5*40+104=304$

iii) LTC or PPA method

This approach   sets the order horizon equal to the number of periods that most closely matches the total carrying cost with the order cost, which is $40 in this problem. Therefore, the absolute value of the difference between the holding and order costs is calculated in each period and the one with the lowest value is found.

Iteration1
Starting period:1
Through holding cost
Period n

$$APP_n = \sum_{i=1}^{n}(i-1)D_i \qquad C_h APP_n \qquad |C_h APP_n - C_o|$$

1	0	0	40
2	1×12	12	28
3	$1 \times 12 + 2 \times 4$	20	20
4	$20 + 3 \times 8$	44	4 ← (closest)
5	$44 + 4 \times 15$	104	64

Iteration2
Starting period:2
Through holding cost
Period n

$$\underline{n} \qquad APP_n = \sum_{i=1}^{n}(i-1)D_i \_ \qquad |C_h APP_n - C_o|$$

5	40	0
6	25	15 ← closest
7	65	25

Iteration 3: starting period:7

$$\underline{n} \qquad APP_n = \sum_{i=1}^{n}(i-1)D_i \_ \qquad |C_h APP_n - C_o|$$

7	40	0
8	5	35
9	25	15 ← closest

10   85   45

Iteration 4: starting period:10

$$\underline{n} \qquad APP_n = \sum_{i=1}^{n}(i-1)D_i \underline{} \qquad\qquad |C_h APP_n - C_o|$$

10   40   0
11   5   35
12   45   5 ← closest

Solution of LUC: = (2, 12, 4, 8, | 15, 25, | 20, 5, 10, | 20, 5, 20)

i.e.

1st order occurs in Period 1 with size2+12+4+8=26

2nd order occurs in Period 5 with size25+15=40,

3rd order occurs in Period 7with size 35,

Final order occurs in Period 10 with size 20+5+20=45

The calculations are given below:

Calculations of PPA=LTC algorithm applied to Example 4-18			
Iteration	Included Periods	Demand	|holding cost − ordering cost |
1	1	2	$\|0 - 40\| = 40$
	1.2	14	$\|12 - 40\| = 28$
	1.2.3	18	$\|20 - 40\| = 20$
	1.2.3.	26	$\|44 - 40\| = 4$
	1.2.3.	41	$\|104 - 40\| = 64$
2	5	15	$\|0 - 40\| = 40$
	5.6	40	$\|25 - 40\| = 15$
	5.6.7	60	$\|65 - 40\| = 25$
3	7	20	$\|0 - 40\| = 40$
	7.8	25	$\|5 - 40\| = 35$
	7.8.9	35	$\|25 - 40\| = 15$
	7.8.9.	55	$\|85 - 40\| = 45$
4	10	20	$\|0 - 40\| = 40$
	10.11	25	$\|5 - 40\| = 35$

Results of PPA or LTC algorithm												
t	1	2	3	4	5	6	7	8	9	10	11	12
D_t	2	12	4	8	15	25	20	5	10	20	5	20
Q_t	26	-	-	-	40	-	35	-	-	45	-	-
t	24	12	8	0	25	0	15	10	0	25	20	0

Cost
$C_h = 1$ and $C_o = 40$

$$TVC = 4C_o + C_h \sum_{t=1}^{12} I_t = 160 + 1(24 + 12 + \cdots + 20 + 0) = 299$$

iv) EOI or POQ Algorithm

$$\overline{D} = \frac{2+12+4+8+15+25+20+5+10+20+5+20}{12} = 12.16$$

$C_0 = 40 \qquad C_h = 1$

$$T = \sqrt{\frac{2C_o}{\overline{D} \times C_h}} \qquad\qquad T = \sqrt{\frac{2 \times 40}{12.16 \times 1}} = 2.56 \cong 3$$

t	1	2	3	4	5	6	7	8	9	10	11	12
D_t	2	12	4	8	15	25	20	5	10	20	5	20
Q_t	18	-	-	48	-	-	35	-	-	45	-	-
I_t	16	4	0	40	25	0	15	10	0	25	20	0

$$TVC = 4C_o + C_h \sum_{t=1}^{12} I_t = 4 * (C_0) +$$

$$C_h * (16 + 4 + 40 + 25 + 15 + 10 + 25 + 20) = 160 + 155 = 315$$

v) PPB   Algorithm :

Calculation of PPB algorithm applied to Example 4-18			
Iteration	Included Periods		$App_n = \sum_{i=1}^{n} (i-1)D_i$
1	1		0
	1.2		$0 + 12$
	1.2.3		$12 + 8$
	1.2.3.4		$20 + 24 = 44$
Since APP4 exceedsEPP $= \frac{40}{1}$ an order of size Q=8+15=23is placed for the 3 previous periods i.e.1,2&3			
2	4		0
	4.5		$0 + 15 = 15$
	4.5.6		$15 + 50 = 65$

Since APP6 exceedsEPP $= \frac{40}{1}$ an order is placed for Periods 4&5 of size Q=8+15=23			
3	6		0
	6.7		$0 + 20 = 20$
	6.7.8		$20 + 10 = 30$
	6.7.8,9		$30 + 30 = 60$
Since APP9 exceeds EPP $= \frac{40}{1}$ an order of size Q=5+20+25=50 is placed for the 3 previous periods i.e.6,7&8			
4	9		0
	9.10		20
	9.10.11		$20 + 2(5) = 30$
	9.10.11		30+60=90
Since APP12 exceedsEPP $= \frac{40}{1}$ an order of size Q=10+20+5=35 is placed for the 3 previous periods i.e.9,10&11. Furthermore an order is placed for Period 12 with size 20 the final order.			

The summary of PPB algorithm												
t	1	2	3	4	5	6	7	8	9	10	11	12
D_t	2	12	4	8	15	25	20	5	10	20	5	20
Q_t	18	-	-	23	-	50	-	-	35		-	20
$_t$	16	4	0	15	0	25	5	0	25	5	0	0

Cost
$C_h = 1$ $C_o = 40$
TVC =
$5C_o + C_h \sum_{t=1}^{12} I_t = 200 + 1(16 + 4 + \cdots + 5 + 0 +$
$0) =$200+95=295.

$TVC = 5C_o + C_h \sum_{t=1}^{12} I_t = 200 + 1(16 + 4 + \cdots + 5 + 0 + 0)$ =200+95=295
vi)POS Method

Assume the inventory before the horizon begins is 4 units , $T_L = 2$ months, POS=5, safety stock =3

Results of POS							
Period(t)	-2	-1	1 Jan	2 Feb	3 Mar	4 Apr	5 May
Net Requirement(D_t)			2	12	4	8	15
Available inventory (I_t)		4	42	30	26	18	3
Received order			40= 41-4+3				
Planned order	40					80	

Results of POS(continued)								
t	6 June	7 July	8 Aug	9 Sep	10 Oct	11 Nov	12 Dec	sum
Net Requirement(Dt)	25	20	5	10	20	5	20	146
Available Inventory(I_t)	58	38	33	23	3	23	3	
Received Order	80					25		146
Scheduled order				25				

Note That since POS=5, the lot size is derived from the summation the requirement of 5 consecutive periods

$$C_0 = 40, \quad C_h = 1, TVC = 3C_o + C_h \sum_{t=1}^{8} I_t = 120 + 300 = 420$$

Vii) Incremental part Period Algorithm(IPPA)

The calculations are given in the following table.

The sign * in the table means that the iteration has not arrived at the stop point i.e to the period for which $IPP_n \geq EPP$

where $EPP = \dfrac{C_O}{C_h} = \dfrac{40}{1}$ and $IPP_n = (n-1)D_n$

Calculations of Example 4-18 by IPPA										
Q	Iteration	t	1	2	3	4	5	6	7	
		D_t	2	12	4	8	15	25	20	
	1	n	1	2	3	4				
		$IPP_n = (n-1)D_n$	0	*12	*8	*24	60			
Q_1			2+12+ 4+8=26							
	2	n					1	2	3	
		$IPP_n = (n-1)D_n$					0*	*25	40	

Calculation of IPPA (continued)											
	Iter.	t	5	6	7	8	9	10	11	12	
		Dt	15	25	20	5	10	20	5	20	
Q_2			15+ 25+ 20 60								
	3	n				1	2	3			
		$IPP_n = (n-1)D_n$				0*	*5	40			
Q_3						35					
	4	n							1	2	
		$IPP_n = (n-1)D_n$							0	20	

Results of IIPA algorithm applied to Example 4-18												
t	1	2	3	4	5	6	7	8	9	10	11	12
D_t	2	12	4	8	15	25	20	5	10	20	5	20
Q_t	26	-	-	-	60	-	-	35	-	-	25	
I_t	24	12	8	0	45	20	0	30	20	0	20	0

$C_0 = 40$ $C_h = 1$

$TVC = 3 * C_0 + C_h * (24 + 12 + 8 + 45 + 20 + 30 + 20 + 20) = 120 + 209 = 329$

viii) With Lingo software

The model of Example 4-18 typed in Lingo environment:

min =
40*(z1+z2+z3+z4+z5+z6+z7+z8+z9+z10+z11+z12)+1*(i1+i2+i3+i4+i5+i6+i7+i8+
i9+i10+i11+i12);
   i0+Q1=i1+2;
   i1+Q2=i2+12;
   i2+Q3=i3+4;
   i3+Q4=i4+8;
   i4+Q5=i5+15;
   i5+Q6=i6+25;
   i6+Q7=i7+20;
   i7+Q8=i8+5;
   I8+Q9=i9+10;
   I9+Q10=i10+20;
   I10+Q11=i11+5;
   I11+Q12=i12+20;
   Q1<=146*z1;
   Q2<=146*z2;
   Q3<=146*z3;
   Q4<=146*z4;
   Q5<=146*z5;
   Q6<=146*z6;
   Q7<=146*z7;
   Q8<=146*z8;
   Q9<=146*z9;
   Q10<=146*z10;
   Q11<=146*z11;
   Q12<=146*z12;
   @BIN(z1);@BIN(z2);@BIN(z3); @BIN(z4);@BIN(z5);@BIN(z6);
   @BIN(z7);@BIN(z8);
   @BIN(z9);@BIN(z10);@BIN(z11);@BIN(z12);
   i0=0;i1>=0;i2>=0;i3>=0;i4>=0;i5>=0;i6>=0;i7>=0;i8>=0;I9>=0;I10>=0;
   I11>=0;I12=0;end
   Global optimal solution found at iteration: 507
    Objective value: 295.0000

Variable	Value	Reduced Cost
Z1	1.000000	40.00000
Z2	0.000000	-106.0000
Z3	0.000000	-252.0000
Z4	1.000000	40.00000

.………..

I0	0.000000	0.000000
Q1	**18**.00000	0.000000

Q2	0.000000	0.000000
Q3	0.000000	0.000000
Q4	**23**.00000	0.000000
Q5	0.000000	0.000000
Q6	**50**.00000	0.000000
Q7	0.000000	0.000000
Q8	0.000000	0.000000
Q9	**35**.00000	0.000000
Q10	0.000000	0.000000
Q11	0.000000	0.000000
Q12	**20**.00000	0.000000

	Results of Lingo			$C_h = 1$			$C_o = 40$						
t	1	2	3	4	5	6	7	8	9	10	11	12	sum
D_t	2	12	4	8	15	25	20	5	10	20	5	20	146
Q_t	18	0	0	23	0	50	0	0	35	0	0	20	
I_t	16	4	0	15	0	25	5	0	25	5	0	0	

Costs
>>   i1=16;   i2=4;   i3=0;   i4=15;   i5=0;   i6=25;
i7=5;i8=0;i9=25;i10=5;i11=0;i12=0;i0=0;
>>z1=1; z2=0; z3=0; z4=1; z5=0; z6=1; z7=0; z8=0;z9=1;z10=0;z11=0;z12=1;
>>TVC=
40*(z1+z2+z3+z4+z5+z6+z7+z8+z9+z10+z11+z12)+1*(i1+i2+i3+i4+i5+i6+i7+i8+
i9+i10+i11+i12)

$TVC = 5 * C_0 + C_h(16+,,,+5) = 200+95 = 295$

The costs of the solution of the algorithms applied to Example 4-18
are inserted in the following Table for comparison

TVC of Algorithms' solution for Example 4-18								
Method	Silver Meal	LUC	PPA= LTC	POQ= EOI	PPB	Wagner whitin	IPPA	Lingo
TVC	295	304	299	315	295	295	329	295

Exercises

1-What is meant by dynamic lot sizing?
2-Compare POQ and EOQ methods.
3-What is the difference between PPA and IPPA methods.
4-The requirements for the next 6 months are as follows:

t	1	2	3	4	5	6
D_t	0	10	30	40	60	20

The holding cost per unit product for each period is $5. The ordering cost for the first period is $70 and for the other periods is $200. The lead time is one month. Use the LUC method and another approach to find the order lot sizes. Which method is better? Why?

5-(Tersine, 1994p199) An item has a unit purchase price of $45,an ordering cost of $110 and the carrying cost fraction per period is 2.5%. Determine the order sizesusing PPB , IPPA and Silver-Meal algorithms. Which method is better? Why?

period	1	2	3	4	5	6	7	8	9	10
D_t	10	3	30	100	7	0	80	50	0	90

6-The requirements of a 12-period time horizon are given in the following table.   The holding cost fraction is 2%. The ordering cost per period is $200. Determine the order sizes using LTC, LUC & Silver-Meal algorithms.   Also solve this problem by Lingo software. Which method is better? Why?

t	1	2	3	4	5	6	7	8	9	10	11	12
D_t	10	0	0	120	70	180	250	270	0	40	0	10

Solve example 4-1 with Lingo, assuming that 8 units is necessary after the last period

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God is the light of the heavens and the earth.

Light is in your heart, you will find your way

Chapter5
Inventory Control

under

Uncertainty

Chapter 5

Inventory Control under Uncertainty

Aims of the chapter

This chapter deals addresses the problem of inventory control under uncertainty which is an important issue in supply chain management across industrial and commercial firms. In this regard such inventory models as single period inventory,(R,T) and(r,Q) are introduced . The end of chapter deal with the application of decision making in complete uncertainty in inventory control.

5.1 Introduction

As mentioned in chapter 1, the uncertainty condition could be divided into complete uncertainty conditions and risk conditions. The so-called completer uncertainty condition in inventory planning will be dealt at the end of this chapter. In risk conditions there is some records of past data which enable us to calculate the occurrence probability of the occurrence of the inventory model parameters. In what follows you will find inventory models such as single period inventory, FOI=(R,T) and FOS =(r,Q) models under risk conditions.

5.2 Single Period Inventory Model with Probabilistic demand

The single period inventory model described here are used in situations where a kind of raw material or a finished product is

ordered based on the probabilistic demand for it. The demand is a random variable where occurs in only a single period. The objective of the problem is to find that level of inventory before the start of the period(R) which maximizes profit. This model which is often called the newsboy problem or Christ- mas tree problem is used for perishable or seasonal items that could be ordered once   or have a short period of consumption such as bread , flower, fruit, vegetable, newspaper, new year cards, deteriorating items, the items that are produced once and cannot be carried in inventory and sold in future periods.

In this model

-The demand is a random variable,

-The period of consumption is relatively short, and one order for purchase or production is placed to be received at the beginning of the period.

-The salvage price is very low compared to the initial price.

- The objective in this problem is to determine an optimal level for the maximum inventory which maximizes profit.

Symbols

A	The position of inventory before placing an order
$X = D$	Demand
$f(x) =$ $f_D(x)$	The probability density function of variable demand
$F(x) =$	$\Pr(X \le x)$
$G_U(k)$	Unit loss normal integral
H $= H'$ $- L$	The actual holding cost of one item not sold during the period
H'	The cost of disposal of one unit at the end of period

$K(R)$ $\quad P(R - I) + HR + (V + \pi + H) \int_R^\infty (x - R)f(x)dx$

L $\qquad$ Salvage or sale value of one unit

p $\qquad$ Service level $(\Pr(X \le R))$

P $\qquad$ Unit price or unit cost of production

$P_D(x)$ $\qquad$ The probability function for discrete demand

R $\qquad$ The level of inventory after receiving the order

R^* $\qquad$ Optimal R

U $\qquad$ The sales revenue during the period

V $\qquad$ The value earned per unit sold

Y $\qquad$ The cost during the period(purchase/production ,holding &shotage)

Z $\qquad$ The profit during the period

π $\qquad$ Unit shortage cost(lost profit not included)

Note that:

It is assumed the cost of holding for the units sold during the period is ignorable .

H, the actual holding cost for each unsold unit at the end of the period, is equal to the differencebetween the disposal cost(H') and the salvage or sale price(L) i.e.

$$H=H'-L \qquad\qquad (5\text{-}1).$$

H' and L could be zero or positive; therefore H could be negative, zero or positive.

Let the shortage cost of one unit be denoted by π. In this model there is no time-depended shortage cost because there is only one period. By the way if in the case of shortage it is said that there is only lost profit per each shortage unit during the period then let $\pi=0$.

 As mentioned before in this model we would like to determine the inventory level after receiving the order (R) in such a way the the profit is maximized. For the period let

Y denotes the purchase, holding and shortage cost

U denotes sales revenue

Z =the profit during the period or $Z = U - Y \implies$
$$E(Z) = E(U) - E(Y). \tag{5-2}$$
It is obvious if the demand is more than R the sales amount is R :

Sales volume random variable= $= \begin{cases} x & D < R \\ R & D \geq R \end{cases}$

To deal with the model two cases are distinguished

      a)The order cost or setup cost (C_O) is ignorable,
      b) C_O is considerable.

5.2.1 Single Period Inventory Model –order/setup cost ignorable

Let us denote the revenue per unit sold be V then
      the average revenue =$V \times$ average sales volume

5.2.1.1 Single Period Inventory Model : $C_o \cong 0$ & continuous demand

If the order / setup cost (C_O) is ignorable and the demand is a continuous   random variable with probability density function $\mathbf{f(x)}$ then:

Average sale volume=

$$= \int_0^\infty (\text{sale volume}) f(x) dx = \int_0^R x f(x) dx + \int_R^\infty R f(x) dx \implies$$

    Average   sales   volume=$\int_0^\infty x f(x) dx - \int_R^\infty x f(x) dx + \int_R^\infty R f(x) dx = E(D) + \int_R^\infty (R - x) f(x) dx$

Average sales revenue= $E(U) = VE(D) + V \int_R^\infty (R - x) f(x) dx$ or

Average sales revenue = $E(U) = VE(D) - V \int_R^\infty (x - R) f(x) dx$ $(5-3)$

The total cost Y was defined as:

Y=purchase/production cost +holding cost+ shortage cost.

The unsold units at the end of the single period is a function of the demand :

$$\text{unsold units} = \ g(x) \qquad\qquad (5-4)$$

If the actual holding cost per unsold unit is H, then:

The average holding cost=

$$H \int_0^\infty g(x)f(x)dx = H\left[\int_0^R (R-x)f(x)dx + \int_R^\infty 0 f(x)dx\right] \Longrightarrow$$

The average holding cost of the period= $H \int_0^R (R-x)f(x)dx.$

Let the shortage which is a function of the demand be denoted by b(x):

$$b(x) = \begin{cases} 0 & D < R \\ x - R & D > R \end{cases} \qquad\qquad (5-5)$$

For continuous demand, the average   shortage volume for the period denoted by $\bar{b}(R)$ is equal to :

$$\bar{b}(R) = \int_0^\infty b(x)f(x)dx = \int_0^R 0 f(x)dx + \int_R^\infty (x-R)f(x)dx$$

This relationship after simplification is inserted in the following table as well as a similar relationship for the discrete demand.

Demand	Average shortage volume for the period	
continuous	$\bar{b}(R) = \int_R^\infty (x-R)f(x)dx$	(5-6)
Discrete	$\bar{b}(R) = \sum_{x=R+1}^\infty (x-R)P_D(x)$	(5-7)

Where $f(x)$ is the probability density function for continuous demand and $P_D(x)$ is the probability function for discrete demand.

If the cost per unit shortage is π then:

Average shortage cost for the period $= \pi\bar{b}(R) = \pi \int_R^\infty (x - R)f(x)dx$.

Let the position of inventory before placing an order be A. If the unit price is P then

Production /purchase cost $= P(R - A)$

Average total cost $= E(Y) = P(R - A) + H \int_0^R (R - x)f(x)dx + $

$$\pi \int_R^\infty (x - R)f(x)dx \implies$$

$$E(Y) = P(R - A) + H \int_0^\infty (R - x)f(x)dx$$
$$- H \int_R^\infty (R - x)f(x)dx + \pi \int_R^\infty (x - R)f(x)dx$$
$$\implies$$

$$E(Y) = P(R - A) + HR \int_0^\infty f(x)dx - H \int_0^\infty xf(x)dx$$
$$- H \int_R^\infty Rf(x)dx + H \int_R^\infty xf(x)dx$$
$$+ \pi \int_R^\infty xf(x)dx - \pi R \int_R^\infty f(x)dx \implies$$

$$E(Y) = P(R - A) + HR - HE(D) - (\pi + H) \int_R^\infty Rf(x)dx$$
$$+ (\pi + H) \int_R^\infty xf(x)dx \implies$$

Finally:

$$E(Y) = P(R - A) + H\big(R - E(D)\big) + (\pi + H) \int_R^\infty (x - R)f(x)dx.$$

Average profit is given by:

$$E(Z) = E(U) - E(Y)$$

$$E(U) = VE(D) + V \int_R^\infty (R - x)f(x)dx$$

$$E(Z) = VE(D) + V \int_R^\infty (R - x)f(x)dx - P(R - A) - HR$$
$$+ HE(D) -$$

$$(\pi + H) \int_R^\infty (x - R)f(x)dx$$

$$E(Z) = \underbrace{(V + H)E(D)}_{\text{does not depend on } R}$$
$$- \underbrace{\left[P(R - A) + HR + (V + \pi + H) \int_R^\infty (x - R)f(x)dx\right]}$$

Now let

$$K(R) = P(R - A) + HR + (V + \pi + H) \int_R^\infty (x - R)f(x)dx \quad (5\text{-}8)$$

Then

$$E(Z) = \underbrace{(V + H)E(D)}_{\text{does not depend on } R} - K(R) \quad (5 - 9)$$

Our objective is to determine a value for R which maximizes E(Z) or equivalently minimizes K(R) which plays a significant role in the cost of

this model. Note that $\frac{\partial^2 K(R)}{\partial R^2} = (V + \pi)Hf(R)$ is the product of 3 non- negatives then $\frac{\partial^2 K(R)}{\partial R^2} \geq 0$. Therefore $K(R)$ has minimum.   Figure 5.1 shows a typical function K(R) and its minimum
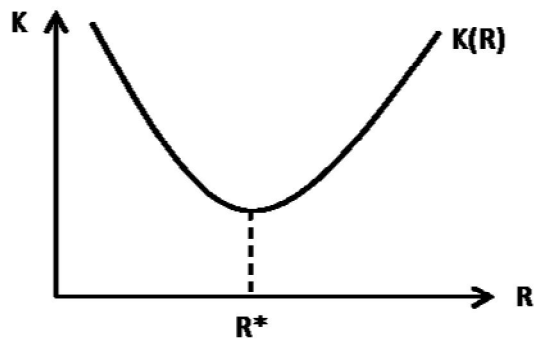
Fig 5.1 A typical function K(R)

Example 5.1

In a single period decision model $P = 0.2, A = 0, V = 2, \pi = 0,$ $H = 0.1$ and

If the demand for the period is uniformly distributed over [10,20], draw the function K(R) , $10 < R < 45,$

If the demand for the period is normally distributed with mean 20 and variance 9, draw the function K(R) , $0 < R < 20,$

Solution
a)

$K(R) = P(R - A) + HR + (V + \pi + H) \int_R^\infty (x - R)f(x)dx$

$P = 0.2, A = 0, V = 2, \pi = 0, H = 0.1, f(x) = \frac{1}{20-10}, x \in [10\ 20]\,,$

$$K(R) = 0.2(R - 0) + 0.1R$$
$$+ (2 + 0 + 0.1) \int_{x=R}^{20} (x - R) \frac{1}{10} dx$$
$$= 0.3R + \frac{2.1}{40} (30 - R)^2$$

The following command in MATLAB draws Fig 5.2:
R=10:.01:45;K=.3*R+2.1*(30-R).^2/40;plot(R,K)
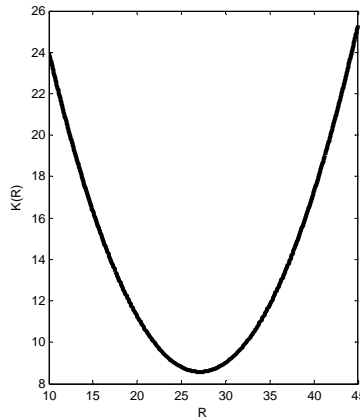
Fig 5-2 Function K(R) for Example 5.1 (uniform demand)

b)

   Substituting the data yields

$$K(R) = 0.3R + 2.1 \int_{R}^{\infty} (x - R)f(x)dx$$

Where $f(x)$ is the pdf of a normal distribution with $\mu = 20 \& \sigma = 3$.
According to Eq. 5-1 in Sec. 1.5.1 we could write

$$\int_{R}^{\infty} (x - R)f(x)dx = \sigma G_U(k) \qquad k = \frac{R - \mu}{\sigma}$$

Where

   $G_U(k)$ is given by Table A at the end of the book or by the
following MATLAB command:

   $G_U(k)$ =exp(-k.^۲/۲)/sqrt(۲*pi)-k.*(۱-normcdf(k))

Then $K(R) = 0.3R + 2.1\sigma G_U \left(\frac{R-20}{3}\right)$.

Fig 5.3 is the plot of K® versus R   drawn by the following MATLAB commands:

R=0:.01:20; k=(R-20)/3;

KR=.3*R+2.1*3*exp(-k.^2/2)/sqrt(2*pi)-k.*(1-normcdf(k));plot(R,KR)
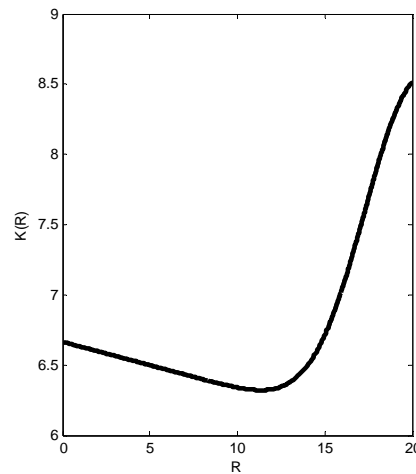
Fig 5-3 Function K(R) for Example 5.1(normal demand)

5-2-1-1-1 Optimal value of maximum inventory(R^*)

We are in search of that value of maximum inventory (R) which maximizes the profit E(Z) or that value of R which satisfy $\frac{dE(Z)}{dR} = 0$.

$$\frac{dE(Z)}{dR} = 0 \implies -P - H + (V + \pi + H)\int_R^\infty f(x)dx = 0 \implies$$

$$P + H - (V + \pi + H)[1 - F(R^*)] = 0 \implies$$

If demand is continuous, the optimal value of R is derived from:

$$F(R^*) = \frac{V+\pi-P}{V+\pi+H} \qquad\qquad (5\text{-}10)$$

The answer exists if $0 \le \frac{V+\pi-P}{V+\pi+H} \le 1$ and shortage is allowed.

Note that

-The differentiation under integral sign has used Leibniz's Rule. According to this rule if $F(y) = \int_{g(y)}^{h(y)} f(x,y)dx$, then

$$F'(y) = h'(y)f(h(y),y) - g'(y)f(g(y),y) + \int_{g(y)}^{h(y)} \frac{\partial f(x,y)}{\partial y}dx.$$

-the difference V-P is the profit of one unit,

-If the distribution of consumption during the period is denoted by X then

Shortage probability for the period= $\Pr(X > R^*) = 1 - F(R^*)$.

-What is sometimes called service level is equal to:

$$\text{Service level } p = \Pr(X \le R^*)$$

5-2-1-1-2 Optimal strategy in single period model

If $A \ge R^*$ i.e. the inventory level before placing an order is greater than or equal to R^*, no order is placed; and if $A < R^*$ an order is placed with the quantity

$$Q^* = R^* - A \qquad\qquad (5\text{-}11)$$

Needless to say that A is deducted from R^* only if the units are usable for the period ad are not things such as newspaper which is not usable for the coming period .

Some comments:

-When $\pi = 0$ we have $F(R^*) = \frac{V-P}{V+H}$. In this case it obvious that there exists an answer for R^* only if $V \geq P$ which is economically true.

- When the range of the demand is restricted to interval [a   b], if $F(R^*) = 1$ then set $R^* = b$; if a negative value was calculated for $F(R^*)$ set $R^* = a$ and if shortage is not permitted in the model ($\pi = \infty$)then $F(R^*) = 1$ and $R^* = b$.

5-2-1-1-3 average shortage cost in the single period model

Shortage occurs when the demand over the period (X) exceeds R; other wise we would not face with shortage and we have no cost incurred due to shortage.

$$\text{unit shortage cost} = \begin{cases} \pi & \Pr(X > R) \\ 0 & \Pr(X \leq R) \end{cases}$$

The expected value of shortage cost =

$$\pi \times Pr(X > R) + 0 \times Pr(X \leq R) = \pi \times Pr(X > R)$$

Example   5-2

The weekly demand of a kind of liquid follows a Weibul distribution with parameters A=0, B=1000 lit C=2. If the liquid is not consumed within a week ,it would be considered salvage and no one buys it and its cost of dis- posal is $0.1 per one liter unsold. There no shortage cost except the lost profit. The liquid is bought $0.2 per liter and sold $2 per liter. Find the opti- mal order quantity.

Solution

Weekly $D \sim \text{weib}(B = 1000, C = 2)$ $P = \frac{0}{2}, V = 2$, $\pi = 0,$

$$H = H^{'} - L = 0.1 - 0 = 0.1$$

$$F(R^*) = \frac{V + \pi - P}{V + \pi + H} = \frac{2 + 0 - 0.2}{2 + 0 + 0.1} = \frac{1.8}{2.1} \quad \Longrightarrow$$

$$1 - e^{-(0.001R^*)^2} = \frac{1.8}{2.1} = 0.8571 \quad \Longrightarrow R^* \cong 1392.$$

Or with MATLAB:

$$R^* = wblinv(.8571,1000,2) = 1392$$

$$Q^*{=}R^*{-}A \quad = 1392{-}\cdot = 1392 \quad \blacktriangle$$

Example 5-3

In a one period model an item is sold $20 per unit where the unit pur- chase price is $12. Shortage incur no cost except the lost profit. The unsold units have no value and cost at the end of the period. there is 5 units available at the beginning of the period. Find the optimal order quantity for the following cases:

a)The demand of the item in the one period model   follows a uniform distribution over(0, 100)

b) The demand is exponentially distributed with parameter $\lambda = 0.01$.

Solution

In this problem there is no shortage cost i.e. $\pi = 0$, since there is no cost except the lost profit and H'=0&L=0   since there is not any cost and revenue for the unsold units

$$F(R^*) = \frac{V + \pi - P}{V + \pi + H} = \frac{20 + 0 - 12}{20 + 0 + 0} = 0.4$$

a)For the uniform distribution:

$$F(x) = \frac{x - 0}{100 - 0}$$

$$F(R^*) = 0.4 \Longrightarrow \frac{R^* - 0}{100 - 0} = 0.4 \quad \Longrightarrow \quad R^* = 40$$

optimal order quantity $=Q^* =(\text{R}^*\text{-A})= 40\text{-}5\text{=}35.$

b)

$1 - e^{-0.01R^*} = 0.4 \qquad R^*=\text{expinv}(0.4,100) = =51$

Optimal order quantity $=51\text{-}5\text{=}46$ ▲

5.2.1.2 Single period Inventory model : $C_o \cong 0$ & discrete demand

In this section the above single-period model is retreated under assumption that the demand for the period is not continuous and the setup/order cost is negligible. In this case relationships similar to those developed for continuous demand case are obtained .   The difference lies on the use of sigma sign (Σ)sign instead of integral sign($\int$):

$$K(R) = P(R - I) + HR + (V + \pi + H) \sum_{x=R+1}^{\infty} (x - R)P_X(x)$$

$$\Delta K(R) = K(R + 1) - K(R) \implies$$

$$\Delta K(R) = P + H + (V + \pi$$
$$+ H)\left\{\left[\sum_{x=R+2}^{\infty} (x - R - 1)P_X(x)\right]\right.$$
$$\left.- \left[\sum_{x=R+1}^{\infty} (x - R)\, P_X(x)\right]\right\} \implies$$

$\Delta K(R)= P + H + (V + \pi + H)\{ \ [P_X(R + 2) + 2P_X(R + 3) + 3P_X(R + 4) + \cdots] - [P_X(R + 1) + 2P_X(R + 2) + 3P_X(R + 3) + \cdots]\}$

$\Delta K(R)=$P+H+$(V + \pi + H) \{-P_X(R + 1) - P_X(R + 2) - P_X(R + 3) - \cdots\}$
$\implies$

$$\Delta K(R) = P + H - (V + \pi + H)P_X(X > R)$$

We would like to minimize the discrete function K(R). Assuming ΔK(R)≥0 we could write:

$$\Delta K(R) = P + H - (V + \pi + H)P_X(X > R) \geq 0$$

$$P_X(X > R) \ \ or \ Pr(D > R) \leq \frac{P + H}{V + \pi + H} \ \ \Rightarrow$$

$$1 - Pr(D > R) \geq 1 - \frac{P+H}{V+\pi+H} \ \ or \ F(R) \geq 1 - \frac{P+H}{V+\pi+H}$$

Where F(R)   is the cumulative distribution function of demand.
   The best value of R denoted by R^* is the smallest R value which satisfies the following   inequality(based on Peterson &Silver, 1991 page 395)

Discrete demand $F(R) \geq \frac{V+\pi-P}{V+\pi+H}$ (5-12).

This R^* minimizes the cost function K(R).

Optimal Policy

R^* is the smallest R value which satisfies

If A≥R* i.e. the inventory level before placing an order is greater than or equal to R*, no order is placed; and if A<R* an order is placed with the quantity Q*=R*-A .

Example 5-4

  A single-period item is bought $3000 per unit and sold $5000 per unit; There is no shortage cost except the lost profit. The actual holding cost of one unsold item is H'≈0 i.e. negligible. The sale cost at the end of the period is: L=2000.

 The demand is discrete with the probabilities given below:

demand	6	7	8	9	10	11	12	13	14
Prob.	0.05	0.05	0.1	0.2	0.2	0.2	0.1	0.05	0.05

Find the optimal order quantity and the probability of shortage.

Solution

D or X	6	7	8	9	10	11	12	13	14
Probability	0.05	0.05	0.1	0.2	0.2	0.2	0.1	0.05	0.05
$F_D(x)$	0.05	0.1	0.2	0.4	0.6	0.8	0.9	0.95	1

R^* is the smallest value which satisfies $F_D(R) \geq \frac{V+\pi-P}{V+\pi+H}$

Since there is no shortage cost then $\pi = 0$.

$$H = H' - L = 0 - 2000$$

Shortage probability $=$

$$\Pr(X > R^*) = 1 - F_D(R^*) = \frac{P+H}{V+\pi+H} = \frac{3000-2000}{5000-2000} = \frac{1}{3}$$

$$F_D(R^*) \geq \frac{V+\pi-P}{V+\pi+H} = \frac{5000+0-3000}{5000+0-2000} = 0.66$$

     The smallest value which satisfies $F_D(R^*) \geq 0.66$ is the answer. According to the table $R^* = 11$.

5.2.2 Single Period Model –order/setup cost (C$_O$) considerable

   In this section the single-period model is studied subject to nonzero order/setup cost

Symbols

A   inventory level at the beginning of the period
R   inventory level after receipt of the order
r_0 The smallest root of $P r_0 + L(r_0) - C_O - PR_0 - L(R_0) = 0$

$L(R)$ $L(R) = HR + (V + \pi + H) \int_{R}^{\infty} (x - R)f(x)dx$

$K'(R)$ $K'(R) = PR + L(R)$

R_0 The point where functions $K(R), K'(R)$ are minimized derived from $F(R_0) - \frac{V+\pi-P}{V+\pi+H} = 0$

L(A)   The cost during the period if no order is placed

In the previous section where order/setup cost   was negligible ($C_o \cong 0$). K(R) in the relationship given for profit sometimes equals the cost which we want to minimize. In this section the cost including the order/setup cost C_o would be:

If R<A no order is placed no calculations is needed.

If R $\geq$ A,the cost of period equals C_o as well as the cost in the previous section i.e. $K(R) = P(R - A) + L(R)$ where

$$L(R) = HR + (V + \pi + H) \int_R^{\infty} (x - R)f(x)dx$$

Then the cost of the period:$C_O + K(R) = C_O + P(R - I) + \underbrace{HR + (V + \pi + H) \int_R^{\infty}(x - R)f(x)dx}_{L(R)}$

If we let $K'(R) = P \times R + L(R)$ then the cost of period =

$C_O + P(R - A) + L(R) = C_O + P \times R + L(R) - PA = C_O + K'(R) - PA$

Now let focus on function $K'(R)$ which plays a major role in the cost

$$K(R) = K'(R) - PA \qquad \rightarrow \qquad K'(R) = K(R) + PA$$

The product of the unit price and the inventory at the beginning of the period(A) is positive, then if R_0 is the point at which the minimum of K(R) occurs, the minimum of function $K'(R)$ occurs at the same point R_0. Now note that when R=A, no order is placed i.e. $C_O = 0$. Substituting $C_O = 0$ &$R = A$ in the above relationship yields the cost of the inventory system when $R = A$.

the cost of period $= C_O + P(R - A) + L(R)$

The cost for (R=A)=

$$0 + P(A - A) + HI + (V + \pi + H) \int_{I}^{\infty} (x - A)f(x)dx$$

Denoting the above cost with L(I), we could write:

$$L(A) = HA + (V + \pi + H) \int_{I}^{\infty} (x - A)f(x)dx$$

The following figure shows an example of the function

$K'(R) = PR + L(R)$. R is on the horizontal axis and $K'(R)$ on the vertical axis.
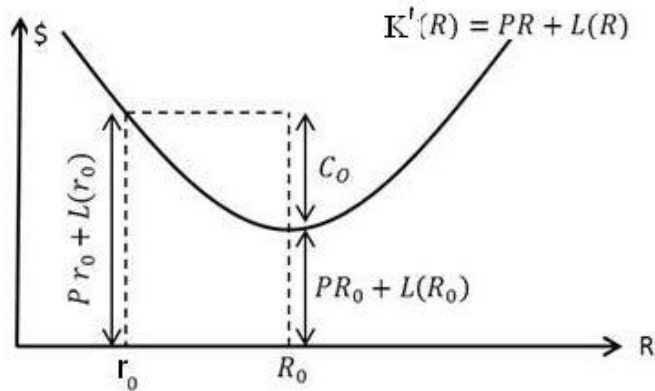
Fig. 5.4 A typical plot of function $K'(R) = PR + L(R)$

The minimum of function $K'(R)$ happens at the same point where K(R) is minimized i.e. a point such as R_0 derived from $F(R_0) - \frac{V + \pi - P}{V + \pi + H} = 0.$

Assume point r_0 be that value of R that minimizes "$C_O + K'(R)$". The minimum of $K'(R)$ is $K'(R_0)$ then the" minimum of $C_O + K'(R)$ " *is* " $C_O + K'(R_0)$". As Fig 5-4 shows this value on the vertical corresponds to 2 values   on the horizontal axis; however , the smaller value is of our interest in this case. Another words r_0 the smallest value ($r_0 < R_0$) which satisfies:

$$C_O + PR_0 + L(R_0) = K\left(r_0\right) = P\,r_0 + L\left(r_0\right).$$

r_0 and the inventory at the beginning of the period(A) play a role in determining the optimal policy in this case. 3 states are distinguished here:

State I: A>R_0

Substituting $R = A$ in K' yields $PA + L(A)$.   Referring to Fig 5.5 it is obvious that $K'(R) = PR + L(R) > PA + L(A)$. Adding the positive number C_O to the both sides does not change the direction of the inequality symbol: $C_O + PR + L(R) > PA + L(A) \Rightarrow C_O + P(R - A) + L(R) > L(A)$

The right had side of the inequality is the cost of the inventory system when no order us placed and the left hand of the inequality when A<R and an order of size R-A is placed. Since the latter cost is greater the former cost, then we have to place no order.
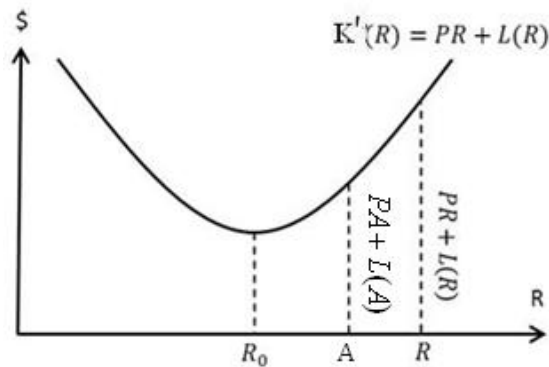
Fig 5.5 Single –period model $C_O \neq 0$ and A>R_0

State II: $r_0 \leq A \leq R_0$ for R>A

With the assumption $C_O + K'(R_0) > K'(A)$

For any R in the interval $r_0 < I < R \leq R_0$ (Fig. 5.6) we could write:
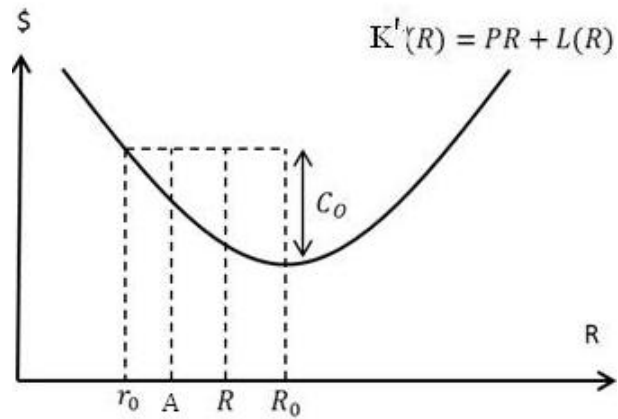
Fig 5-6 Single period model $C_O \neq 0 \, \& \, r_0 \leq A \leq R_0$

$$C_O + PR + L(R) > PA + L(A) \Rightarrow C_O + P(R - A) + L(R) > L(A)$$

The right hand side of the lat inequality the cost of the inventory system if no order is placed. Again here ($r_0 < A < R_0$) the cost of the inventory system if no order is placed is less than the cost if an order is placed; then we have to place no order.

Note since practically $R \nless A$ the state $r_0 < R < A \leq R_0$ is not applicable.

State III $A < r_o$ (Fig 5.7)

$$K'(R) = PR + L(R)$$

$$C_O$$
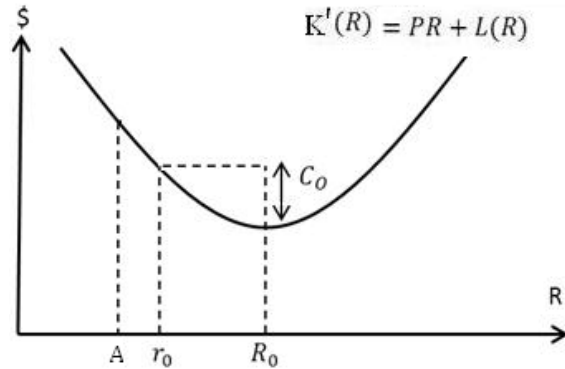
Fig 5-7 State III(A<r_o in single period model having C_o

Remembering the definition of r_o , in this state $K'(A) > K'(r_0)$.
Referring to Fig 7-5 we could write:

$$P R_0 + L\left(R_0\right) = K'\left(R_0\right) \ , \quad P r_0 + L\left(r_0\right) = K'\left(r_0\right)$$

$$K'(r_0) < K'(A)$$

$$C_O + P R_0 + L\left(R_0\right) = P r_0 + L\left(r_0\right)$$

$$P r_0 + L\left(r_0\right) < PI + L(A) \ \Rightarrow$$

$$C_O + P R_0 + L\left(R_0\right) < PA + L(A)$$

$$C_O + P\left(R_0 - A\right) + L\left(R_0\right) < L(A)$$

The right hand side of the lat inequality is the cost of the inventory
system when no order is placed which was previously calculated. The
left hand side is the cost when an order is placed with size $R_o - A$.
Therefore if we place an order our cost decreases .

Optimal strategy for single period Model having order cost

If A ≥r_0 place no order;
where
    A is the inventory at the beginning of the period,
    r_0 is the smallest root of the following equation solved for r_0:

$$P\, r_0 + L(\,r_0)-C_O - PR_0 - L(R_0) = 0, \qquad\qquad (5\text{-}12)$$

R_0 is the point where the function $K(R) + PA\ or\ K'(R) = PR + L(R)$ is minimized.; it is obtained from:

$$F(R_0) - \frac{V+\pi-P}{V+\pi+H} = 0 \qquad\qquad (5\text{-}13),$$

$$L(R_0) = HR_0 + (V + \pi + H)\int_R^\infty (x - R_0)f(x)dx \quad (5-14)$$

$f(x)$ is the probability density function of the demand.

If A <r₀, place an order of size
$$Q= R_0\text{-}A \qquad\qquad (5\text{-}15)$$
This   is a kind of   the so-called continuous review   policy denoted by (r, Q) which is frequently used in industry.

Example 5-5

An item is sold in a single period. The unit purchase and selling prices are $12 and $20 respectively. Shortage cause no cost except lost profit. The unsold units at the end of the period have no cost and no revenue. The demand for the period is uniformly distributed over (0,100). The initial inventory is 5 useable units. Find the optimal order strategy if the fixed order cost is a) C_O=160   b) C_O=200.

 Solution

We have to find R_0 and r_0 :

$$F(R_0) = \frac{V + \pi - P}{V + \pi + H}$$

$\pi = 0$ since no shortage cost is incurred.

$V = 20, P = 12,\ \mathrm{L} = 0, H' = 0$

$H = H' - L = 0 - 0 = 0$

$$F(R_0) = \frac{V + \pi - P}{V + \pi + H} = \frac{20 + 0 - 12}{20 + 0 + 0} \Rightarrow F(R_0) = 0.4$$

Since the demand is uniformly distributed on the interval [0 100] then

$$F(R_0) = \frac{R_0 - 0}{100 - 0} \Longrightarrow 0.4 = \frac{R_0}{100} \Longrightarrow R_0 = 40.$$

To find r_0 for part (a)   we have to solve the following equation for r_0:

$$P\, r_0 + L(\, r_0) = C_O + PR_0 + L(R_0)$$

$$R_0 = 40 \quad P = 12 \quad C_O = 160 \quad , L(R_0):$$

$$L(R) = HR + (V + \pi + H) \int_R^\infty (x - R)f(x)dx$$

The probability distribution function of a uniformly distributed demand is $\frac{1}{100}$ ove {0 100)

$$L(R_0) = (0)(40) + (20 + 0 + 0) \int_{40}^{100} (x - 40)\left(\frac{1}{100}\right) dx = 360$$

To derive r_0 we substitue $R = \, r_0 \ in \ L(R):$

$$L(r_0) = (0)(r_0) + (20 + 0 + 0) \int_{r_0}^{100} (x - r_0)\left(\frac{1}{100}\right) dx$$

$$P\, r_0 + L(\, r_0) = C_O + PR_0 + L(R_0) \Longrightarrow$$

$$12r_0 + 20 \int_{r_0}^{100} (x - r_0)\left(\frac{1}{100}\right) dx = 160 + 4 * 12 + 360 \Rightarrow$$

$$12\, r_0 + 1000 - 20 r_0 + \frac{r_0^2}{10} = 1000 \Longrightarrow r_0 = 0 \,, 80$$

We have to choose the smallest root i.e. $r_0 = 0$.

$r_0 < A = 5 \Longrightarrow$ no order is placed.

Solution of part b is similar to part a:

$$P\, r_0 + L(\,r_0) = C_O + P R_0 + L(R_0) \Longrightarrow$$

$$12 r_0 + 20 \int_{r_0}^{100} (x - r_0)\left(\frac{1}{100}\right) dx = 100 + 40 * 12 + 360 \Rightarrow$$

$$12 r_0 + (1000 - 20 r_0 + \frac{r_0^2}{10}) = 100 + 480 + 360 \Rightarrow \frac{r_0^2}{10} - 8 r_0 + 70 = 0$$

MATLAB $\Rightarrow$ r$_0$ $= roots([0.1 \quad -8 \quad 70]) \Rightarrow r_0 = 8.4, \ 71.68$

The smallest root is 8.4

Since $A < r_0 = 8.4$, an order of size $Q = 40 - A = 35$ has to be placed. End of example ▲

Exercises

1.(Tersine, 1994 page 327)

The Parker Flower shop promises its customers to deliver within 4 hours on all flower orders. All flowers are purchased on the previous day and delivered to Parker by 8 a.m. in the next morning. Parker's daily demand for roses is as follows:

Dozens of roses	7	8	9	10
Probability	0.1	0.2	.4	0.3

Parker purchases roses for $ 10 per dozen and sells them at $ 30 All unsold roses are donated to a local hospital. How many dozens of roses should parker order each evening to maximize its profits? What is the optimum expected profit?

2.(Tersine,1994 page 228)

You are having a new furnace installed. The dealer offers to sell you spare fuel pumps at $20 each if you buy them during installation. The pumps sell for $50 retail. Manufacturer records indicate the following probability of fuel pump failures during the furnace's lifetime.

Failures	0	1	2	3	4
Probability %	10	30	40	10	10

Ignoring installation and holding cost, how many spare fuel pumps should be purchased during installation? What is the expected purchase cost?

Hint:

   Solve the problem with Single period model; treat the failures as demand and substitute $P = 20, V = 50$.

3.(Extracted from Peterson &Silver,1991 page 418)

   A local vendor of newspapers feels that dissatisfaction of customers leads to future lost sales. In fact, he feels that the average demand (μ) for a particular newspaper is related to the service level(p) as follows: $\mu = 100 + p$.   The demand is normally distributed and the standard deviation (per period) s equal to 200, independent of the service level. The ordering cost is negligible and the other (possibly) relevant factors are:

   Cost per paper (for vendor)=P=$0.07

   Selling price per paper=V=$0.15

   Salvage value per paper=L=$0.02, $H' = 0$

If shortage has no cost except lost profit,

   a)What is the optimal value for maximum inventory(R)

   b)Solve Part a if the unit shortage cost is $2.

   c) What average profit is the vendor losing if he proceeds as in (a) instead of as in (b)?

   Hint: estimate of service level $= \hat{p} = \Pr(X \leq R)$.

4. In a single period model similar to that of Example 5-5 The following data is available :

The setup cost	$C_O = 5$
The demand is uniformly distributed over [0 100]	$f(x) = 0.01$
The actual holding per unit remained at the end of the period	$H = 3$
The production cost per unit	P=1
Unit shortage cost (lost profit not included)	$\pi = 2$
The unit selling price	V=5

Find r_0 و R_0.

Ans : $r_0 = 5.9, R_0 = 60$

5. Given the following data in a single-period model, Find r_0 &R_0. What is the optimal strategy,

The ordering cost	$C_O = 800$
The demand is exponentially distributed with mean 10000 units	$f(x) = 0.01$
The actual holding cost per unit unsold at the end of the period	$H = -9$
The purchase cost per unit	P=20
Unit shortage cost (lost profit not included)	$\pi =0$
The unit selling price	V=45

Ans : $r_0 = 10674, R_0 = 11856$

   If $A < r_0$ Place an order of size $R_0 - A$ to minimize cost.

   If $A > r_0$ No order is placed.

5.3 Probabilistic Continuous and Periodic review models- introduction

A continuous review system, which is sometimes called a fixed order size system, is one in which inventory is monitored at a continuous rate and whenever the inventory reaches a value such as *r* an order of size say Q is placed. The symbol for this model is FOS   and (r Q). In periodic review model stock is reviewed at fixed and specific intervals of time (say every T days), and an order is placed with the quantity necessary to achieve the desired maximum inven-tory denoted here by R. The later model is denoted by FOI=(R,Q). Some of the applications of these 2 models are:

   -FOS is advised for contingency stocks as demand is
   usually highly unpredictable and also may be used for
   expensive items and those which need precise control.

- FOI may be applied to items with more regular demand.

- whenever several items have to be ordered from the same provider, FOI system is advised.

Note that:

   -Shortage probability in FOI policy is less than that in FOS.

   At a fixed service level (p=1- shortage probability) the safety stock, the average shortage level and the average inventory level in FOI. policy is more than those in FOS and also the shortage cost.

   -Classic EOQ model is both (R,T) and (r ,Q).

   -Due to more safety stock, the holding cost in FOI policy is more than that in FOS policy.

In (R,T) policy the order quantity is more than that in (r Q); therefore when the ordering cost (C_o) is high it is advised to use (R,T) policy and when C_o is low, (r Q) is advised.

- In (r Q) policy ,the order quantity is fixed and the cycle time (T) is variable while in (R,T) policy the cycle time is fixed and the order quantity is variable.

Before giving more details about the two probabilistic models, some definitions are reminded below.

Definitions

5-3-1 Safety stock

Safety stock is an extra quantity held in the inventory by a retailer or a manufacturer to cope with unexpected increase of demand and the variation of lead time.

5-3-2 Service Level

The service level represents the desired probability of not getting a stock-out during the lead time(TL) in other words the probability that the amount of stock during the TL is sufficient to meet expected demand. The more this probability which is dented by p, the less the probability of stockout, which equals 1-p and sometimes called risk level.

At a fixed, the following values in FOS policy are less those in FOI system: the average shortage level, the holding cost and the shortage cost.

Theorem 5-1:The relationships for mean and variance of the lead time demand

If in FOS policy, the demand(D) and lead time(T_L) are independent random variables with mean and variance (μ_D

$,\sigma_D^2),(\mu_L \ ,\sigma_L^2)$ respectively then, regardless of their statistical distributions, the following relationship hold:

$$Var(D_L) = \mu_D^2 \sigma_L^2 + \mu_L \sigma_D^2$$

Furthermore if   D and T_L are independent or at least un correlated, then $E(D_L) = \mu_D \mu_L$.

Proof of the first relationship

Let divide D_L the consumption during the lead time(L), into L elements $D_i, i = 1,2,\dots.L$, with mean $E(D_i) = \mu_D$ and variance $Var(D_i) = \sigma_D^2$. Then $X = D_L = \sum_{i=1}^{L} D_i$. If the lead time is a random variable with mean μ_L = E(L) & variance $Var(L) = \sigma_L^2$ then assuming $D_i's$ are independent and using the equality Var(X) = E(Var(X|Y) + Var[E(X|Y)]) we could write:

$$Var(X) = Var\left(\sum_{i=1}^{L} D_i\right) = E\left[Var\left(\sum_{i=1}^{n} D_i|n = L\right)\right] + Var\left[E\left(\sum_{i=1}^{n} D_i|n = L\right)\right]$$
$$= E(L\sigma_D^2) + Var(L\mu_D) \implies$$

Now assuming the demand (D)and the lead time(L $=T_L$) are independent

$$Var(D_L) = \mu_L \ \sigma_D^2 + \mu_D^2 \ \sigma_L^2 \quad \text{or} \quad \sigma_{D_L} = \sqrt{\mu_D^2 \sigma_L^2 + \mu_L \sigma_D^2} \ .$$

End of proof ■.

Note that

- the above relationship is valid regardless of the statistical distributions of the demand and the lead time.

-when either the demand (D) or the lead time (L) is not random variable zero is substituted for the its standard deviation.

Theorem 5-2:

If in FOI policy, the demand(D) and lead time($L=T_L$) are independent random variables with mean and variance (μ_D ,σ_D^2),(μ_L,σ_L^2)   respectively then, regardless of their statistical distributions,   the following relationships are hold for the variance and mean of the quantity consumed during $T + L$:

$$Var(D_{L+T}) = \mu_{T+L}\sigma_D^2 + \mu_D^2 \; \sigma_{T+L}^2$$

The proof is similar to that presented in Theorem 5-1.

Furthermore if D and$L =T_L$ are independent or at least uncorrelated, then $E(D_{L+T}) = (\mu_D)(\mu_{L+T})$.

End of theorem ■

Note the above two relationship are valid, regardless the type of the statistical distributions of D and L+T.

5.4 Continuous Review Inventory Model

or (r, Q) policy or FOS system

This section deals with continuous review inventory systems which is denoted by (r,Q) or FOS.

Symbols
$b(x)$ Bereft function in each cycle
$\bar{b}(r)$ Average shortage in each cycle
$\bar{B}(r)$ Average shortage per year
$X =$ The demand(consumption) during T_L
D_L
$E(D_L)$ Average consumption) during T_L
$f_{D_L}(x)$ pdf of consumption during T_L
$G_U(k)$ Normal loss integral
m Number of cycles per year
N_b Average number of cycles with shortage per year
p Service level, probability of lack of shortage

P	Purchase price
r	Reorder point
$r*$	Optimal reorder point
T	Cycle time
T_b	The mean time between "2 successive cycles with shortage"
V	Selling price
$1-p$	Shortage probability in each cycle
π	Total shortage cost per unit
π_0	unit shortage cost (lost profit not included)

In continuous review policy denoted by (r,Q) or FOS, whenever the inventory reaches say r m an order or quantity Q is placed.

5.3.1 Order quantity in (r,Q) system

In continuous review system, the order quantity might be determined based on the experience and judgment or from Wilson-Harris formula $\mathbf{Q} = \sqrt{\dfrac{2DC_0}{C_h}}$. If annual demand (D) is a random variable, its average i.e. E(D) replaces D in the formula. Take note not to confuse E(D) with $E(D_L)$, the average demand during the lead time.

5-3.2 Safety stock in (r,Q) system

Let D_L denote the demand during the lead time and let r denote the reorder point; stockout occurs when $D_L > r$.   If the reorder point coincides the average demand during the lead time i.e. $r=E(D_L)$ and no safety stock is available, after the time T_L has expired and just before arrival of the quantity ordered, it is expected that 50% of the times we do encounter stockout and 50% do not i.e. p= $\mathbf{Pr}(\mathbf{D_L} < r)$=50% if the consumption during T_L is normally distributed.

if the consumption during TL is exponentially distributed then p = $\Pr(D_L < r = \theta) = 1 - e^{-\frac{\theta}{\theta}} = 0.633, \Pr(D_L > r = \theta) = 0.367$

To reduce the risk of shortage or to increase the safety level (p) an amount known as safety stock(SS) is added to E(D_L), Therefore in this model

Reorder point	r= E(DL)+SS	(5-16)
Safety stock	SS=r-E(D_L)	(5-17)
Max inventory	= r+Q (if T_L =0)	(5-18)

Furthermore, the average holding cost equals $C_h \times \left(\frac{Q}{2} + SS\right)$. The maximum demand that could be satisfied during T_L equals r. Therefore SS is an extra amount of inventory as well as E(D_L) kept in reserve to make sure we satisfy the maximum demand and service level(p) and do not run out of stock i.e. SS=r-E(D_L). Let F_D denote the cumulative distribution function of consumption during T_L and assume the service level is p :

$p = \text{Pr}(\text{no stockout during } T_L)$

$$p = \text{Pr}(D_L \leq r) \qquad (5\text{-}19)$$

Therefore

$$F_{D_L}(r) = p \quad \text{ or } 1\text{-}p = Pr(D_L > r) = 1 - F_{D_L}(r),$$

$$\text{p=} F_{D_L}(r) \rightarrow r = F_{D_L}^{-1}(p) \qquad (5\text{-}20)$$

$$SS = r - E(D_L) - (r, Q) \text{ model} \qquad (5\text{-}21)$$

Note :

Make sure that the variables have the same dimension when being substituted in the relationships. For example if the unit time given for one variable is month and for the other one is year, change both to year or both to month.

Example 5-6

In an FOS policy the average consumption during the one-week lead time is 45 and the desired service level is p=95% . Using the following figure find the reorder point and the necessary safety stock.
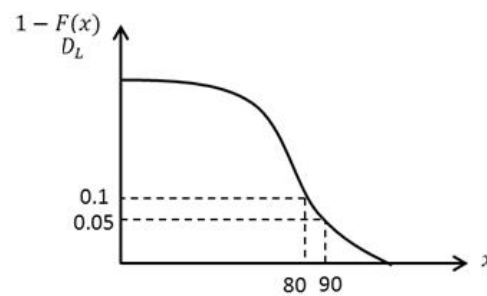
Solution

$$F_{D_L}(r) = p = 0.95 \;,\; 1 - F_{D_L}(r) = 0.05 \xrightarrow{\text{From figure}} r = 90$$

$$SS = r - E(D_L) = 90 - 45 = 45 \blacktriangle$$

Example 5-7

The demand for a product is uniformly distributed over [50 150]. Using a service level 90% find the reorder and the safety stock.

Solution

If X is uniformlly dustributed over [a b]then $F_X(x) = \dfrac{x-a}{b-a}$.

$$Pr(D_L \leq r) = 0.9 = F_{D_L}(r)$$

$$D_L \sim U(50,150) \quad \Rightarrow \quad F_{D_L}(r) = \frac{r-50}{100} = 0.9 \quad \Rightarrow r = 140 \; units$$

$$E(D_L) = \frac{50+150}{2} = 100, \quad SS = r - E(D_L) = 140 - 100 = 40 \blacktriangle$$

Example 5-8

A small shop uses FOS =(r,Q) policy. The demand for a product   during the lead time is approximately Poisson with mean 2 units. With a risk of 2% find the reorder point and the safety stock. Furthermore if the annual demand is uniformly distributed over [0   10] and $C_h = 4$ per year and the ordering cost is $80 per order. Find the optimal order quantity.

Solution

$$1 - p = 0.02, \qquad\qquad\qquad\qquad E(D_L) = 2$$
$$\Pr(D_L \le r) = 0.98 \quad , \lambda = 2$$

Using MATLAB command   r=Poissinv(0.98,2)

or Poisson Table at the end of the book results in $r = 5$.

$$S.S = r - E(D_L) = 5 - 2 = 3$$

$$E(D) = \frac{0+10}{2} = 5 \qquad\qquad Q^* = \sqrt{\frac{2 \times 5 \times 80}{4}} \quad \Longrightarrow \quad Q^* \cong 14 \blacktriangle$$

Example 5-9

Using the data in the table and service level of 87.5% related to an FOS policy, find the safety stock.

i	D_{L_i}	$P_{(D_{L_i})}$	$F_{(D_{L_i})}$
1	30	0.025	0.025
2	40	0.1	0.125
3	50	0.2	0325
4	60	0.35	0.675
5	70	0.2	0.875
6	80	0.1	0.975
7	90	0.025	1.00

Solution

$$\Pr(D_L \leq r) = 87.5\% \implies r = 70$$

$$SS = r - E(D_L) = 70 - \sum_{i=1}^{7} D_{L_i} P_{(D_{L_i})} = 70 - 60 = 10$$

If the service level is not found in the table the greater service level in the table should be chosen.▲

Example 5-10

If the shortage probability in an FOS policy is 30% and the probability of the demand during the lead time (D_L) is as shown in the following table, find the safety stock.

D_L	probability	Cum. Probability
80	0.3	0.3
85	0.2	0.5
90	0.05	0.55
95	0.2	0.75
100	0.15	0.9
105	0.1	1

Solution

$$S.S = r - E(D_L)$$

$$E(D_L) = (80)(0.3) + \cdots + (105)(0.1) = 90$$

Shortage probability $=0.3 \implies p = 0.7$

$$\Pr(D_L \leq r) = 0.7 \implies r = 95$$

$$r = E(D_L) + SS \implies SS = 5$$

Example 5-11
(Asadzade et al ,2006, page 245)

The daily demand for a product is deterministic and equals 20 units. The policy used is FOS and the probability distribution of the lead time follows the data given in the following table. Find the safety stock for a service level of 0.85

T_L or L	1	2	3	4	5	6
Probability	0.05	0.1	0.15	0.35	0.25	0.1

Solution

Since $D_L = D \times T_L$ then we have the following probabilities:

$D_L = D \times L$	probability	Cumulative probability
20	0.1	0.1
40	0.25	0.35
60	0.35	0.7
80	0.15	0.85
100	0.1	0.95
120	0.05	1

$p = \Pr(D_L \leq r) = 0.85 \implies r = 80$. That is whenever the inventory level reaches 80 units an order is placed.

$$SS = r - E(D_L)$$

$$E(D_L) = 20 \times 0.1 + \cdots + 120 \times 0.05 = 61$$

$$\implies SS = 80 - 61 = 19 \blacktriangle$$

Example 5-12

The demand during the lead time in a FOS policy is uniformly distributed over[0 100], the order quantity is 40 units, the average demand is 400 units per year and the service level is 90% . Find SS.

Solution

$$F_{D_L}(x) = \Pr(D_L \leq x) = \begin{cases} \dfrac{x - 0}{100 - 0} & 0 \leq x \leq 100 \\ 0 & \text{otherwise} \end{cases}$$

$$Pr(D_L \le r) = 0.9 \implies \frac{r}{100} = 0.9 \implies r = 90$$

$$E(D_L) = \frac{0+100}{2} = 50 , \quad r = E(D_L) + S.S \implies S.S = 40 \ \blacktriangle$$

5-4-4 Reorder point and safety stock for normally distributed D_L in FOS Policy

If D_L, the demand during the lead time in a FOS policy, is normally distributed with mean and standard deviation μ_{D_L} & σ_{D_L} then

$$p = Pr(D_L \le r) \ \text{ or } Pr(D_L > r) = 1\text{- p} \implies Pr\left(Z > \frac{r-\mu_{D_L}}{\sigma_{D_L}}\right) = 1\text{-p} \implies \frac{r-\mu_{D_L}}{\sigma_{D_L}} = Z_{1-p} = k.$$

$$r = E(D_L) + Z_{1-P}\,\sigma_{D_L} \qquad \textbf{D}_\textbf{L} \ \text{نرمال}: normal \qquad (5\text{-}22)$$

Since $r = E(D_L) + SS$ then if D_L is normally distrusted :
$$SS = Z_{1-P}\,\sigma_{D_L} \qquad (\,5\text{-}23)$$

norminv(p) gives the values of Z_{1-P} in MATLAB. Also the following table gives the value of Z_{1-P} for some values of service level *p*

p (%	50	55	60	65	70	75	80	82	84	86	88
Z_{1-p}	0	0.12	0.253	0.385	0.524	0.675	0.842	0.915	0.995	0.108	1.175
p (%	90	92	94	95	96	97	98	99	99.5	99.9	99.99
Z_{1-p}	1.282	1.405	1.555	1.645	1.751	1.888	2.054	2.326	2.576	3.09	3.719

In what follows we would like to deal with the cases in FOS policy where the service level p and the distribution of demand and/or that of T_L are known to determine reorder point and safety stock.

5.4.5 Determining safety stock and reorder point in (r,Q) system when demand and/or lead time is probabilistic

The aim of this section is to distinguish the cases in which the demand per unit time or the lead time or both are probabilistic in

order to calculate their mean and standard deviation and then to calculate the reorder point and safety stock in an FOS system.

Again it is reminded not to use demand per unit time(D) whose mean and variance are $\mu_D = E(D)$ & $Var(D) = \sigma_D^2$, instead of the demand during the lead time (D_L) whose mean and variance are denoted by $E(D_L)$ & $Var(D_L)$.

To calculate the mean and variance of D_L, assuming D and T_L are independent, consider the 4 following cases:

5-4-5-1: Case 1: Demand and lead time (D &L=T$_L$) probabilistic and independent

Suppose the demand (per year, month…) D is a random variable with $E(D) = \mu_D$ & $Var(D) = \sigma_D^2$ and the lead time (L=T$_L$) is also probabilistic with mean μ_L & variance $Var(L) = \sigma_L^2$. If these 2 variables are independent, then

$$E(D_L) = \mu_D \mu_L \qquad \text{(5-24-1)}$$

And according to theorem 1-5:

$$\sigma_{D_L} = \sqrt{\mu_D^2 \sigma_L^2 + \mu_L \sigma_D^2}. \qquad \text{(5-24-2)}$$

In the special case in which the demand during the lead time(D_L) is normally distributed, given service level (p):

$$p = \Pr(D_L \leq r) = \Pr\left(Z \leq \frac{r - \mu_D \mu_L}{\sigma_{D_L}}\right),$$

Since $\frac{r - \mu_D \mu_L}{\sigma_{D_L}} = Z_{1-p}$ then

$$r = \mu_D \mu_L + Z_{1-p} \sigma_{D_L} \qquad (5 - 25 - 1)$$
$$SS = r - \mu_D \mu_L = Z_{1-p} \sigma_{D_L} \qquad (5 - 25 - 2)$$

Where Z_{1-p} is a number related to standard normal distribution with probability greater than 1-p: $\Pr(Z > Z_{1-p}) = 1 - p$.

Example 5-13

The annual demand for a product has a mean of 3600 tons and a standard deviation of 30 tons. The lead time is normally distributed with mean 15 days and standard deviation 1 day.   If there   are 360 working days in a year, what is the mean and standard deviation of the demand during the lead time?

Solution

The mean of the lead time is $\frac{15}{360}$ in year and the standard deviation is $\frac{1}{360}$ yr; then

$$E(D_L) = \mu_D\mu_L = 3600 \times \frac{15}{360} = 150$$

$$\sigma_{D_L} = \sqrt{\mu_D^2\sigma_L^2 + \mu_L\sigma_D^2} = \sqrt{(3600)^2 * \left(\frac{1}{360}\right)^2 + \left(\frac{15}{360}\right) * (30)^2} = 11.73$$

End of example▲

Note that since in the unit conversion of some parameters such as σ_{D_L} we could write;

$$\sigma_{D_L} = \sqrt{(3600)^2 * \left(\frac{1}{360}\right)^2 + \left(\frac{15}{360}\right) * (30)^2} =$$

$$\sqrt{\left(\frac{3600}{360}\right)^2 \times (1)^2 + (15) \times \left(\frac{30}{\sqrt{360}}\right)^2},$$

Then the following point has to be mentioned.

5-4-5-1-1 Some points on the unit conversion of demand's variance and standard deviation

When the variance of demand i.e. $\text{Var}(D)$ is expressed in $(\frac{\text{units}^2}{\text{unit time}})$ and σ_D in $(\frac{\text{unit}}{\sqrt{\text{unit time}}})$ then to convert the standard deviation of monthly demand to that of yearly demand, multiply it by $\sqrt{12}$, because:

$$\sigma_D = a \ \left(in:\frac{\text{units}}{\sqrt{\text{month}}}\right) = a \ \frac{units}{\sqrt{\text{year} \times \frac{1}{12}}} = \sqrt{12} \times a \ \left(in:\frac{\text{units}}{\sqrt{\text{year}}}\right).$$

e.g. $\sigma_D = 10$ units/month is equivalent to $\sigma_D = 10\sqrt{12}$ units per year.

To covert the variance of monthly demand to that of yearly demand, multiply it by 12 ; also to convert the variance of daily demand to that of yearly demand, multiply it by N= no. of working days in a year. To covert the standard deviation of daily demand to that of annual demand, multiply it by $\sqrt{N}$.

To covert the standard deviation of annual demand to that of daily or monthly demand, divide it by $\sqrt{N} \ or \ \sqrt{12}$ respectively.

For calculating σ_{D_L}, it is easier to state the mean and standard deviation of the lead time(L) in terms of the time units given for the demand D. For example if we have annual demand and the mean and standard deviation of L is given in units/(day or month); divide the mean and the standard deviation by 12 or N.

Example 5-14

A warehouse uses an FOS policy with the service level p=%97. The monthly demand is estimated to be 300 tons on average with a standard deviation of 8.67. The unit price per ton of the product is $8000, the ordering cost is $3000 per order, the insurance+ tax _ money blockade +… is calculated in interest rate of 20%. The lead time is normally distributed with mean 15 days and standard deviation of 1 day. D and T_L are independent and D_L is normally distributed Find a)the reorder point and SS b)the quantity for each order. There are 360 working days and 12 30-day month in a year.

Solution

$$SS = Z_{1-p}\sigma_{D_L} = Z_{0.03}\sigma_{D_L}$$

$$\sigma_{D_L} = \sqrt{\mu_D^2\sigma_L^2 + \mu_L\sigma_D^2} == \sqrt{300^2 * \left(\frac{1}{30}\right)^2 + \left(\frac{15}{30}\right) * (8.66)^2} = 11.73$$

Based on Section 5-4-5-1-1 we have the following unit conversion:

$\sigma_D = 8.67 \times \sqrt{12} = 30$ tons   per yr

$\mu_D = 300 \times 12 = 3600$ tons/yr .

Since according to theorem 5-1

$$\sigma_{D_L} = \sqrt{\mu_D^2 \sigma_L^2 + \mu_L \sigma_D^2}$$

$$\sigma_{D_L} = \sqrt{(300 \times 12)^2 * \left(\frac{1}{360}\right)^2 + \left(\frac{15}{360}\right) * (30)^2} = 11.73$$

$$E(D_L) = E(D)E(T_L) = 3600 \times \frac{15}{360} = 150.$$

The variable D_L here is the product of 2 normally distributed variables i.e. D   and T_L. If the   distribution of D_L be approximated with $D_L \sim N(150, 11.73)$ then:

$$S.S = Z_{0.03} \times \sigma_{D_L} = 1.88 \times 11.73 \cong 22$$

$r = E(D_L) + SS = 172$

Furthermore the following value is proposed for the order quantity:

$$Q = \sqrt{\frac{2\mu_D \times C_O}{C_h}} = \sqrt{\frac{2 \times 3600 \times 3000}{0.2 \times 800}} = 367$$

That is whenever the inventory reaches r=172, place an order of quantity 372 units. ▲

5-4-5-2   Case 2: Demand(D) Deterministic but lead time (L=T_L) probabilistic

In this case:

$$\mu_D = E(D) = D \qquad , \sigma_D = 0$$
$$D_L = DT_L$$
$$E(D_L) = \mu_D \times \mu_L = D\mu_L \quad (5-26)$$

$$\sigma_{D_L} = \sqrt{\mu_D^2 \sigma_L^2 + \mu_L \sigma_D^2} = \sqrt{D^2 \sigma_L^2 + \mu_L \times 0} \Rightarrow$$

$$\sigma_{D_L} = D\sigma_L \qquad\qquad (5-27)$$

In the special case where T_L has the normal distribution $N(\mu_L, \sigma_L)$

We have:

$$D_L = D \times T_L \sim N(D\mu_L, D\sigma_L).$$

Calculation of reorder point:

$$r = E(D_L) + SS$$

$$Z_{1-P} = \frac{r - E(D_L)}{\sigma_{D_L}}$$

$$r = E(D_L) + Z_{1-p} \times \sigma_{D_L}$$

Then

$$r = D\mu_L + Z_{1-P} \times D \times \sigma_L \quad (5-28)$$

and

$$SS = r - E(D_L) = Z_{1-P}\, D\sigma_L . \qquad (5-29)$$

Example 5-15

A shop uses an FOS policy with the service level p=%97.5.

The annual dement for a product is 1000 units and the lead time is normally distributed mean 1 month and standard deviation 0.2 month. Find reorder point and the required safety stock.

Solution

$$T_L \sim N(\mu_L = 1\,month\,, \sigma_L = 0.2\,month)$$

$$\sigma_{D_L} = D\sigma_L = 1000 \times \frac{0.2}{12} = 16.67 \quad or = \frac{1000}{12} \times \frac{2}{10} = 16.67$$

$$SS = Z_{1-p}\, \sigma_{D_L} = Z_{0.025} \times 16.67, Z_{0.025} = norminv(1 - 0.025) = 1.96$$

$$SS = 1.96 \times 16.67 = 32.66$$

$$r = E(D_L) + \ SS \ = r = 1000 \times \frac{1}{12} + 32.66 = 116\blacktriangle$$

5-4-5-3 Case 3: Demand(D) probabilistic but lead time deterministic

If   the (monthly , annual,…)demand is a random variable with mean μ_D and standard deviation σ_D but the lead time is either fixed or has a small variations compared to its mean then

$$\mu_L = E(T_L) = T_L \qquad , \qquad \sigma_L \cong 0$$

$$D_L = DT_L$$

$$E(D_L) = T_L\mu_D \qquad (5-30)$$

$$\sigma_{D_L} = \sqrt{\mu_D^2\sigma_L^2 + \mu_L^2\sigma_D^2} = \sqrt{\mu_D^2 \times 0 + T_L\sigma_D^2} \implies$$

$$\sigma_{D_L} = \sigma_D\sqrt{T_L} \qquad (5-31)$$

In special case in which D~$N(\mu_D , \sigma_D)$ andT_L is fixed then

$$D_L \sim N(\mu_{D_L} = T_L \mu_D , \sigma_{D_L} = \sigma_D \sqrt{T_L})$$
$$r = E(D_L) + \text{ SS}$$
$$\text{SS} = Z_{1-P} \sigma_D \sqrt{T_L} . \qquad\qquad (5\text{-}32)$$

Example 5-16

A distributer uses FOS policy with a service level of p= 97.5% for a product whose annual demand is normally distributed with mean 8000 and standard deviation of 1000. The lead time is approximately one half of a month. Find the safety stock and the reorder point.

Solution

The problem satisfies the conditions Eq. 5-32 i.e.

$$\text{SS} = Z_{1-P} \sigma_D \sqrt{T_L} = Z_{1-0.975} \times 1000 \sqrt{0.5/12}$$
$$= 1.96 \times 204.12 = 400$$
$$r = E(D_L) + \text{ SS} \qquad E(D_L) = E(D)E(T_L) = \frac{8000}{12} \times \frac{1}{2} = 333$$
$$r = 333 + 400 = 733 \blacktriangle$$

5-4-5-4 Case 4: Both demand and lead time deterministic

When both D and T_L are deterministic
$$E(D_L) = E(DT_L) = DT_L \qquad\qquad \sigma_{D_L} = 0$$
$$r = E(D_L) + \text{ SS}$$
In chapter 2 we saw if both D and T_L are fixed:
r=ROP= DT_L, then   SS=0 .
In fact we have a classic EOQ model

5-4-6 On Lost sale and stockout in FOS systems

In continuous review system, shortage occurs when the demand during the lead time exceeds the reorder point. Given a service level of p in a continuous review system, the shortage probability equals $\Pr(D_L > r) = 1 - p$.   In fact in every n cycles, the   ratio of "number of cycles encountered   with stockout" to the total number of the cycles i.e. n, equal to 1-p; e.g. if p=0.88. On the average there are 12 cycles (out of 100 cycles) in which a lost sale or shortage occurs.

Let b(x) denote the shortage function in each cycle of our FOS system then:

$$b(x) = \begin{cases} 0 & x \le r \\ x - r & x > r \end{cases} \quad \text{(5-33)}$$

where

x = the demand during the lead time

The average of shortage function:

If the demand is continuous with density function of $f_{D_L}(x)$ then

$$E[b(x)]=\int_{-\infty}^{\infty} b(x)f_{D_L}(x)dx = \int_{-\infty}^{r} 0 f_{D_L}(x)dx + \int_{r}^{\infty}(x-r)f_{D_L}(x)dx$$

Since this value depends on r, The average of the shortage function is denoted by $\bar{b}(r)$, then

$$\bar{b}(r) = \int_{r}^{\infty}(x-r)f_{D_L}(x)dx \qquad (\ 5\text{-}26)$$

Where

$f_{D_L}(x)$ is the pdf of the demand during the lead time

$\bar{b}(r)$ is the average shortage during each cycle(b stands for bereft).

if the demand is discrete with probability function of $p_{D_L}(x)$ then

$$\bar{b}(r) = \sum_{x>r}(x-r)p_{D_L}(x). \quad \text{(5-27)}$$

If the order quantity in each cycle is Q and the annual demand for the product I is D then annual average shortage denoted by $\bar{B}(r)\ is$:

Annual average shortage:

$$\bar{B}(r) = \frac{D}{Q}\bar{b}(r) = \frac{\bar{b}(r)}{T} = m\bar{b}(r), \quad \text{(5-28)}$$

where $m = \frac{1}{T}$ is the number orders per year.

Therefore as much as $\frac{\bar{B}(r)}{D} \times 100$ percent of the annual demand the inventory system encounters lost sale or shortage. If p is given as the service level , the shortage probability is 1-p and according the concept of probability, the average of the annual number of the cycles in which shortage occurs is:

$$N_b = \frac{D}{Q}(1-p) = m(1-p) \quad \text{(5- 29)}$$

Then on the average every $T_b = \frac{1}{N_b}$ years a shortage occurs.

The above results are summarized in the following table:

Type of demand distribution	$E(D_L > r)$ьₑ $\bar{b}(r)$		(N_b)
continuous	$\bar{b}(r) =$ $\int_r^\infty (x-r)f_{D_L}(x)dx$ (5- 26)	$\bar{B}(r)=$ $m\bar{b}(r) = \frac{\bar{b}(r)}{T}$ (5- 28)	$N_b=$ $\frac{D}{Q}(1-p)$ $= m(1-p)$
discrete	$\bar{b}(r) =$ $\sum_{x>r}(x-r)p_{D_L}(x)$ (5- 27)		(5- 29)
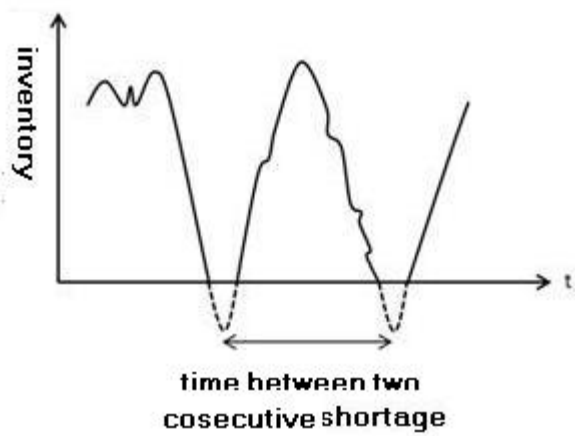
Fig. 5-9 The time between 2 consecutive shortages in an FOS system

Figure 5-9 illustrates the time between two consecutive shortages. The mean of this time , denoted by T_b ,is derivable from:

$$T_b = \frac{1}{N_b} = \frac{Q}{D(1-p)} \qquad (5-30)$$

There fore the shortage probability equals:

$$T_b = \frac{1}{N_b} = \frac{Q}{D(1-p)} \qquad (5-31)$$

therefore

$$\text{shotage probability } | = 1 - p = \frac{N_b}{\frac{D}{Q}} \qquad (5\text{-}31)$$

The service level i.e. the probability of the lack of shortage is estimated from

$$\hat{p} = 1 - \frac{QN_b}{D} \qquad (5\text{-}32)$$

Note that since $Q = \sqrt{\frac{2\mu_D C_O}{C_h}}$ then:

-An increase in ordering cost (C_o) *will* increase Q and will decrease average annual shortage in FOS pliocy i.e. $\bar{B}(r)$

--A decrease in holding cost (C_h) will decrease Q and will increase $\bar{B}(r)$.

What will be the effect of an increase in demand on $\bar{B}(r)$.?

Example 5-17

An FOS inventory system reports   2 shortages per year on the average. The quantity per order is 800 and the average annual demand is 8000. Estimate the service level ?

Solution

$$\hat{p} = 1 - \frac{QN_b}{E(D)} = 1 - \frac{800 \times 2}{8000} = 0.80 \text{ End of example } \blacktriangle$$

Example 5-18

  The demand during the lead time in an FOS system is uniformly distributed over [0 100]. If the survive level is 90% and order of 40 units are placed, what is the ratio of " annual average shortage " to " annual demand" ?

Solution

$$\frac{\bar{B}(r)}{D} = ?$$

$$\bar{B}(r) = \frac{D}{Q}\bar{b}(r), \quad \bar{b}(r) = \int_r^\infty (x-r)f_{D_L}(x)dx$$

$$f(x) = \frac{1}{100} \quad 0 \le x \le 100 \quad F(x) = \frac{r-0}{100-0}$$

$$Pr(D_L \le r) = p = 0.9 \quad \frac{r-0}{100-0} = 0.9 \quad \Rightarrow \quad r = 90$$

$$\bar{b}(r) = \int_{90}^{100}(x-90)\frac{1}{100}dx = 0.5$$

$$\frac{\bar{B}(r)}{D} = \frac{\bar{b}(r)}{Q} = \frac{0.5}{40} = 1.25\% \blacktriangle$$

Example 5-19

The demand during the lead time in an FOS system is according the data in the following table, If the safety stock is three tons . What is the average shortage per cycle **?**

D_L(ton)	6	7	8	9	10	11	12	13	14	15	16
Probability(%)	5	5	5	5	2	2	2	5	5	5	5

Solution

$$\bar{b}(r) = \sum_{x>r}(x-r)p_X(x) \qquad , r = E(D_L) + SS \ ,$$

$$E(D_L) = 6 \times 0.05 + \ldots + 16 \times 0.05 = 11 \quad r = 11 + 3 = 14 \ \text{tons} \quad \Rightarrow$$

$$\bar{b}(r) = \sum_{x>14}(x-14)P_X(x) = (15-14)(0.05) + (16-14)(0.05) = 0.15 \blacktriangle$$

5-4-6-1 Calculation of average shortage in FOS systems when D_L is normally distributed using normal loss integral

If the demand during the lead time (D_L) is normally distributed with mean μ_{D_L} and standard deviation σ_{D_L} and density function

$$f_{D_L}(x) = \frac{1}{\sigma_{D_L}\sqrt{2\pi}} e^{-\frac{(x-\mu_{D_L})^2}{2\sigma_{D_L}^2}}$$ then the average shortage per cycle

which is derived from

$$\bar{b}(r) = \int_{x=r}^{\infty} (x-r)\frac{1}{\sigma_{D_L}\sqrt{2\pi}} e^{-\frac{(x-\mu_{D_L})^2}{2\sigma_{D_L}^2}} \, dx$$

Is calculabled from(see Sec. 1.5.1):

$$\bar{b}(r) = \sigma_{D_L}G_U(k) \qquad k = \frac{r-\mu_{D_L}}{\sigma_{D_L}} \qquad (5\text{-}33)$$

where

$$k = Z_{1-P} = \frac{r-\mu_{D_L}}{\sigma_{D_L}}.$$

The function $G_U(k)$ which is called unit loss normal integral is a function of k=Z_{1-P} , known some times as safety coefficient; the more this coefficient the less $G_U(k)$ and the less the shortage. The values of this function could be calculated using MATLB command exp(-k^2/2)/sqrt(2*pi)-k*(1-normcdf(k)); some its values are given below:

Some values of $G_U(k)$			
p(%)	1-p	k	$G_U(k)$
99.9	0.001	3.45	0.00007127
99	0.01	2.33	0.003352
97.5	0.025	1.96	0.009445
95	0.05	1.64	0.02114
0.93	0.07	1.48	0.03070
92.5	0.075	1.44	0.03356
90	0.1	1.28	0.04750

Example 5-20

In an FOS system, the average demand is 200 units, orders are placed with quantity Q= 30 units. The consumption during

the lead time is normally distributed :

$D_L \sim N\left(\mu_{D_L} = 58.3, \sigma_{D_L} = 13.1\right)$. Find $\bar{b}(r)$, ROP, SS , T_b.

Solution

$$T_b = \frac{Q}{D(1-p)} = \frac{30}{200(1-0/925)} = 2 \text{ yr}$$

This means that on average every 2 years the systems encounter a shortage and the average number of shortages is

$$N_b = \frac{1}{T_b} = \frac{1}{2} \text{ yr.}$$

$$\text{ROP} = r = E(D_L) + Z_{1-p}\sigma_{D_L} =$$

$$58.3 + z_{(1-0/925)} \times 13.1 = 58.3 + 1.44 \times 13.1 = 77.16$$

$$\text{SS} = 1.44 \times 13.1 = 18.9$$

Since D_L is normally distributed:

$$\bar{b}(r) = \sigma_{D_L} \times G_U(k) \qquad\qquad k = \frac{r - \mu_{D_L}}{\sigma_{D_L}} = 1.48$$

k=1.48;exp(-k^2/2)/sqrt(2*pi)-k*(1-normcdf(k)) $\Longrightarrow$ 0.0307

$$\bar{b}(r) = (13.1)(0.0307) = 0.44$$

Example 5-21

In An FOS system, the demand during the lead time is normally distributed with mean 58.3 and standard deviation 13.1.   Assuming a service level of 90%, find the average shortage per cycle. What is the reorder point?

Solution

$k = Z_{1-p} = Z_{0.1} = norminv(1 - .1) = 1{,}2816$

$\bar{b}(r) = \sigma_{D_L} G_U(k) = 13.1 * G_U(1.28) = 13.1 \times 0.04750 = 0.62$

$$r = E(D_L) + k\sigma_{D_L} = 58.3 + (1.28)(13.1) = 75.07\blacktriangle$$

5-4-7 Average inventory in FOS system

The inventory average($\bar{I}$) and the mean of holding cost in continuous review systems are as follows:

$$\bar{I} = \frac{Q}{2} + S.S \qquad\qquad (5\text{-}34)$$

$$\text{averagge hoding cost} \ = \ \bar{I} \times C_h \quad \text{(5-35)}$$

Example 5-22

In an FOS inventory model, as well as the data in the following table , we know that the average demand is 4000 per year, the order size is fixed, the annual unit holding cost is \$10, the service level is 90% and the ordering cost is \$50. Find the optimal order quantity, the reorder point, the safety stock holding cost, the average inventory and its annual holding cost

Solution

i	$D_{L_i} = x_i$	p_i	Cumulative probability
1	11	0.10	0.1
2	13	0.20	0.3
3	15	0.40	0.7
4	17	0.20	0.9
5	19	0.10	1

$$Q^* = \sqrt{\frac{2C_o E(D)}{C_h}} = \sqrt{\frac{2 \times 50 \times 4000}{10}} = 200$$

$$\Pr(D_L \leq r) = 0.9 \Rightarrow ROP = r = 17$$

$$SS = r - E(D_L)$$

$$E(D_L) = \sum_{i=1}^{5} x_i p_i = 11 \times 0.1 + ... + 19 \times 0.1 = 15$$

$$SS = r - E(D_L) = 17 - 15 = 2$$

$$\bar{I} = \frac{Q^*}{2} + SS = \frac{200}{2} + 2 = 102$$

SS annual holding cost $= SS \times SS \times C_h = 2 \times 10 = 20$

average annual holding cost $= 10 \times \bar{I} = 1020$ ▲

5-4-8 Other ways for determining reorder point in FOS systems

To determine the reorder point, in an FOS inventory model where demand (D) and/or the lead time(T_L) are probabilistic, as well as

i-using Eq. 5-19 i.e. $p = \Pr(D_L \leq r)$which uses the service level and the probability distribution of lead time consumption,

there are 2 other was as follows

ii-using average lead time and maximum annual demand

$r = max(D) \times E(T_L)$ (5-35)

iii- using maximum lead time and average demand

$r = max(T_L) \times E(D)$.   (5-36)

In any case $SS = r - E(D_L)$.

The above 3 ways are illustrated below.

Determining reorder point given the service level and lead time consumption distribution

Example 5-23

Given the following table of frequencies and a fixed weekly demand of 6 units, determine the safety stock of .95 (or more) service level in an FOS system.

T_L(week)	4	5	6	7	um
frequency	14	18	12	6	50
probability	0.28	0.36	0.24	0.12	1

Solution

D_L	24	30	36	42
probability	0.28	0.36	0.24	0.12
Cum. Prob	0.28	0.64	0.88	1

$$SS = r - E(D_L)$$

$$E(D_L) = 0.28 \times 24 + ... + 0.12 \times 42 = 31.2$$

$$Pr(D_L \le r) \, ^3 \, 0.95 \, Þ \, r = 42$$

$$SS = 42 - 31.2 = 10.8 \blacktriangle$$

Determining reorder point given the average consumption and the maximum of lead time

Example 5-24

The demand for a product in an FOS model is fixed and equal to 12 per 6-day week. The following frequencies of the lead time is also available.

T_L(day)	4	5	6	7
frequency	14	18	12	6

Find the reorder point and the safety stock

i) based on the service level of at least 95%.

ii) based on the maximum of the lead time if

Solution

i)Since the consumption during T_L is given by $D_L = D \times T_L$ then we have:

D_L	frequency	relative frequency	Cum. frequency
8	14	0.28	0.28
10	18	0.36	0.64
12	12	0.24	0.88
14	6	0.12	1

$$Pr(D_L \le r) = 0.95 \implies r = 14$$

ii)

$r = \max(T_L) \times E(D)$

$ROP = r = T_{L_{max}} \times D = 7 \times \frac{12}{6} = 14$

$E(D_L) = 8 \times 0.28 + \cdots + 14 \times 0.12 = 10.4$

$S.S = r - E(D_L) = 14 - 10.4 = 3.6$▲

Example 5-25

In an FOS model the lead time and the demand are independent. The following data are available. Find the safety stock based on the maximum of the lead time and the average demand.

period	1	2	3	4	5	6	7	8
demand	30	60	50	30	60	50	70	50
T_L (day)	6	5	7	3	6	5	4	4

Solution

$SS = r - E(D_L) = r - E(D) \times E(T_L), \quad \hat{E}(D) = \bar{D} = \dfrac{30 + \dots + 50}{8} = 50$

$\hat{E}(T_L) = \dfrac{6 + \dots + 4}{8} = 5$

$r = \max(T_L) \times E(D) = 7 \times 50 = 350$

$SS = 350 - 5 \times 50 = 100$ ▲

Determining reorder point given the demand maximum and the lead time average

Example 5-26

Solve the previous example again using $r = \max(D) E(T_L)$

Solution

$SS = r - E(D_L) \qquad r = \max(D)E(T_L) = 70 \times 5 = 350,$

$$E(D_L) = E(D)E(T_L) = 50 \times 5 = 250$$

$$SS = r - E(D_L) = 350 \text{ - } 250 = 100 \blacktriangle$$

It worth mentioning that the maximum inventory in FOS model is r+Q; e.g. in the previous example if the order quantity is 520 the maximum of the inventory would be 870.

Example 5-27

The frequencies of T_L in an FOS system in given below. The demand is fixed and equal to 6 per week. Find the safety stock and reorder based on maximum demand . There 6 working days in a week.

T_L(day)	4	5	6	7
frequency	14	18	12	6

Solution

$$r = \max(D)E(T_L), \quad \max(D) = \frac{6}{6} = 1,$$

$$\hat{E}(T_L) = \frac{4 \times 14 + 5 \times 18 + 6 \times 12 + 7 \times 6}{50} = \frac{260}{50} = 5.2$$

$$r = 1 \times 5.2 = 5.2.$$

Since the demand per day is equal to one, the lead time consumption(D_L) and the related frequency would be

D_L	4	5	6	7
frequency	14	18	12	6

$$SS = r - E(D_L) \qquad \hat{E}(D_L) = \frac{4 \times 14 + 5 \times 18 + 6 \times 12 + 7 \times 6}{50} = 5.2$$

$$SS = 5.2 - 5.2 = 0 \blacktriangle$$

At the end of this section some useful relations used in contiguous review model are given below:

Some relations related to (r,Q)=FOS model	
Desired order quantity	$Q^* = \sqrt{\dfrac{\gamma C_o E(D)}{C_h}}$
annuual average shortage	$\overline{B}(r) = D \times \dfrac{\overline{b}(r)}{Q^*}$
ratio of shortage to demand	$\dfrac{\overline{B}(r)}{D} = \dfrac{1}{Q^*}\overline{b}(r)$
Average Inventory	$\overline{I} = \frac{Q^*}{2} + SS$
Average number of shortages per year	$N_b = \dfrac{D}{Q^*}\Pr(D_L > r) = \dfrac{D(1-p)}{Q^*}$
Safety stock in lost sale FOS	$SS = r^* - E(D_L) + \overline{b}(r)$
Safety stock in FOS-- D_L normally distributed	$SS = Z_{1-p}\sigma_{D_L} = k\sigma_{D_L}$
Average shortage per period-D_L normally distributed	$\overline{b}(r) = \sigma_{D_L} \times G_U(k)$

5-5 Two-bin or max-min policy

A special case of continuous review (r,Q) model is what is called two-bin or max-min model. In this model $T_L < T$ and there are two bins either physically or virtually; one is used for supplying current demand and the other for satisfying demand during the lead time. When the first bin which is greater is depleted; an order is placed as much as the capacity of this bin. The demand during the lead time is satisfied from the small bin. When the order quantity arrives, the small bin is filled at the beginning and the rest is poured into the great bin. It is possible to use one bin with a sign on it as the reorder point(Fig5-10). An application of this policy is for the goods with low price and small lead time.
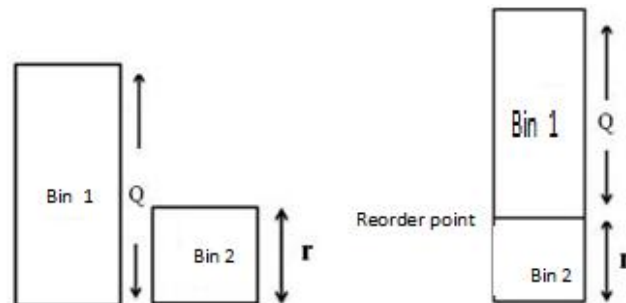
Fig. 5. 10   Two-bin inventory system

In this system whenever the inventory reaches r an order is placed with a quantity equal to Q. The size of the small bin is the average demand during the lead time as well as the safety stock i.e $r = E(D_L) + SS$

where $E(D_L)$ is the consumption during the lead time and SS is the safety stock.

An   advantage of this policy to the general FOS model is preventing running out of stock and saving time and money.

Shortage in inventory systems

It is important in inventory control to determine what to do when a costumer arrives and there is no inventory temporarily. Two possible alternative are available either (Peterson, Silver,1991,p209)

-Complete backordering i.e.to permit shortage in the system. The demands during out stock are backordered and filled as soon as new replenishment arrives.

-Complete Lost sale i.e. the demands during out stock are lost and we incur costs due to lost sale.

The above two alternatives are investigated below for continuous   review model (r,Q).

5.6 Back ordering in FOS system

In continuous review systems with back-ordering, the demands during this time are not lost but are backordered and filled as soon as adequate-sized order arrive.   This policy is more common in industry[1].

The order quantity (Q)   could be calculated from Wilson formula. To determine the optimal   reorder point (r*), we assume r is not dependent on Q and distinguish   two cases(Tersine, 1994 page 218):

-the stockout cost per unit is known

- the stockout cost per outage is known

5-6-1 Backordered (r Q) - Stockout cost/ unit (π)known

In   (r Q) or FOS systems with back-ordering, shortage happens   when X or D_L i.e. the consumption during the lead time, exceeds r. In this case, when π ,i.e. the cost per each time the stockout happens, is fixed and known, the expected annual safety stock cost   (TC_{ss}) is:

TC_{ss}= holding cost + stockout cost or:

$$TC_{SS} = (C_h)(ss) + \pi(\frac{D}{Q}) \times \bar{b}(r) \qquad (5\text{-}37)$$

Where π is the cost per outage, SS is the units of safety stock and $\bar{b}(r)$ is the average stockout units (backordered units) per cycle. SS and $\bar{b}(r)$ are calculated from:

Backordered FOS

$$ss = r - E(D_L) \qquad\qquad (5\text{-}38)$$

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[1] *https://www.sciencedirect.com/science/article/pii/S0377221711001354*

$$\bar{b}(r) = \int_r^\infty (x-r)f_{D_L}(x)dx \quad or \quad \sum_{x>r}(x-r)p_{D_L}(x) \qquad (5\text{-}39)$$

where x is the demand during the lead time.

By taking derivative of Eq. 5-37 with respect to r,the following optimizing relationship results(Tersine, 1994, Proof in Johnson&Montomeri,1974 p59)L

$$\frac{\partial TC_{ss}}{\partial r} = 0 \Rightarrow \Pr(D_L > r^*) = \frac{C_h Q}{\pi D} \qquad\qquad Q = Q_w = \sqrt{\frac{2DC_0}{C_h}}$$

or:

FOS: Back-order case(π known)

$$F_{D_L}(r^*) = \Pr(D_L \le r^*) = 1 - \frac{C_h Q}{\pi D} \qquad (5\text{-}40)$$

The above relationship is valid for both continuous and discrete probability distributions of lead time demand (Tersine, 1994 p 219).

It is obvious if $\dfrac{C_h Q}{\pi D} > 1$ or (if $Q = Q_w$) $TC_w > \pi D$ there is no solution to the equation. this means that the cost of stockout is very low such that we prefer to have always backorder!

Notice not to mistake $E(D_L > r)$ for $E(D_L)$. $E(D_L)$ is the expected

demand during the lead time computable from:

$$E(D_L) = \int_0^\infty xf_{D_L}(x)dx \qquad or \quad \sum_x xp_{D_L}(x).$$

$\bar{b}(r) = E(D_L > r)$ is derived from:

$$\bar{b}(r) = E(D_L > r) = \int_r^\infty (x-r)f_{D_L}(x)dx \qquad or$$

$$\bar{b}(r) = \sum_{x>r}(x-r)p_{D_L}(x)$$

where

f_{D_L} is the pdf of continuous lead time demand

p_{D_L} is the probability function of discrete lead time demand. In this Policy, safety stock is derived from(Winston,1994 p917):

$$SS = r^* - E(D_L) \qquad (5\text{-}41)$$

Example 5-28(Winston,1994 page 917)

The annual demand is normally distributed with a mean of 1000 units and $\sigma_D = 40.8$. The ordering cost is c_h =10. The backorderin

incur a cost of π =\$20 per unit. Find the reorder point r^* and safety stock if

T_L is fixed and equal to 2 weeks

T_L is a random variable with $E(T_L)$=2 weeks $\sigma_{T_L} = \dfrac{1}{52}$ yr

In each case determine the service level.

Solution

$$Q^* = \sqrt{\frac{2(1000)(50)}{10}} = 100$$

$\pi D = 20 \times 1000 > C_h Q^* = (10)(100)$;therefore the problem has a solution.

$$Pr(D_L > r^*) = \frac{(10)(100)}{(20)(1000)} = 0/05$$

Part i

Since the lead time is constant and equal to $T_L = \dfrac{2}{52} = \dfrac{1}{26}\ yr$ and the annual demand is normally distributed, therefore the lead time demand $(D_L = D \times T_L)$ is also normally distributed with mean and variance :

$$\begin{cases} E(D_L)=E(DT_L)=E(D)\times E(T_L)=1000\times\dfrac{1}{26}=38/46 \\[4mm] Var(D_L)=\sigma_D^2\mu_L+\sigma_{T_L}^2\mu_D^2 \ \Rightarrow Var(D_L)=(40/8)^2\times\dfrac{1}{26}+0\Rightarrow\sigma_{D_L}=8 \end{cases}$$

Then

$$Pr(D_L > r^*) = \Pr[Z > \frac{r^* - E(D_L)}{\sigma_{D_L}}] = 0.05 \Rightarrow$$

$$\frac{r^* - E(D_L)}{\sigma_{D_L}} = z_{0.05} = 1.65 \Rightarrow r^* = 38.46 + 8 \times (1.64) = 51.58$$

Whenever the inventory reaches 51 units an order of 100 units is placed. This reorder point assures a service level of p=1-0.05=0.95.

$$SS = r^* - E(D_L) \overset{normal}{=} z_{1-p} \times \sigma_{D_L}$$

$$SS = z_{1-p} \times \sigma_{D_L} = (8)(1.65) = 13.12$$

$$SS = r^* - E(D_L) = 51.58 - 38.46 = 13.12$$

Part ii

$$E(T_L) = 2 \text{ weeks} \qquad \sigma_{T_L} = \frac{1}{52} \text{ yr}$$

$$\sigma_{DL} = \sqrt{\mu_D^2 \sigma_L^2 + \mu_L \sigma_D^2}$$

$$\sigma_{D_L} = \sqrt{(1000)^2 (\frac{1}{52})^2 + (40.8)^2 (\frac{1}{26})} = 20.43 .$$

Now suppose the lead time demand(D_L)is normally distributed:

$$Pr(D_L > r) = 0.05 \Rightarrow$$

$$r = E(D_L) + z_{1-p} \sigma_{DL} = \underset{1000 \times \frac{1}{26}}{E(D)E(T_L)} + \underset{\substack{norminv(0.95) \\ =1.6449}}{z_{0.05}} \times (20.43)$$

r=38.46+33.60 ≅ 72

Whenever the inventory reaches 72 units an order of 100 units is placed. Backordering is allowable . This reorder point assures a service level of p=0.95. ▲

5-6-2 Backordered (r Q) - Stockout cost/ outage (g)known

To determine r and Q in back-order case of continuous review model service level if the cost per outage(g) and the

probability density function of lead time demand, $f_{D_L}(x)$ are known, then the total cost of safety stock is(Martin&Miller,1962 page 63):

$$TC_{ss} = C_h \times SS + g\left(\frac{D}{Q}\right)\Pr(D_L > r) \qquad \frac{\partial TC_{ss}}{\partial r} = \cdot \Rightarrow$$

FOS: Backorder case , cost per outage known

$$f_{D_L}(r^*) = \frac{C_h Q}{gD} \qquad (5\text{-}42)$$

$$SS = r - E(D_L) \qquad (5\text{-}43)$$

To compute SS r^* replaces r in Eq. 5-43. Q is the order quantity at the reorder point. Eq.5-42 is developed for a continuous distribution m but bfrquently integer values of inventory are possible. When the optimum reorder point lises between 2 integer values, the integer with the larger $f_{D_L}(r^*)$ is selected(Tersine, 1994 page 219).

Example 5-29

Weekly demand is normally distributed with mean 20units and standard deviation of 4 units. Back ordering is applied when shortage occurs. When- ever shortage occurs it incurs $ 10. The annual holding cost is $ 5 per unit. The ordering quantity is 26units per order. Find the optimal reorder point and safety stock if the lead time is 1 week in a 52-week year.

Solution

The model is a continuous review with back-ordering. Since T_L is fixed and

D is normally distributed, then $D_L = DT_L$ is also normally distributed with a mean and standard deviation as follows:

$E(D_L) = E(DT_L) = T_L E(D) = 1 \times 20 = 20 \ per \ week$

$\begin{cases} Var(D_L) = \sigma_D^2 \mu_L + \sigma_L^2 \mu_D^2 \Rightarrow Var(D_L) = (4)^2 \times 1 + 0 \Rightarrow \\ \sigma_{D_L} = 4 \ per \ week \end{cases}$

Needles to say, that   we did not need to do the above calculations; because

D_L is the demand for one week and we have the weekly demand. The density function of the lead time in point r^* is:

$f_{D_L}(r^*) = \dfrac{C_h Q}{gD} = \dfrac{5 \times 26}{10(52 \times 20)} = 0.0125 \Rightarrow$

$\dfrac{1}{\sqrt{2\pi}\sigma_{D_L}} e^{-\frac{1}{2}\left(\frac{r^* - \mu_{D_L}}{\sigma_{D_L}}\right)^2} = 0.0125 \Rightarrow$

$\dfrac{r^* - \mu_{D_L}}{\sigma_{D_L}} = \pm \sqrt{Ln \dfrac{1}{2\pi\sigma_{D_L}^2 \times 0.0125^2}} = \pm 2.038$

$ROP = r^* = \mu_{D_L} \pm 2.038\sigma_{D_L} = 20 \pm 2.038 \times 4 \cong 28.15 \ , \ 11.85$

Choosing $r^* = 28$ is more cautious than the other answer. Therefore whenever the inventory reaches 28 an order of size 26

is placed. $SS = r^* - E(D_L)$ $SS = 28\text{-}20 = 8$ ▲

Example 5-30(Tersine,1994 page222)

Weekly demand for a product follows a Poisson distribution with mean of   5 units. The annual holding cost is \$5.   The backorder cost is \$5 per outage. What is the optimum reorder point if T_L is 1 week and the order quantity is 13 units.

Solution

D_L is the demand for one week and we have the distribution of weekly demand. Then D_L is poison distributed with $\lambda = 5$. r^* is calculated as follows:

$$p_{D_L}(r^*) = \frac{C_h Q}{gD} \;\; \rightarrow \;\; \frac{5^{r^*} \times e^{-5}}{r^*!} = \frac{\frac{5}{52} \times 13}{5 \times 5} = 0.05$$

The left hand side of the equation is the probability function of poisson distribution denoted by posspdf in MATLAB:

poisspdf(rstar, 5)=0.05

The   answer to the   above   equation could be found graphically . Using running the following MATLAB command, plots the a figure which is helpful to find the solution.

x=0:1:10;forI=1:length(x); pd(I)=poisspdf(x(I),5);end;plot(x,pd)
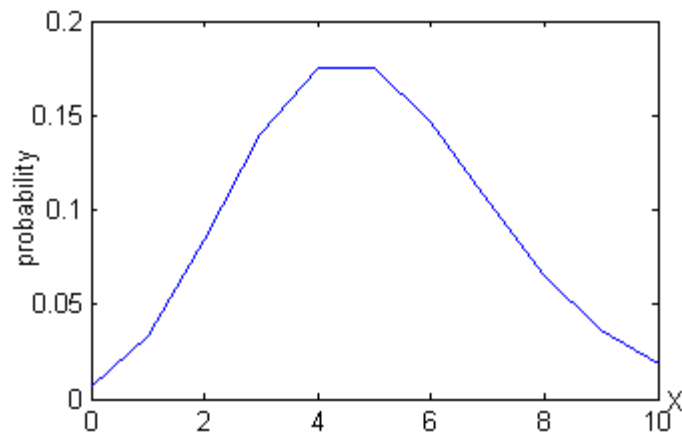
This figure gives two values (near $x \cong 2$ and $x \cong 9$) for the probability 0.05.

The second answer r*= 9 is chosen.

5-7   Lost sale case in FOS system

As it is clear in the lost dale case all stock-outs are lost and not satisfied   later.   The order quantity is determined empirically or by $Q^* = \sqrt{\dfrac{2C_o E(D)}{C_h}}$.

To determine the optimum reorder point here , two case are distinguished i.e. lost sale cost expressed per unit or lost sale cost per outage. These two treated as follows, assuming r and Q are independent.

5-7-1 Lost sale (r Q) - Stockout cost/ unit (π)known

In continuous review systems   when we have complete lost sale and the cost per unit lost   (π) is known, then r^* is calculated from:

$$\Pr(D_L > r^*) = \frac{C_h Q^*}{C_h Q^* + \pi E(D)} \qquad (5\text{-}45)$$

In which

$$\pi = \pi_0 + V - P \qquad\qquad (5\text{-}44)$$

Where

P	Purchase price per unit
V	Sale price perunit
$\pi_\circ$	Lost sale cost/unit (other than lost profits)

If D is constant, D replaces E(D).

5-7-1-1 Safety Stock in (r Q) - Lost Sale case

When the lead time demand (D_L or x) is less than the reorder point, the quantity of product left is $r - x$ with mean

$$= \int_0^r (r - x) f_{D_L}(x)\,dx =$$

$$\int_0^\infty (r - x) f_{D_L}(x)\,dx - \int_r^\infty (r - x) f_{D_L}(x)\,dx$$

$$= r \int_0^\infty f_{D_L}(x)\,dx - \int_0^\infty x f_{D_L}(x)\,dx - \int_r^\infty (r-x) f_{D_L}(x)\,dx \Rightarrow$$

$$SS = r - E(D_L) + \int_r^\infty (x-r) f_{D_L}(x)\,dx$$

<center>Therefore in the optimum state</center>

$$SS = r^* - E(D_L) + \int_r^\infty (x-r) f_{D_L}(x)\,dx = r^* - E(D_L) + \bar{b}(r) \qquad (\text{۵-۴۶})$$

for (r Q) systems with lost sale when the lost sale cost per unit is known, $SS = r^* - E(D_L)$ has also been introduced (Winston 1994, page 917); however the first one is more accurate because it takes shortage into consideration.

Example 5-31(Winston,1994 pagep17)

Annual demand for a product which independent from the lead time is normally distributed: $D \sim N(\mu_D = 1000, \sigma_D = 40)$.

Continuous review model with lost sale is used and we have:

$T_L = 2$ weeks $C_o = 50$ $C_h = 10/yr$ $V = 50$ $P = 30$ $\pi_0 = 20$.

Find the order quantity, the optimal reorder point, and the safety stock.

Solution

$\pi = \pi_0 + V - P = 20 + 50 - 30 = 40$

$Q^* = \sqrt{\dfrac{2C_o E(D)}{C_h}} = 100$

$Pr(D_L > r^*) = \dfrac{C_h Q^*}{C_h Q^* + \pi E(D)} = \dfrac{10 \times 100}{1000 + (20 + 50 - 30)(1000)} = 0.024$

$\mu_D = 1000/yr \rightarrow \mu_{D_L} = \dfrac{1000 \times 2}{52} = 38.46$

$\sigma_{D_L} = \sqrt{\sigma_D^2 \mu_L + \sigma_L^2 \mu_D^2} = \sqrt{\sigma_D^2 \times \mu_L + 0 \times \mu_D^2}$

$\sigma_D = 40/yr \Rightarrow \sigma_{D_L} = \sigma_D \sqrt{\mu_L} = \sigma_D \sqrt{T_L} = 40\sqrt{\tfrac{2}{52}} \simeq 8$

$D_L = DT_L \sim Normal(38.46, \sigma_{D_L} = 8)$

$Pr(D_L > r^*) = Pr(Z > \dfrac{r^* - 38.46}{8}) = 0/024 \Rightarrow \dfrac{r^* - 38.46}{8} = Z_{0.024}$

Using MATLALB command norminv:

$$Z_{0.024} = \text{norminv}(1\text{-}0.024) = 1.9774 \Rightarrow r^* = 54.3$$
.

Table D could be used instead of MATLAB command norminv.

$r^* = 54.3$ states Whenever the level of inventory   reaches 54 units,place an order of 100 units.

Calculation of safety stock:

$$SS = r^* - E(D_L) = ۵۴/۳ - ۳۸/۴۶ = ۱۵/۸۴ \cong ۱۶$$

More accurately :

$$SS = r* - E(D_L) + \bar{b}(r)$$

Since the lead time demand is normally distributed:

$$\bar{b}(r) = \sigma_{D_L} \times G_U(k)$$

$$k = Z_{.024} = 1.98 \quad \bar{b}(r) = 8 \times \overset{\text{From Table A } = 0.009}{G_U(1.98)} = 0.072$$

$$SS = r* - E(D_L) + \bar{b}(r) = 54.3\text{-}38.46 + 0.07 \simeq 15.91 \blacktriangle$$

Example 5-32

 The annual demand and order quantity are fixed and equal to

D=420 and Q=60. $\pi = \$10$ per unit. The lead time demand (D_L) Is as follows:

D_L	10	11	12	13	14	15	16	17
Probabil.	0.1	0.2	0.2	0.15	0.15	0.1	0.07	0.03

A continuous review system with lost sale is used. Which of the following choices do you recommend to use as a reorder point in order to have an average annual shortage cost near 25?

a)14   b)15   c)16   d)17

 Solution

     If the reorder point is taken 14 and D_L equals 10,11, 12,13,14 we do not encounter shortage. But if it equals 15,16, 17 shortage will happen. The following table shows $\bar{b}(r)$, and

its cost for this case and cases $r_=13$, 15,16 and 17. Note that the number of lead time in a year is approximately $\frac{D}{Q} = \frac{420}{60} = 7$.

$ROP(r) \rightarrow$	13	14	15	16	17
D_L causing shortage	14,15,16,17	15,16,17	16,17	17	-
Average shortage in 1 lead time $\bar{b}(r)$	1×0.15+ 2×0.1+ 3×0.07+ 4×0.03=0.68	1×0.1+ 2×0.07+ 3×0.03=0.3	1×0.07+ 2×0.03 =0.13	.1×0.03	0
Average shortage cost $\pi \times \bar{b}(r)$	10×0.68 =6.8	10×0.33 =3.3	10×0.13 =1.3	10×0.03 =0.3	0
Average annual shortage cost $\pi \times \bar{b}(r) \times D/Q$	6.8×7 =47.6	3.3×7 =23.1	7×1.3 =9.1	7×0.3= 2.1	0

Average annual shortage costs are shown in the last row of the table. The average annual shortage cost near 25 belongs to $r_=14$. Therefore choice "a" is the right choice ▲

5-7-2 Lost sale (r Q) - Stockout cost/ outage (g) known

In continuous review systems with lost sale if the shortage cost per outage and the pdf of the lead time density function is known, the relationship for optimum reorder point is(Tersine, 1994, page 225):

$$\frac{f_{D_L}(r^*)}{F_{D_L}(r^*)} = \frac{C_h Q}{gD} \qquad (5\text{-}47)$$

Proof:

Let *a* denote the expected number of shortages occurring in a year and the TC_{ss} denote the cost related to SS and shortage; then $TC_{ss}=C_h \times SS$ mean$+g \times a$.

Although the average number of cycles in a year is $\dfrac{D}{Q+\bar{b}(r)}$

(Tersine, 1994 page 22)but usually it is approximated with $\dfrac{D}{Q}$.
Therefore

$$a = \frac{D}{Q}\Pr(D_L > r) = \frac{D}{Q}\int_r^\infty f_{D_L}(x)dx \qquad (5\text{-}48)$$

$$TC_{ss} = C_h \times \left\{ r - E(D_L) + \int_r^\infty (x-r)f_{D_L}(x)dx \right\} + g \times \frac{D}{Q}\int_r^\infty f_{D_L}(x)dx$$

$$\frac{dTC_{ss}}{dr} = \cdot \quad \Rightarrow C_h - C_h \Pr(D_L > r) - g\left(\frac{D}{Q}\right)f_{D_L}(r) = \cdot \Rightarrow$$

The optimum reorder point (r^*) has a value which satisfies the following relationship[1]:

$$\frac{f_{D_L}(r^*)}{F_{D_L}(r^*)} = \frac{C_h Q}{gD} \qquad (5\text{-}49)$$

Example 5-33
  The annual demand for product is normally distributed with mean 200 and standard deviation of 4. A continuous review system with lost sale is used. The lead time demand is exponentially distributed with mean 50. The shortage cost per outage is g=\$1. The order quantity is   Q=26 and C_h=\$0.077/yr. Find safety stock and optimal reorder point.

Solution

---

[1]The differentiation under integral sign used Leibniz's Rule.

$$\frac{f_{D_L}(r^*)}{F_{D_L}(r^*)} = \frac{C_h Q}{gD} \Rightarrow \frac{\frac{1}{50}e^{-\frac{1}{50}r^*}}{1-e^{-\frac{1}{50}r^*}} = \frac{0.077 \times 26}{1 \times 200} \Rightarrow r^* = 55$$

$$SS = r - E(D_L) + \int_r^\infty (x-r)f_{D_L}(x)\,dx$$

$$SS = 55 - 50 + \int_{55}^{\yen} (x\text{-}55)\frac{1}{50}e^{-\frac{1}{50}x}\,dx$$

$$\int_{55}^{\yen}(x\text{--}55)\frac{1}{50}e^{-\frac{1}{50}x}\,dx = \int_{55}^{\yen}\frac{x}{50}e^{-\frac{x}{50}}\,dx - 55\int_{55}^{\yen}-\frac{1}{50}e^{-\frac{1}{50}}\,dx$$

$$55\int_{55}^{\yen}-\tfrac{1}{50}e^{-\frac{1}{50}}\,dx = 55e^{-\frac{55}{50}}$$

$$\frac{x}{50}=u \qquad e^{-\frac{x}{50}}\,dx=dv \qquad \int_{55}^{\yen}\frac{x}{50}e^{-\frac{x}{50}}\,dx = \int_{55}^{\yen}u\,dv=$$

$$\frac{x}{50}(\text{-}50)e^{-\frac{x}{50}}\Big]_{55}^{\yen} - \int_{55}^{\yen}\text{-}e^{-\frac{x}{50}}\,dx = 105e^{-\frac{55}{50}}$$

$$\Rightarrow SS = 5 + \int_{55}^{\yen}(x\text{-}55)\tfrac{1}{50}e^{-\frac{1}{50}}\,dx = 8 + 105e^{-\frac{55}{50}} - 55e^{-\frac{55}{50}} = 21.64$$

▲

Example 5-34

The inventory system for a product is continuous review with lost sale. The weekly demand is uniformly distributed over [0 100]. The lead time is 2 weeks and Q=26. Find the optimum reorder point if g=$1and the annual holding cost per unit is $7.

Solution

Using moment generating it could easily be shown that the product of a constant number c and a uniform random variable over the interval [a b] has a uniform distribution on [ca   cb]. Therefore $D_L = DT_L$ is uniformly distributed over [0   100].

$$\frac{f_{D_L}(r^*)}{F_{D_L}(r^*)} = \frac{C_h Q}{gD} \Rightarrow \frac{\frac{1}{200}}{\frac{r^*-0}{200}} = \frac{7 \times 26}{1 \times \frac{100+0}{2} \times 52} \Rightarrow r^* \simeq 14$$

$$SS = r^* - E(D_L) + \int_{r^*}^{\infty} (x - r^*) f_{D_L}(x) dx$$

$$SS = 14 - 50 + \int_{14}^{100} (x-14) \tfrac{1}{200} dx \simeq 19$$

5.8 Periodic Review Inventory Model
or (R, T) policy or FOI system

   This section is concerned with continuous review inventory systems which is denoted by (R,T) or FOI. Figure 5-11 shows this model shematically

Symbols	
A	The inventory level at reorder point
$\bar{b}(R)$	Average shortage in each cycle
$\bar{B}(R)$	Average shortage per year
D_{L+T}	The demand(consumption) during $T_L + T$
$f_{D_{T+L}}(.)$	pdf of consumption during $T + T_L$
g	Shortage cost per outage
L	Lead time
$G_U(k)$	Normal loss integral
p	Service level, probability of lack of shortage during T+T_L
P	Purchase price
Q_t	order quantity at time t
$R=E=Q_m$	Desired maximum level of inventory
T	the review interval (cycle time), the time between 2 successive orders
T_L	Lead time
π	shortage cost per unit
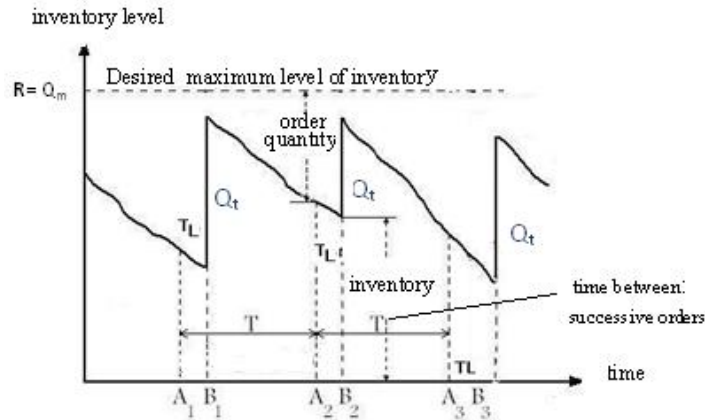
Fig 5-11 Periodic review(R, T) or FOI model

In this system every T time an order is placed in such a way that the order quantity makes the inventory level to a predetermined value denoted by R or Q_m. R has is equal to a value that is sufficient for time T; however when w place an order at the beginning of the lead time as much the lead time demand is deducted from the inventory at the time of placing the order. Therefore the predetermined value R is such that it covers the demand during the review interval (cycle time) and the lead time(T +T_L). With minor modification, the relationships given in the previous section for continuous review system can be used here. Since by definition, the service level is $p = \Pr(D_{T+L} \leq R)$, then given a service level p. the dished R is calculated from:

Demand		
Continu.	$F_{D_{T+L}}(R) = p$	(5-50)
Discrete	$F_{D_{T+L}}(R) \geq p$	(5-51)

Where $F_{D_{T+L}}(.)$ is the cumulative distribution function of D_{T+L}, i.e. the demand during T +T_L.

The safety stock in this system is:

$$SS = R - E(D_{L+T}) \qquad (5-52)$$

Later it will be shown that if D_{L+T} is normally distributed with mean $\mu_{D_{T+L}}$ and standard deviation $\sigma_{D_{T+L}}$ then:

$$R = Q_m = \mu_{D_{L+T}} + Z_{1-p} \times \sigma_{D_{L+T}} \quad (5\text{-}53)$$

The ordering Quantity is given by:

$$Q_t = Q_m - A = R - A \qquad (5\text{-}54)$$

Where

A   The inventory level at reorder point

Q_t The ordering quantity

R=E=Q   Maximum inventory level

The inventory level will never reach the maximum unless the lead time is negligible.

5-8-1 Determination of review interval(T) in (R,T) model

The review interval (cycle time) is often set to:

$$T = \frac{Q_W}{E(D)} \qquad (5\text{-}55)$$

or may be determined empirically.   Note to set *Co* when equal to the ordering cost plus the per cycle cost of reviewing the level of inventory .

Example 5-35

In a period inventory   system, it costs \$500 to review the inventory and 5000 dollars to place an order for a kind of product. The average annual demand is 990 units. The holding cost per unit is \$100 annually. What is your suggestion for the review interval.

Solution

$$T = \frac{Q_W}{\mu_D} \qquad Q_W = \sqrt{\frac{2\mu_D C_0}{C_h}}$$

$$Q_w = \sqrt{\frac{2 \times 990 \times (5000+500)}{100}} = 330 \qquad T = \frac{330}{990} = \frac{1}{3} \text{ yr}$$

$$\text{or } T = \sqrt{\frac{2C_0}{C_h \mu_D}} = \sqrt{\frac{2 \times 5500}{100 \times 990}} = 0.33 \text{ yr} \quad \blacktriangle$$

5-8-2 Calculation of maximum inventory(R)

Given some service level (p) and the distribution of the demand during the lead rime plus the review time (D_{L+T}), the maximum inventory(R) is calculated from the following relationship:

$$\Pr(D_{L+T} \leq R) = p \Rightarrow R = F^{-1}(p) \qquad (5\text{-}56)$$

5-8-3 Mean and Variance of L+T demand (D_{L+T})

To deal with the mean and variance of consumption during the lead time plus the review time, 4 cases are distinguished as follows

Case 1: Demand(D) and the lead time(L $=T_L$) are independent random variables,

Case 2: Demand(D) constant, the lead time(L $=T_L$) random variable,

Case 3: Demand(D) random variable , the lead time(L $=T_L$) constant,

Case 4: Demand(D) and the lead time(L $=T_L$) constant,

When using the relationships given in each case, be careful to differentiate between the mean and the variance of "annual or daily or weekly" demand and the mean and   the variance of "T+L" demand.

5-8-3-1 Demand(D) and the lead time(L =T_L) independent random variables

According to Theorem 5-2, if the "annual or daily or weekly" demand denoted by D and the lead time denoted by L=T_L are independent random variables, then the mean and variance of D_{L+T} the demand   related to $T+L$, are :

$$E(D_{L+T}) = E(D) \times E(T+L) \qquad (5\text{-}57)$$

$$Var(D_{L+T}) = \mu_{T+L}\sigma_D^2 + \mu_D^2\ \sigma_{T+L}^2 \qquad (5\text{-}58)$$

Note that

T is not probabilistic, then $\sigma_{T+L}^2 = \sigma_L^2$,

T could determined from

$$T = \frac{Q_w}{D \ or \ E(D)} = \sqrt{\frac{2C_0}{C_h \mu_D}} \qquad (5\text{-}59)$$

Furthermore in this model

$$R=E(D_{L+T})+SS \qquad\qquad (5\text{-}60)$$

$$SS=R-E(D_{L+T}) \qquad\qquad (5\text{-}61)$$

$$Var(T+L)=Var(L) \qquad\qquad (5\text{-}62)$$

$$E(T+L)=T+\mu_L \qquad\qquad (5\text{-}63)$$

Special case: D_{T+L} normally distributed

If D_{T+L} is normally distributed, the for a given service level

$$\Pr\left(Z \le \frac{R - E(D_{L+T})}{\sigma_{D_{T+L}}} \right) = p \ \text{ then}$$

Relationship for maximum inventory

$$R \ \ or \ Q_m = \mu_{D_{L+T}} + Z_{1-p}\sigma_{D_{L+T}} \qquad (5\text{-}64)$$

Where

$$\sigma_{D_{L+T}} = \sqrt{\mu_{T+L}\mathrm{Var}(D) + \mu_D^2 \mathrm{Var}(T+L)} \ .$$

Relationship for safety stock

Since $SS = R - E(D_{L+T})$ then according to Eq. (5.64):

$$SS=Z_{1-p}\sigma_{D_{L+T}} \qquad (5\text{-}65)$$

Note: when replacing the values of the parameters in the equations be sure to have the same dimensions.

Example 5-35

In a periodic review inventory system, the lead time(L) is normally distributed :Normal(1 week, half week), the weekly demand is also normally distributed: Normal(400, 25) and independent from the lead time. The annual holding cost per unit is $C_h = 0.65$. Find the maximum inventory for a review time of 4 weeks and 95% service level.

Solution

$$D_{L+T} = D(T+T_L), \ \ T_L \sim N(1,\tfrac{1}{2}) \Rightarrow T+T_L \sim N(5,\tfrac{1}{2})$$

$$F_{D_{T+L}}(R) = p$$

According to Theorem 5-2

$$E(D_{L+T}) = E[D \times (T + T_L)] = E(D)E(T + T_L) = 400 \times 5 = 2000$$

$$\sigma_{D_{T+L}} = \sqrt{\mu_{T+L}\text{Var}(D) + \mu_D^2 \text{Var}(T+L)}$$

$$\sigma_{D_{L+T}} = \sqrt{5 \times 25^2 + 400^2 \times \tfrac{1}{4}} = 207.7$$

Since both $T + T_L$ and D are normally distributed, Sec. 1-6-1
allows us to   approximate D_{L+T} with $D_{L+T} \sim N\,(2000,207.7)$
then

$$\Pr(D_{L+T} \leq R) = p \qquad \Pr\left(Z > \frac{R - E(D_{L+T})}{\sigma_{D_{L+T}}}\right) = 1\text{-}0.95$$

$$R = E(D_{L+T}) + Z_{\%o}\sigma_{D_{L+T}} \Rightarrow R = 2000 + 1.6445 \times 207.7 = 2342 \blacktriangle$$

5-8-3-2 Demand(D) random variables and the lead time(L $=T_L$) constant

$D_{L+T} = D(T + T_L)$. If D and L=T_L are independent:

$$E(D_{T+L}) = D\,E(T + L) \qquad (5\text{-}66)$$

$$\text{Var}(D_{T+L}) = D^2 Var(T + L) \qquad (5\text{-}67)$$

5-7-3-3 Demand(D) constant and the lead time(L $=T_L$) random variables

$$Var(T + L) = 0 \qquad ,$$

If D and L=T_L are independent:

$$E(D_{L+T}) = E(D) \times (T + L) \qquad (5\text{-}68)$$

$$Var(D_{L+T}) = (T + L)\sigma_D^2 \qquad (5\text{-}69)$$

Special case: D   normally distributed

If demand is normally distributed, the   consumption
during T+L
Will be normally distributed:

$$D_{L+T} \sim Normal\,(\mu = (T + L)\mu_D \quad , \quad \sigma = \sigma_D \times \sqrt{T + L})$$

and

$$Q_m = R = (T + L)E(D) + Z_{1-p} \times \sigma_D \sqrt{T + L} \quad (5\text{-}70)$$

$$SS = R - E(D_{T+L}) \quad\quad\quad (5\text{-}71)$$

$$SS = Z_{1-p} \times \sigma_D \sqrt{T + L} \quad\quad\quad (5\text{-}72)$$

Note

As mentioned in Sec 5-4-5-1-1,the variance of demand i.e. Var(D) is expressed in $(\frac{\text{units}^2}{\text{unit time}})$ and σ_D in $(\frac{\text{unit}}{\sqrt{\text{unit time}}})$ then to convert the standard deviation of monthly demand to that of yearly demand, multiply it by $\sqrt{12}$. For example $\sigma_D = 10$ units/month is equivalent to $\sigma_D = 10\sqrt{12}$ units per year. To convert the variance of monthly or daily demand to that of yearly demand, multiply it by 12 or N= no. of working days in a year respectively.*5-8-3-4 Demand(D) and the lead time(L =T_L) constant*

If demand and T_L are non-probabilistic then

$$SS = R - E(D_{L+T}) = D \times (T + L) - D \times (T + L) = 0.$$

Let A denote the inventory level at the time of placing an order. R has to cover the demand during T+L, then

$$Q_t = DT - (A - DT_L) = D(T + T_L) - A \quad (5\text{-}73)$$

5-7-4 Average shortage

Let $f_{D_{T+L}}(x)$ denotes the pdf of continuous X *or* D_{T+L} (the demand during T+L) and $p_{D_{T+L}}(x)$ denotes the probability function of discrete X *or* D_{T+L}. $\overline{b}(R)$,the average shortage related to one cycle is calcu-
lated from the following integral or summation depending on the continuity or discreteness of D_{T+L}.

$$\overline{b}(R) = \begin{cases} \displaystyle\int_R^{\infty} (x - R)f_{D_{T+L}}(x)dx \\[2em] \displaystyle\sum_{x>R} (x - R)p_{D_{T+L}}(x) \end{cases}$$

The annual amount of shortage is derived from $\overline{B}(R) = \overline{b}(R) \times \dfrac{1}{T}$

.

where T is the review time in year.

5-7-4-1 Average shotage, maximum inventory, safety stock when D_{L+T} is normal

If the demand during L+T is normally distributed with mean and standard deviation $E(D_{T+L})$, $\sigma_{D_{T+L}}$ then the average shortage in a cycle is:

$$\overline{b}(R) = \int_{R}^{\infty}(x-R)\frac{1}{\sigma_{L+T}\sqrt{2\pi}}e^{-\frac{\left(x-\mu_{D_{L+T}}\right)^2}{2\sigma_{D_{L+T}}^2}}\ dx$$

Using   normal Loss integral mentioned in Sec 1-5-1:

$$\overline{b}(R) = \sigma_{D_{L+T}}G_U(k)\quad k = \frac{R-\mu_{D_{L+T}}}{\sigma_{D_{L+T}}}\quad(5\text{-}76)$$

Where

$$k = Z_{1-p} = \frac{R-\mu_{D_{T+L}}}{\sigma_{D_{T+L}}}\ and\ G_U(k)\ \text{is the loss normal intergral}$$

whose value is obtained from Table A or the following command

GUk=exp(-k^2/2)/sqrt(2*pi)-k*(1-normcdf(k)).

The   following   table   summarized   some   of   the   above relationships.

Some relationships used in (R T) =FOI system		
Review time	=	$T^* = \dfrac{Q_W}{\mu_D} = \sqrt{\dfrac{2C_0}{C_h\mu_D}}$
Annual shortage	=	$\overline{B}(R) = \dfrac{\overline{b}(R)}{T^*}$
The ration of annual shortage to annual demand	=	$\dfrac{\overline{B}(R)}{D} = \dfrac{\overline{b}(R)}{DT^*}$
Average inventory	=	$\overline{I} = \dfrac{DT^*}{2} + SS$
Average no. of shortages in a year	=	$N_b = \dfrac{1}{T^*}\Pr(D_{L+T} > R) = \dfrac{1-p}{T^*}$
Average time between 2 successive shortages	=	$\dfrac{T^*}{1-p} = \dfrac{1}{N_b}$

Safety stock	=	$SS = R - E(D_{L+T})$
SS if D_{L+T} is normal	=	$Z_{\backslash -p} \sigma_{D_{L+T}}$
Average shortage in a cycle if D_{L+T} is normal	=	$\overline{b}(R) = \sigma_{D_{L+T}} \times G_U(k)$

Note

-If safety stock is not requited in a periodic review system then $R = E(D_{T+L}) + SS = E(D_{T+L})$

- In FOI model the order quantity for all cycles is not the same.

- Using the following   transforms FOS relationships are converted into FOI ones(Sabahno,2008, page 4):

FOS =(r Q)		FOI =(R T)
L	$\longrightarrow$	L+T
r	$\longrightarrow$	R
Q	$\longrightarrow$	DT

Example 5-37

The annual demand for a product is 18000, each order costs $5000, annual holding cost per unit is $25 and the lead time is 2 days for a kind of product which is ordered every fixed time T . Assuming a 90% service level in (R T) model, find economic T, maximum inventory, average inventory, average shortage per cycle and per year. The demand during t days is approximated with $N(\mu = 15t, \sigma = 4\sqrt{t})$. There are 360 working days in a year. Calculate safety stock as well.

Solution

$$T^* = \sqrt{\frac{2C_o}{DC_h}} = \sqrt{\frac{2 \times 5000}{25 \times 18000}} \Rightarrow$$

$$T^* = \frac{150}{1000} \text{ yr} = \frac{150}{1000} \times 360 \cong 50 \, days$$

$$k = \frac{R - E(D_{L+T})}{\sigma_{D_{L+T}}} = Z_{0.1}$$

$$\frac{R\text{-}1300}{4\sqrt{52}} = 1.28 \Rightarrow R = Q_m \cong 1337$$

Note in this problem the lead time is constant as well as the review time; then T+L is constant and fixed.

$$D_{L+T} \sim \text{Normal}(\mu=15(L+T), \sigma=4\sqrt{L+T})$$

$$E(D_{L+T})=15(2+50)=1300$$

$$\sigma_{D_{L+T}} = 4\sqrt{L+T} = 4\sqrt{52} \simeq 28.8$$

$$\Pr(D_{L+T} \le R) = p = 0.90 \rightarrow \Pr\left(Z > \frac{R\text{-}E(D_{L+T})}{\sigma_{D_{L+T}}}\right) = 1\text{-}p = 0.1$$

Every 50 days an order with the following quantity has to placed

$$Q_t = Q_m - A = R - A \qquad R = Q_m = 1337$$

$$SS=? \quad D_{L+T} \sim \text{Normal} \Rightarrow$$

$$SS = k\sigma_{D+L} = Z_{\cdot/\backslash}\,\sigma_{D+L} = 1.28 \times 4\sqrt{52} \cong 37$$

$$\bar{b}(R) = \sigma_{D_{T+L}} \times G_U(k) \quad k = Z_{1\text{-}p} = Z_{0.025} = 1.28$$

$$G_U(1.28) = 0.0475 : \text{Table A}$$

$$\bar{b}(R) = 4\sqrt{52} \times 0.0475 \cong 1.37$$

$$\overline{B}(R) = \overline{b(R)} \times \frac{1}{T^*} = 1.37 \times \frac{1}{\frac{50}{360}} \cong 10 \blacktriangle$$

Example 5-38[1]

A product is ordered every T time to reach the inventory to its maximum R. If the monthly demand(D) is variable with mean E(D)and the lead time is deterministic, find an expression for the mean inventory($\overline{I}$):

Solution

$$\overline{I}=\frac{Q}{2}+SS=\frac{DT}{2}+SS \qquad SS=R-E(D_{T+L})=R-(L+T)\times E(D)$$

$$\overline{I}=\frac{T\times E(D)}{2}+R-L\times E(D)-T\times E(D), \overline{I}=R-L\times E(D)-\frac{T}{2}\times E(D)=R-E(D)[L-\frac{T}{2}E(D)]$$

End of example▲

Example 5-39[2]

A kind of product is ordered every 3 months. The lead time is one month. The demand during t days is approximated with $N(\mu = t, \sigma = 10\sqrt{t})$. With a service level of 90%, calculate the maximum inventory.

Solution

In this problem the lead time is constant as well as the review time; then T+L is constant and fixed. Since D is normally distributed D_{L+T} is normally distributed : $D_{L+T} \sim Normal(E(D_{L+T}), \sigma_{D_{L+T}})$:

$$\begin{cases} E(D_{L+T})=(3+1)(100)=400 \\ \sigma_{D_{L+T}}=10\sqrt{3+1}=20 \end{cases}$$

$p = \Pr(D_{L+T} \leq R) = 0.9$ then

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[1] Iranian Universities entrance Exam (from Asadzadeh et al(2006) page 226
[2] Asadzadeh et al(2006) page 233

$$\Pr\left(Z > \frac{R - E(D_{L+T})}{\sigma_{D_{L+T}}}\right) = 1 - p = 0.1$$

$$\Rightarrow \frac{R - E(D_{L+T})}{\sigma_{D_{L+T}}} = Z_{0.1} = 1.28$$

$$R = E(D_{L+T}) + Z_{1-p}^{=Z_{0.1}} \sigma_{D_{L+T}}$$

$$R = 400 + 1.28 \times 20 \simeq 425.6$$

If the inventory level at the time of ordering is A the order quantity would be $Q = 425 - A$ ▲

Example 5-40

A kind of product is ordered every 3 months. The lead time is two weeks. The service level is 90% and he demand during T+L is given in the following table, Find shortage probability, average shortage in each cycle and the safety stock

$X = D_{L+T}$	50	60	70	80	90	100
Prob.	0.1	0.1	0.2	0.3	0.2	0.1
Cum.	0.1	0.2	0.4	0.7	0.9	1

Solution
shortage probability =1-0.9=0.1
$$\Pr(D_{L+T} \leq R) = 0{,}9 \Rightarrow R = 90$$

$$\bar{b}(R) = \sum_{x > R}(x - R)p_{D_{L+T}}(x) =$$

$$\sum_{x>90}(x\text{-}90)p_X(x)=(100\text{-}90)\times0.1=1$$

$$SS = R - E(D_{L+T}) = 90\text{-}(50\times0.1+60\times0.1+....+100\times0.1)$$

SS=13 ▲

Shortage in periodic review systems

When the demand(D) is greater than the maximum inventory(R) some policies including the following ones might be adopted to remedy this situation:

complete backordering,

complete lost sale.

These two are dealt in detail below.

5-9 Back ordering in FOI system

In this section complete backordering is assumed in periodic review inventory systems and 2 cases are distinguished : either the shortage cost per unit or the shortage cost per outage is k.nown.

5-9-1 Backordered (R T) - Stockout cost/ unit (π)known

If the stockout cost per unit (π)is known R^* , the maximum inventory in its optimum state, is calculated from (Tersine, 1994 p 244):

$$\Pr(D_{L+T} > R^*) = \frac{C_h T^*}{\pi} \qquad (5\text{-}77)$$

If $\dfrac{C_h T^*}{\pi} > 1$,there would not be an answer for R^* ,

Example 5-41

The lead time for ordering a product is normally distributed with mean of one week and variance of $\frac{1}{4}$ and the weekly demand has a normal distribution $N(\mu = 400, \sigma = 25)$ and is independent of the lead time. Shortage is backord at the cost of one dollar per unit. The a holding cost per unit is $0.65. Find the optimal value of the maximum invean FOI mode used with a 4-week period review,

Solution

$$D_{L+T} = D(T + T_L)$$
$$T_L \sim N(1, \tfrac{1}{2}) \Rightarrow T + T_L \sim N(5, \tfrac{1}{2})$$

According to Sec 1-6-1 D_{T+L} is approximately normally distributed   with
$$E(D_{L+T}) = E[D(T + T_L)] = E(D)E(T + T_L) = 400 \times 5 = 2000$$

$$\sigma_{D_{T+L}} = \sqrt{\mu_{T+L}\mathrm{Var}(D) + \mu_D^2\mathrm{Var}(T + L)} = 207.7$$

$$\Pr(D_{L+T} > R^*) = \frac{C_h T^*}{\pi}$$

$$\Pr\left(Z > \frac{R^* - E(D_{L+T})}{\sigma_{D_{L+T}}}\right) = \frac{C_h T^*}{\pi} = \frac{0.65 \times \dfrac{4}{52}}{1} = 0.05$$

$$R^* = E(D_{L+T}) + Z_{0.05}\sigma_{D_{L+T}} \Rightarrow R^* = 2000 + 1.6445 \times 207.7 = 2342$$

5-9-2 Backordered (R T) - Stockout cost/ outage (g) known

If the cost per outage is known then the optimal value of the maximum inventory is calculated from (Tersine,1994,page244):

$$f_{D_{L+T}}(R^*) = \frac{C_h T}{g} \qquad\qquad (5\text{-}78)$$

Example 5-42

Solve the previous example supposing g=\$200 fot the cost per outage and ignore π.

Solution

$$annual\ C_h = 0.65 \quad \sigma_{D_{L+T}=207.7} \quad \mu_{D_{L+T}=2000} \quad T = \frac{4}{52} yr$$

$$f_{D_{L+T}}(R^*) = \frac{C_h T^*}{g} \Rightarrow \quad \frac{1}{\sigma_{D_{L+T}}\sqrt{2\pi}} e^{-\frac{\left(R^* - \mu_{D_{L+T}}\right)^2}{2\sigma_{D_L+T}^2}} =$$

$$\frac{C_h T}{g} = \frac{0.65 \times \frac{4}{52}}{200} \Rightarrow$$

$$\frac{R^* - 2000}{207.7} = \pm 2.0194 \rightarrow R^* = 2419 \& 1580 \ \blacktriangle$$

5-10   Lost sale case in FOS system

In this section complete lost sale is assumed in periodic review inventory systems and 2 cases are distinguished : either the shortage cost per unit or the shortage cost per outage is k.nown.

5-10-1 Lost sale (R T) - Stockout cost/ unit (π)known

If the stockout cost per unit (π)is known R^* , the maximum inventory in its optimum state, is calculated from (Tersine, 1994p244):

$$\Pr(D_{L+T} > R^*) = \frac{C_h T}{\pi + C_h T} \quad (5-79)$$

5-10-2 Lost sale (R T) - Stockout cost/ outage (g)known

If the cost per outage is known then the optimal value of the maximum inventory is calculated from (Tersine,1994,page244):

$$\frac{f_{D_{L+T}}(R^*)}{F_{D_{L+T}}(R^*)} = \frac{C_h T}{g} \qquad (5\text{-}80)$$

Example 5-43

The demand during L+T is approximately   exponentially distributed with mean 500 units, the cost per outage is $20. A periodic review system is used with four-week review time. Find the optimal value of the maximum inventory if the annual holding cost per unit is$0.65.

Solution

$$\frac{f_{D_{L+T}}(R^*)}{F_{D_{L+T}}(R^*)} = \frac{C_h T^*}{g} \Rightarrow$$

$$\frac{\frac{1}{500}e^{-\frac{R^*}{500}}}{1-e^{-\frac{R^*}{500}}} = \frac{0.65 \times \frac{4}{52}}{20} \Rightarrow \frac{\frac{1}{500}e^{-\frac{R^*}{500}}}{1-e^{-\frac{R^*}{500}}} = 0.0025 \Rightarrow$$

$$e^{-\frac{R^*}{500}} = \frac{25}{45} \quad \Rightarrow R^* \cong 294 \quad \blacktriangle$$

5-11Inventory control under complete uncertainty

Since dealing with inventory control under complete uncertainty and ambiguity needs some knowledge   of decision making under uncertainty,   a short description of   the subject with emphasis on its application to inventory follows.

Decision theory can indirectly assist in defining the problem and in identifying alternatives, while directly helping to evaluate the alternatives(McKenna,1980).

Definitions

Action space

The set of alternative actions from which a decision maker could choose an action to cope with a situation which needs a decision.

States of the real world or states of the nature

The set of the events that may happen after an alternative action is chosen and performed by the decision maker is often referred to as "states of the nature" or "states of the world" and is beyond the control of the decision maker. In this section the set is denoted by $\Theta = \{\theta_1, \theta_2 \ldots\}$ An example of it in inventory control is the level of demand for a particular product.

Objective function

In decision making a decision situation can involve one objective or more objectives. The objective function could be a desired quantity such as profit or an undesired one like cost or loss. We focus here on minimizing the objective function of the inventory cost as a single objective decision-making problem in inventory control under uncertainty.

To evaluate the alternative actions and choosing the appropriate one, a table similar to the following could be prepared. The possible actions being considered by the decision maker and the states of the real world are inserted in the table. Also the cost or the loss for "each action and each real world state" is calculated and inserted in the table.

Loss function for action a_i and natural state θ_j		
Action	States of the nature	
	θ_1	θ_2
a_1	The cost of action a_1 if θ_1 happens	The cost of action a_1 if θ_2 happens
a_2	The cost of action a_2 if θ_1 happens	The cost of action a_2 if θ_2 happens
a_3	The cost of action a_3 if θ_1 happens	The cost of action a_3 if θ_2 happens

Then one of the actions is selected using the rules or criteria discussed below.

5-11-1 Decision criteria in minimization problems

The process of selecting an action when an objective function   is to be optimized is   usually done using some common sense rules or criteria including the minimax decision criterion(rule), the minimin decision rule and the expected value decision criterion (Bayes method). In the following discussion the objective function is assumed to be cost.

The minimax decision criterion(rule)

For each action   determine the worst outcome, the minimax rule chooses the action with the "best" worst outcome When the objective function is the cost or loss, the minimax decision maker examines the possible cost for each alternative and takes particular note of the greatest cost for each alternative . He then chooses the alternative that yields the smallest of those maximum costs. The decision maker who chooses this criterion is more a pessimist than an optimist (based on Wiston,1994 page 728 and McKenna ,1980 chap4)

The minimin decision rule

For each action determine the best outcome, the miniin rule chooses the action with the "best" best outcome. When the objective function is the cost or loss, the minimin decision maker examines the possible cost for each alternative action and takes particular note of the minimum cost for each alternative . He then chooses the alternative that yields the smallest of those minimum costs. The decision maker who chooses this criterion is more an optimist than a pessimist .

The expected value criterion (Bayes method)

If there is some basis for believing that one state of nature is more likely than the others, a weighted average of the function is preferable to a straight average . The weighted average, in which the probabilities arc the weights, is called the expected value criterion(McKenna,1980). When the objective function is the cost or loss. The expected cost is the sum of the products of probability times cost for each of the decision alternatives . The expected value decision maker chooses the alternative with the best expected cost.

Example 5-44

Suppose the demand for a product is 10 or 30 or 50 or 70. We could order 20 or 40 or 60 units . The loss due each combination of demand and order size is given the following table. Determine the order size separately for the product using the above rules.

		The loss for each combination[1]			
		States of the Nature			
		D=10	D=30	D=50	D=70
The order size	20	50	270	1150	2030
	40	480	100	380	1280
	60	900	520	200	480
	80	1045	665	345	250
Probability		0.2	0.4	0.4	0.1

Solution
a)MiniMax criterion
For each action the worst loss is determined:

        alternative   Maximum Loss
        Ordering 20 units   2030=max{50,270,1150,2030}
        Ordering 40 units   1280
        Ordering 60 units   900
        Ordering 80 units   1045

the minimax decision maker chooses the alternative action with the "best" outcome i.e. chooses to order 60 units.

---

[1] Note that this is an example and the costs are not real.

b)MiniMin criterion

For each action the minimum loss is determined:

alternative	Minimum Loss
Ordering 20 units	50= min{50,270,1150,2030}
Ordering 40 units	100
Ordering 60 units	200
Ordering 80 units	250

the minimin decision maker chooses the alternative action with the "best" outcome i.e chooses to order 20 units.

The minimax rule chooses the action with the "best" worst outcome.

c)Expected value criterion

The following table shows the sum of the products of probability times cost loss for each of the decision alternatives . This value is average loss for the corresponding action.

Order size	Average Loss
20	50*.2+270*.4+1150*.3+2030*.1=666
40	480*.2+100*.4+380*.3+1280*.1=378
60	900*.2+520*.4+200*.3+480*.1=496
80	1045*.2+665*.4+345*.3+250*.1=603

The expected value decision maker chooses 40 units as the

order size because it has the best expected cost. ▲

Exercises[1]

5.1   An industrial distributor sells water pumps and other related supplies. A particular water pump is purchased for 60$ from the manufacturer. The average sales per day are 5 units, and ihe annual holding cost is 25% of the unit cost. The annual demand for the pump is 1500 units, and (he order quantity is 300 units. The backorder cost per unit is $50. and the lead time is 20 days. The demand during lead lime is given in the table below:

D_L	70	80	90	100	110	120	
frequency	3	3	4	80	6	4	Sum=100

---

[1] Problems 1 through 4 are from chapter 5 Tersine(994) p247 problems 1, 2,3,4.   Problems 8,9,12,21 of chapter 5 , Tersine(994) p247 were also given to the students

a)what is the reorder point?

b)How much safety stock should be carried?

c)What is the expected annual cost of the safety stock?

5.2 An automotive parts dealer sells 1200 carburetors a year. Each carburetor costs $25, and the average demand is 4 units/day. The order quantity is 120 units, and the lead time is 25 days. The backorder cost per unit is $20, and annual holding cost is 20% of unit cost. The lead time demand is given in the table below. Determine the safety stock level and the reorder point.

D_L	115	110	105	100	95	90	
frequency	10	15	20	5	25	25	Sum=100

5.3   Solve again Problem 5-1 with the assumption that D_L is normally distributed with mean 100 units and variance 25

5.4 What should be the safety stock in Problem 5.2 if the lost sale cost per unit is $20?

Chapter 6
Introduction
to
Forecating
Methods

Chapter 6

Introduction to Forecasting Methods

Aims of the chapter

This chapter describes some forecasting methods used in inventory management. The emphasis is on quantitative methods such as regression, time series methods, moving average, exponential smoothing . Some criteria such as RMSE are introduced to evaluate methods effectiveness. The application of quality control charts to verify whether a forecaster fits the case or not.

Symbols

$\overline{D}$	Average deviations of the forecasts and the observed data
e	Error random variable
e_t	Forecast error at tome t
MA	Moving Average
MAD	mean absolute deviation
MAE	mean absolute error(error =actual or observed value minus the forecasted value)
MAPE	mean absolute percent error
MBD(MBE)	Mean Between Deviations(Mean Between Errors)
MSE	Mean squatted errors

MLE	Maximum likelihood estimator
m	The number recurring cycles in a year
N	Number of periods in MA method, number periods in Seasonal variations
N	Total number of observed data, Number of observed data in each cycle of seasonal variations
r, R	Correlation coefficient
R_t	The ratio of observed value at time t to the corresponding forecast
$\overline{R}_i$	The ratio for season no. i
RMSE	Root Mean Squared Error
S_D	Standard deviation of the deviation between the observed data and the corresponding forecasts
SEE	Standard Error of estimate
SSE	Sum of squared errors
$X, X_1, X_2, ...$	Independent variables in regression, Radom variables in probability
Y	dependent variable or response variable in regression
y_i	The actual or observed value for i^{th} period
$\hat{y}_i$	the forecasted value for period i
α	The coefficient in exponential smoothing method

6-1 Introduction

Forecasting is to identify the picture of the future events and conditions as close as to what it will happen.   Although forecasting is   rarely perfect and error-free, it cannot be discarded, and is used vastly in many subjects such as engineering and economics (including   demand forecasting for goods and services). It is worth mentioning that forecasting is an art before being a science. In science the input is the rules of the nature, while the input of forecasting is data, analysis, experience and judgment.   There is no rule in the nature giving a relationship between demand , for example and some other variables. It is because factors such as economic conditions, the

actions of the rivals and other social phenomena are complex. It should be emphasized that To find an appropriate method and effective use of it, human judgment will be a complement to the method .

6-2 Classification of Forecasting Methods

Various methods are used for forecasting from a thought or simple statement to mathematical equations.   Forecasting methods could be subjective or objective. The former are based on the opinion of the consumers or experts   and use more intuitive or qualitative approaches .   These methods are used when there is little
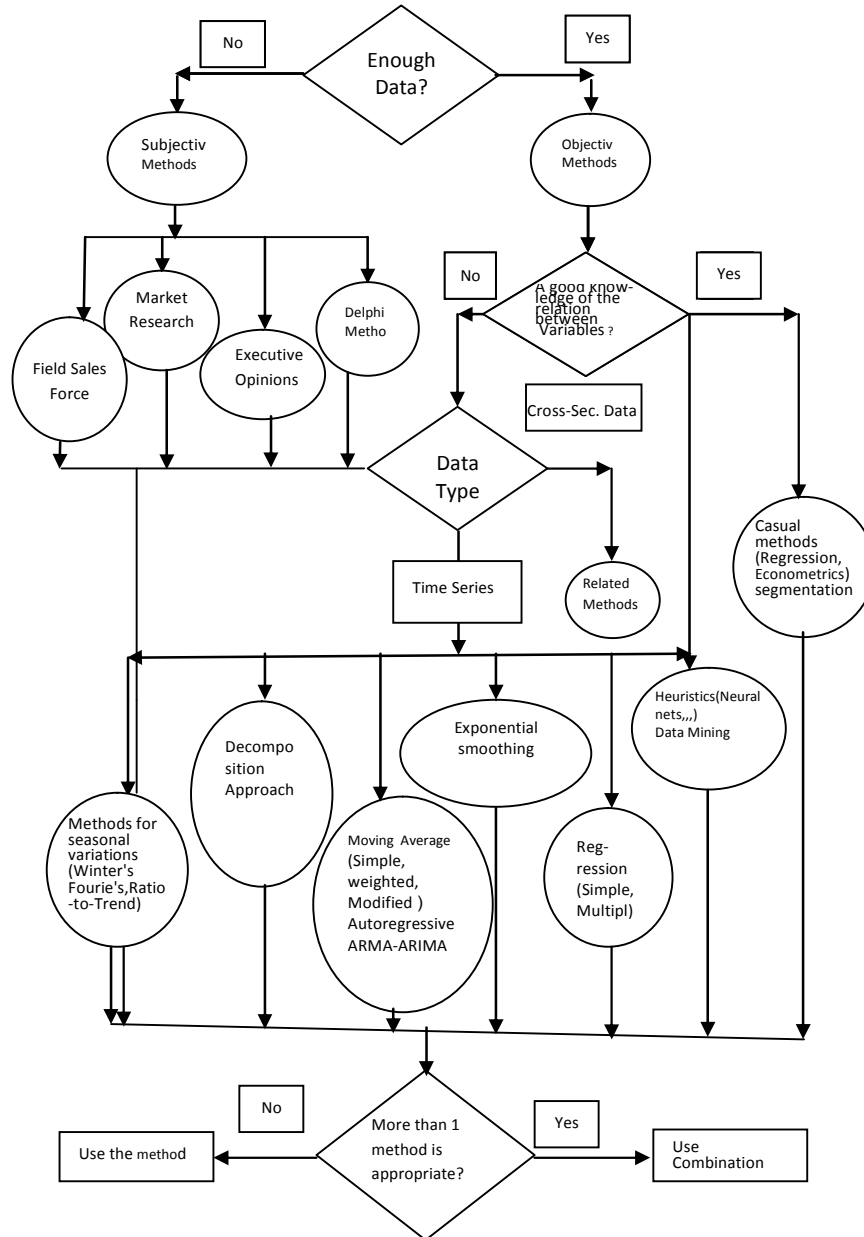
Fig 6.1 A Classification of Forecasting Methods

or no historical data. The former methods use quantitative or mathematical approaches. It is worth mentioning that when an objective method uses a mathematical formula to predict a variable, the method could be called a forecasting model. Figure 6-1 shows more classification of forecasting methods.

6-3 Subjective or qualitative Methods

Subjective forecasting methods are based on common sense. The Forecaster use judgment and self-expertise for forecasting. Some of well-known subjective methods are:

Market research or users' expectation,
 Executive opinions,
 Delphi expertise method,
 Field sales force.
Below Delphi method is described.

6-3-1 Delphi Technique

The Delphi technique is designed to obtain the opinions on a specific topic by means of a questionnaire delivered to selected experts of the subject .

This technique is designed to remedy some of the problems which arise in consensus forecasts . The technique attempts to maximize the advantages of group dynamics while minimizing the problems caused by dominant personalities and silent experts.(Terine, 1994, page 71).   Steps of the method are as follows:
1. Define the problem and the questions for a group of selected experts electronically or physically.
2. Take the group's view as Round 1
3. Explore and discuss the different points of view with the group.
4. Take the groups view again as Round 2
5. Repeat step 2 and 3; ask for Round 3 (if consensus is reached at Round 2, Round 3 is unnecessary)

This is an iterative process and continues until you feel you have reached consensus with your group or sufficient information has been exchanged among the experts.

6-4 Objective or quantitative Methods

Quantitative   methods use a mathematical   model or expression to illustrate the relations between a dependent variable (response) and some independent variables. These methods are used when there is enough historical data. If there is good knowledge of the relation between the dependent and independent variables, then casual models such Quantitative methods use a mathematical model or expression to illustrate the relations between a dependent as regression are used otherwise neural networks and data mining could be used. If the data is given in time series [1] , such model as exponential soothing, moving average, autoregressive auto-regressive moving average (ARMA) , autoregressive integrated moving average(ARIMA) or artificial intelligence algorithms such as neural network modeling might be used.

6-4-1 Regression

In many experiments a variable   varies when the values of some other variables are changed during the exp[merriment. Regression models are used when there exists some inherent relationship among some variables and we want to predict the values of response variable(s) when the values of some independent variables change.

A mathematical equation that allows us to predict values of one dependent variable from known values of one or more independent variables is called a regression equation

---

[1] A time series is a sequence of data points that occur in successive order over some period of time.

(Walpole,1982, page 346). This prediction is in the form of an expected value: $\hat{y} = E\left(Y \mid x_1,...,x_n\right) = \Psi(x_1,...,x_i,...,x_n)$

where

    Y   : response or dependent variable

    $\hat{y}$:predicted value for Y given $x_1,...,x_n$

    $x_i, i = 1,...,n$:independent variables

    $\Psi(x_1,...,x_n$:Regression   function that is a function of $x_i\, 's$
             e.g.:

$$a_\cdot +a_1 x_1 +a_2 x_2 +\ldots+a_n x_n +b_1 x_1 x_2 +b_2 x_1 x_3 +\ldots$$

$$a_1 +b_1 x_1 +b_2 x_2 +\ldots+b_n x_n$$

$$a +b\ x +cx^2$$
$$a +b\ x$$
$$\ln x$$
$$e^{-x}$$

If Ψ is a linear function of $x_i, i = 1,...,n$, the regression model is called linear regression which could be simple or multiple.

6-4-1-1Simple Linear Regression Model

   Simple linear regression is a linear regression model with a single independent (explanatory) variable and one dependent variable., denoted by X, Y respectively.   The mathematical model of simple linear regression is as follows:

$$Y = a+bX +e \qquad (6\text{-}1)$$

Where a and b are the regression coefficients ,

e is the error variable with mean zero .

Given a particular value of X, taking expectation on both sides of Eq. (6.1), yields : $E\left(Y\right)=a+bx+E\left(e\right)$. Then we have:

$$\mu_{Y/x}=E\left(Y/X=x\right)=a+bx \qquad (6\text{-}2)$$

The mean $\mu_{Y/x}$ is considered a predicted value for Y when X=x. The predicted value is denoted in this chapter by $\hat{y}$:

$$\hat{y}=a+bx \qquad (6\text{-}3)$$

6-4-1-1-1 Estimation of model parameters with the method of Least squares

In this section the regression coefficients a and b are estimated with a method often called least squares . in this method the sum of the squares of the residuals   (the difference between results obtained by observation and by computation from   a   formula)   is   minimized.   Given $\left(x_1,y_1\right),...,\left(x_i,y_i\right),...,\left(x_n,y_n\right)$,n pairs of values from the independent and dependent variables X and Y, we would like to estimate a and b in such away that $(\hat{y}_i, y_i)$ i.e. the observed and predicted values for Y become close to each other as much as possible. In other words th aim in estimiating a and bis to minimized the errors e about the regression line (Fig 6-2).
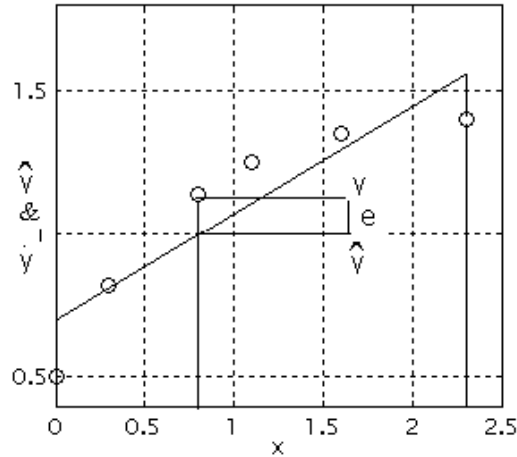
Fig. 6-2 The predicted values ($\hat{y}$), the observed values(y) and
the error(e) in simple regression.

To satisfy this requirement, the sum of   the squares of the
errors(SSE) about the regression line is usually mininimized i.e.
the aim is to minimize:

$$SSE = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \ \textbf{(6-4)}$$

$$SSE == \sum_{i=1}^{n} (y_i - a - bx_i)^2 \qquad \begin{array}{l} \partial SSE / \partial a = 0 \\ \partial SSE / \partial b = 0 \end{array} =====>$$

$$\hat{b} = \frac{\sum_{i=1}^{n} x_i y_i - \frac{1}{n} \sum_{i=1}^{n} x_i \ \sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i^2 - \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right)^2} = \frac{S_{XY}}{S_{XX}}, \qquad (6\text{-}5)$$

$$\hat{a} = \frac{\sum_{i=1}^{n} y_i - \hat{b} \sum_{i=1}^{n} x_i}{n} = \bar{Y} - \hat{b}\bar{X}, \qquad (6\text{-}6)$$

Where

$\overline{Y}$ is the mean of the oberved values $y_1, \dots, y_n$

$\overline{X}$ is the mean of the oberved values $x_1, \dots, x_n$.

The line $\hat{y} = \hat{a} + \hat{b}x$ is called the line of least squared.

Example 6-1

Estimate the regression line for the data of given in the following table

X	Y	x^2	xy
77	5.5	5929	423.5
75	5	5625	375
72	4.7	5184	338.4
71	4.8	5041	340.8
70	4.6	4900	322
$\sum x = 365$	$\sum y = 24.6$	$\sum x^2 = 26679$	$\sum xy = 1799.7$

Solution

Using Eqs. 6-5 , 6-6 and the calcualtion done in the table:

$$\hat{b} = \frac{(5)(1799.7) - (365)(24.6)}{5(26679) - 365^2} = 0.1147 \qquad \hat{a} = \frac{24.6 - \hat{b}(365)}{5} = -3.453$$

Following MATLAB commands give similar results:

```
>> x=[77   75   72   71   70]';   y=[5.5 5 4.7 4.8 4.6]';
>> X = [ones(size(x))  x];
>>ab= regress(y,X)¹
```

-3.4535   0.1147

_____

[1] X\y   could be used instead of regress

The difference between the values obtained for a from Eq. 6-6 and MATLAB is due to the approximations used in the manual calculations .

The equation for the regression line is $\hat{y} = -3.4531 + .1147x$. If the value of the independent variable for   Period 6 is $x_6 = 73$ then the dependent variable fof the period is predicted to be on the average $\hat{y}_6 = (.1147)(73) - 3.4531 = 4.92$ ▲.

6-4-1-1-2 Correlation coefficient

What makes simple linear regression appropriate for predicting Y from X is their degree of their linearity relation. The correlation coefficient   is the specific measure that quantifies the strength of the linear relationship between two variables.   Suppose a sample n pairs of X and Y are available; then the coefficient (r) is defined as follows:

$$r = \frac{n\sum x_i y_i - \left(\sum x_i\right)\left(\sum y_i\right)}{\sqrt{n\sum x_i^2 - \left(\sum x_i\right)^2}\sqrt{n\sum y_i^2 - \left(\sum y_i\right)^2}}. \qquad (6-8)$$

It is proved that $-1 \le r \le 1$ and the more $|r|$ closer to 1 the stronger the linear relation ; and the more $|r|$ closer to zero the weaker the linear relation .   Negative   r denotes that if x increases(decrease) Y will decrease (increase).   Table   6-1 shows the relation between r and the degree of linear ity .

$|r|$	0-0.2	0.2-0.4	0.4-0.7	0.7-0.9	0.9-1
Table 6-1 A classification of correlation of coefficient					
linearity	slight	weak	medium	satisfactory	high

As an example if we calculate the correlation coefficient 0f X, Y in Example 6- 1, we will obtain $r_{xy} = 0.94$ which denotes that there is a strong linear relationship between X and Y. Figures 6-3 through 6-6 shows the linear strength of several sets of data.   It is worth mentioning that such plots are which are

called scatter plot are necessary to   understand the kind of relationship between 2 variables. It is desirable to have at least 30 pairs of data(Kume,1992 page 68) to prepare a scatter plot in order to study the relation between X and Y .
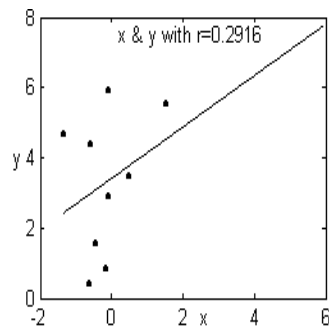
Fig. 6-4 A Scatter plot of a set of data with low positive r
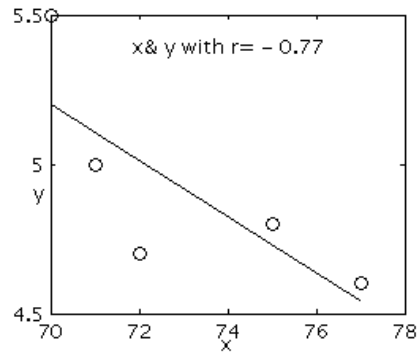
Fig. 6-5 A Scatter plot of a set of data with negative r
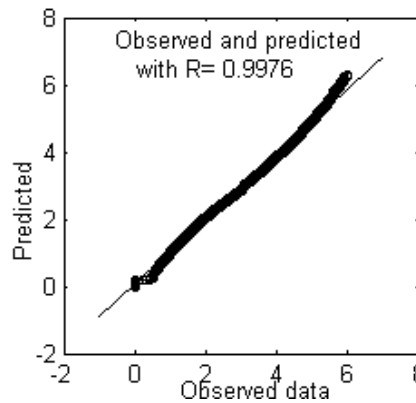
Fig. 6-4 Observed and predicted data with high positive r
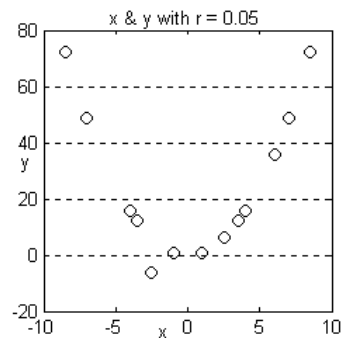
Fig. 6-5 A Scatter plot of a set of data with nearly zero r

6-5 Measures of Model Effectiveness

To verify the validation of   foreecating model, there   are some measures including the ones given in Table 6-2 . In fact the formlus in Table 6-2 measures the forecasting error. **It is**

advised not to use a small set of data to estimate the parameters of the model and valdiating the model.

Measure	Formula	Abb.	Measure	comment
Mean Between Deviation	$\dfrac{\sum\limits_{i=1}^{n}(y_i - \hat{y}_i)}{n}$	MBE	Mean Between Deviation	-Negative MBD: Prediction is greater than actual -Negative MBD: Prediction is less than actual
Mean Absolute Deviation	$\dfrac{\sum\limits_{i=1}^{n}\left| y_i - \hat{y}_i \right|}{n}$	MAD, MAE	Mean Absolute Deviation	
Mean Squared Errors	$\dfrac{\sum\limits_{i=1}^{n}\left(y_i - \hat{y}_i\right)^2}{n}$	MSE	Mean Squared Errors	
Root Mean Squared Errors	$\sqrt{\dfrac{\sum\limits_{i=1}^{n}\left(y_i - \hat{y}_i\right)^2}{n}}$	RMSE	Root Mean Squared Errors	
Standard Error of Estimate	$\sqrt{\dfrac{\sum\limits_{i=1}^{n}\left(y_i - \hat{y}_i\right)^2}{n-f}}$	SEE	Standard Error of Estimate	f is the number parameters to be e4stimated for the Forecaster equation
Mean Absolute Percentage Error	$\dfrac{100}{n}\sum\limits_{i=1}^{n}\left| \dfrac{y_i - \hat{y}_i}{y_i} \right|$	MAPE	Mean Absolute Percentage Error	Gives a dimensionless Measure for error

Table 6 -2 Some measures for evaluation of forecasting models

The corelation coefficient (R)between the obseved (y_i)and predicted($\hat{y}_i$) values from the following relationship is sometimes used in the literature .

$$R = \frac{n\sum y_i \hat{y}_i - \left(\sum y_i\right)\left(\sum \hat{y}_i\right)}{\sqrt{n\sum y_i^{\,2} - \left(\sum y_i\right)^2}\,\sqrt{n\sum \hat{y}_i^{\,2} - \left(\sum \hat{y}_i\right)^2}} \qquad (6\text{-}9)$$

However high R does not necessarily indicate that the prdicted values are appropriate. If R is used another measure such as RMSE has to accompany the coreelation coefficient.

Some researches use   a statistic   called   the **Coefficient of determination** ,denoted by R^2 $(0 \le R^2 \le 1)$,to judge the adequacy of   a regression model.   However   the statistic has several miscionceptions (Montgomery and Rungers, page 510) .

Three other measures of model adequacy are:the coefficient of multiple determination, residual analysis, testing lack of fit using near neighbors. For details refer to Hines& Montgomeri(1990) chapter 15 page 505.

5-6-1 Application of "t-test for paired data" to model effici study

To study the effectve ness of a forecating model, if the difference (D)of the   obseved values and the corresponding predicted values are normally distributed, a special t-test could be used to test;

$H_0 : \mu_D = 0$

$H_1 : \mu_D \neq 0$.

The   test statistic under null hypothes is (Bowker and lieberman,1972 page243):

$$t_0 = \sqrt{\frac{n\overline{D}^2}{S_D^2}} \qquad (6\text{-}10)$$

Where

$$\overline{D} = \sqrt{\frac{\sum\limits_{i=1}^{n} D_i}{n}} = MBD$$

$$S_D = \sqrt{\frac{\sum D_i^2 - n\overline{D}^2}{n-1}} .$$

The test statistic could be calculated equivalently from

$$t_0 = \sqrt{\frac{(n-1)MBD^2}{RMSE^2 - MBD^2}} \qquad (6\text{-}11)$$

Where

$$RMSE = \sqrt{\frac{\overset{n}{\underset{i=1}{\Sigma}}\left(y_i - \hat{y}_i\right)^2}{n}} = \sqrt{\frac{\overset{n}{\underset{i=1}{\Sigma}} D_i^2}{n}} \ .$$

Eqs 6-10& 6-11 are equivalent because:

$$t_0 = \sqrt{\frac{n\overline{D}^2}{S_D^2}} = \sqrt{\frac{n\overline{D}^2}{\dfrac{\sum D_i^2 - n\overline{D}^2}{n-1}}} = \sqrt{\frac{(n-1)\overline{D}^2}{\dfrac{\sum D_i^2 - n\overline{D}^2}{n}}} =$$

$$\sqrt{\frac{(n-1)\overline{D}^2}{\dfrac{\sum D_i^2}{n} - \overline{D}^2}} = \sqrt{\frac{(n-1)MBD^2}{RMSE^2 - MBD^2}}.$$

If t_0 is not greater than the critical value $t_{n-1,\alpha/2}$, then the mean of the observed   values($y_i \, 's$) and the mean of the predicted values ($\hat{y}_i \, 's$) doenot differ significantly.

6-6 Multiple Linear Regression

When we have a case in which one variable depends on several independent variables, multiple regression models which is specific- ally designed to create regressions for such cases may be a good choice. The multiple linear regression with k independent variables (regressors)   is represented by (Montgomeri&rungers,1994page 533):

$$Y = a + b_1 X_1 + b_2 X_2 + ... + b_k X_k + e \qquad \text{(6-12)}$$

where

Y	dependent variable
$X_1, X_2, ..., X_k$	independent variables
$b_k, ..., b_2, b_1, a$	model parameters
e	error random variable with mean zero

Given   some specific values for $X_1, X_2, ..., X_k$, we could take the expection of both sides of Eq. 6-12 as follows:

$$\hat{y} = E\left(Y \mid X_1 = x_1, ..., X_k = x_k\right) = a + b_1 x_1 + b_2 x_2 + ... + b_k x_k + E(e)$$

Since E(e)=o then Given $X_1 = x_1, X_2 = x_2, ..., X_k = x_k$,   the predicted value for the dependent variable ($\hat{y}$) iscalculated from:

$$\hat{y} = a + b_1 x_1 + b_2 x_2 + ... + b_k x_k . \qquad (6\text{-}13)$$

Estimation of the model parameters by the help of   has been dealt in refrences such as Montomeri&Rungers(1994) and softwares such as such as MATLAB   and Minitab.   The following commands might be used in MATLAB to estmate the k-regressor linear model paramters:

$$x_1 = \left[..........\right]' ; x_2 = \left[..........\right]' ; x_k = \left[..........\right]' ; y = \left[..........\right]'$$
>>X=[ones(size(x1)) x1 x2 … xk]; regress(y,X)   or   X\y.

Hines ant Montgomeri () on page 502 mentions that adding an unimportant variable to the model can actually increase the mean square error(MSE),thereby decrease the usefulness of the model. Note that the relations of the form $y = \beta_0 x_1^{\beta_1} \times ... \times x_k^{\beta_k}$ could be transformed to $\log y = \log \beta_0 + \beta_1 \log x_1 + ... + \beta_k \log x_k$ and by setting $\log x_i$'s equal to a new variable, a linear regression model is achieved.

Example 6-2

The following table shows the results of an experiment . Without performing an experiment, could we forecast the result of the experiment if the values of X_1 and X_2 are given.

x_1	1.10	1.00	0.80	0.60	0.50	0.20
x_2	1.40	1.10	0.90	0.400	0.30	0.10
y	0.24	0.27	0.23	0.28	0.26	0.17

Solution

To see if a double linear regression model fits the data or not, at first the parametrs are estimated:

>>x1 = [.2 .5 .6 8 1.0 1.1]'; x2 = [.1 .3 .4 .9 1.1 1.4]'; y = [.17 .26 .28 .23 .27 .24]';

   X = [ones(size(x1))   x1   x2];

     >> regress(y,X)

       0.1018   0.4844   -0.2847

The model is $\hat{y} = 0.1018 + 0.4844 \, x1 \, -0.2847 \, x2$

We do not have any other data for model validation, therefore the above data are inserted in the model as follows:

yhat= 0.1018 + 0.4844 *x1 -0.2847 *x2 or

$\hat{y} = $ [ones(6,1) x_1 x_2] * [0.1018   0.4844 -0.2847]'

The results are given in the following table:

y	0.17	0.26	0.28	0.23	0.27	0.24
$\hat{y}$	0.1702	0.2586	0.2786	0.2331	0.2730	0.2361

The correlation coefficient between the observed and predicted values is $R = 0.9976$ calculated in MATLAB as follows:M=corrcoef(y, yhat);R=M(1,2).

With RMSE = 0.0025 calculated in MATLAB by rmse=sqrt(mse(y-yhat)) .

Before closing this section, a summary of Saffaripour et al(2013) is mentioned below:

The purpose of this investigation is to develop statistical models to estimate the mean daily global solar radiation flux, H, using multiple linear regression models.

The mean daily global solar radiation flux is influenced by astronomical, climatological, geographical, geometrical, meteorological, and physical parameters. This paper deals with the study of the effects of influencing parameters on the mean daily global solar radiation flux.

Saffaripour et al(2013) used multiple linear regression of several parameters in different combinations. The models gave many different correlations to estimate the global solar radiation

fluxes. For example one of the linear regression models they developed was the following relationship:

$$\hat{H} = -17082.9 + 619.68\sin\delta + 0.59H_0 + 3277.15{}^{n}\!/\!_{N}$$
$$+24.34R_h + 64.78T_{max} + 104.25T_{dp(max)} + 14.64P$$

where

$\hat{H}$	Predicted value for the mean daily global solar radiation flux
δ	the solar declination angle
H_0	The extraterrestrial solar radiation flux
n	Hours of measured sunshine
N	the maximum possible sunshine hours from sunrise to sunset
n/N	sunshine duration ratio
R_h	mean daily relative humidity
T_{max}	mean daily maximum air temperature
$T_{dp(max)}$	mean daily maximum dew point temperature
P	mean daily atmospheric pressure

The following table shows the   value for the mean daily global solar radiation flux predicted from the above model($\hat{H}$) and the actual mean values ($\bar{H}$).

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
$\bar{H}$	3583	4602	5358	6473	7491	8192	7956	7656	6827	5440	4050	3421
$\hat{H}$	3630	4813	5267	6457	7485	8257	7999	7511	6823	5611	3907	3308

Saffaripour   et al(2013) calculated the correlation coefficient related to each of the models they created and carried a t-test to choose the appropriate model(s).
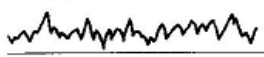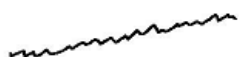
The above model could be used to predict daily global solar radiation flux(H) from atmospheric pressure, air temperature, etc… when the expensive instrument which is used to measure H is not available.

Classical time series forecasting methods

This section introduces some models that are used to predict the future from past data in the form of time series. Some of the models that are used for time series analysis are:

Arithmetic average, simple moving average(MA),weighted moving average, exponential smoothing(single, double, triple), regression, time series decomposition. Decomposition method that splits a time series into several components is suitable for a set of, time series   containing seasonal variation.   The application of some methods for time series analysis have been shown in Table 6-3   with the help of scatter plots.

Table 6.3 Application of forecasting methods(Dilworth,1989 page 131)

APPLICATION OF FORECASTING METHODS

Combination of Components in the Series	Objectives	Models Often Appropriate
Time series models[a]		
No trend (horizontal trend), no seasonal variation; i.e., a stable average with random fluctuation	To average out randomness and find average	Simple moving average; Weighted moving average; Single exponential smoothing
No trend, but seasonal variation	To determine seasonal pattern and project it or to average out seasonality	Time series decomposition; Simple moving average
Trend, but no seasonal variation	To make short-term projection of latest trend estimate; To make longer-term projection of average trend	Double exponential smoothing; Time series decomposition
Trend and seasonal variation	To project trend and seasonal variation around it	Time series decomposition or Winters' triple exponential smoothing[c]
Causal models[b]		
Pattern of changes not related to time	To identify variables that "explain" level of demand	Simple linear regression; Curvilinear regression[c]; Multiple regression[c]

[a] If the series of demand data shows a generally consistent pattern over time and the influencing conditions are expected to continue, a time series model often is adequate.
[b] If demand shows very erratic changes over time so that factors other than time must cause them, then causal models should be investigated to see if they are appropriate.
[c] Not discussed in detail in this chapter.

6-7 Simple Moving Average(SMA)

  Moving average method in its simplest form calculates the forecast for the coming next period by adding up the latest "N" period's observed data and dividing the sum by N as follows:

$$\hat{y}_{t+1} = \frac{\sum_{i=t}^{t+1-N} y_i}{N} \qquad (6\text{-}14)$$

where

$\hat{y}_{t+1}$ is the forecast for Period "t+1" and

y_i is the ith observed value"

For example if the data for the past five periods are $y = [5.5 \quad 5.0 \quad 4.7 \quad 4.8 \quad 4.6]$, The forecast for Period 6 according to SMA would be:

$$\hat{y}_6 = \frac{4.6 + 4.8 + 4.7}{3} = 4.7.$$

    If there is no considerable trend or no considerable seasonal variation, SMA gives an appropriate result. If the N is small random variations affects forecast. Large N smoothes random variations

    The larger the value of N (period of moving average), the smaller is   the effect   of   random   variation and   a higher smoothing effect. The value of N depends upon the speed at which the pattern of demand changes. If the The pattern is not stable, a small value of N should be selected (Telsang [1],1998 page 526). If the variation of   the demand over time is considerable choose a small N (e.g. 3,4,5) ; if it is small choose $12 \le N \le 18$(Hajji,2012 page 173).

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[1] Telsang, M. T , 1998, Industrial Eng;g and Production Manag, S Chand And Co. Ltd

To find the appropriate N for a given case, a short computer code might be prepared to find the N with minimum error. Needless to say that finding an appropriate N for moving average(MA) method does not imply that MA is the most suitable method. As an illustration consider the following 20-period time series:

1.5563   0.8976   0.7482   0.7160   0.3130   0.3617   0.1139   0.1139 -0.2218 -0.1549

0 0   -0.0969   -0.2218   -0.3979   -0.1549   -0.2218   -0.3979   -0.5229   -0.0458

To find the appropriate N for using MA method, a simple computer code gives RMSE for several   periods of moving range method (N) as follows:

Table 6-4 RMSE for several N			
N	RMSE	N	RMSE
1	0.1076	9	0.1095
2	0.3167	10	0.0962
3	0.2703	11	0.0732
4	0.2271	12	0.0444
5	0.1900	13	0.0260
6	0.1597	14	0.0082
7	0.1368	15	0.0116
8	0.1231	16	0.0273

Table 6-4 suggests to choose N=14.   Using this N the predicted value for the following observed value
$y_{17} = -0.2218$ $y_{18} = -0.3979$ $y_{19} = -0.5229$ $y_{20} = -0.0458$

Are:

$$\hat{y}_{17} = -0.2582 \quad \hat{y}_{18} = -0.2582 \quad \hat{y}_{19} = -0.2582 \quad \hat{y}_{20} = -0.2582,$$

6-8 Modified Moving Average

In modified moving range method, The value for Period K from   now $(\hat{y}'_{t+k})$ could be forecasted using the following relationship:

$$\hat{y}'_{t+k} = \hat{y}_t + kb \qquad (6\text{-}15)$$

Where

$$\hat{y}_t = A_t + \frac{6s}{N(N+1)} \qquad A_t = \frac{\sum_{i=t-N+1}^{t} y_i}{N} \qquad b = \frac{12s}{N(N^2-1)}$$

$$s = \frac{N\text{-}1}{2}y_t + \frac{N\text{-}3}{2}y_{t-1} + \frac{N\text{-}5}{2}y_{t-2} + \frac{N\text{-}7}{2}y_{t-3} + ... + \frac{N\text{-}(2N\text{-}1)}{2}y_{t-N+1} =$$

$$\frac{N\text{-}1}{2}y_t + \frac{N\text{-}3}{2}y_{t-1} + ... - \frac{N\text{-}3}{2}y_{t-N+3} - \frac{N\text{-}1}{2}y_{t-N+1}$$

Note that in fact A_t is the forecast for Period t+1 dy simple moving Average(MA) method.

The total sum of the forecasted values for Periods   t+1 through t+L   is given by:

$$\underset{\substack{\text{forecasts sum for}\\ \text{Periods t+1 through t+L}}}{} = L\hat{y}_t + \frac{L(L+1)b}{2} \qquad (6\text{-}16)$$

Example 6-3

The actual demands for January through June are given in the following table:

Period (t)	Jan	Feb	Mar	Apr	May	Jun
demand	90	50	80	64	75	70

Find the forecasts for July and August using 6- period MA technique and also calculate the total   sum of the forecasted values for Periods July through September.

Solution

$$\hat{y}'_7 = \hat{y}_6 + b \qquad \hat{y}_6 = A_6 + \frac{6s}{N(N+1)}$$

$$s = (6-1)(70)/2 + (6-3)(75)/2 + (6-5)(64)/2$$

$$+(6-7)(80)/2 + (6-9)(5)/2 + (6-11)(90)/2 = 20.5$$

$$A_6 = (90 + 50 + 80 + 64 + 75 + 70)/6 = 71.5 \qquad b = \frac{12(20.5)}{6(6^2 - 1)} = 1.17$$

$$\hat{y}_6 = 71.5 + \frac{6(20.5)}{6(7)} = 74.36, \hat{y}'_7 = \hat{y}_6 + b = 74.36 + 1.17 = 75.53 \quad ,$$

$$\hat{y}'_8 = \hat{y}_6 + 2b = 74.36 + 2 \times 1.17 = 76.70$$

Total sum of forecasts for Periods July through September

$$= 3\hat{y}_6 + \frac{3(3+1)}{2} \times 1.17 = 230.11 \quad \blacktriangle$$

6-9 Weighted Moving Average

In simple moving average equal weights were assigned to all N periods; However some- times it is required to assign heavier weighting to more recent data points . This causes the more current data to have heavier effect on the forecast value than the older data. If there is a trend in data, to choose between the weighted moving average(MA) and simple MA, choose the weighted MA.

Weighted moving average(WMA) formula

Mathematically in WMA method the forecast is computed from either of the following formula, depending on the sum of the assigned weights(w_t 's):

$$\hat{y}_{t+1} = w_t y_t + w_{t-1} y_{t-1} + \ldots + w_{t-N+1} y_{t-N+1}, \qquad \sum w = 1. \quad (6-17)$$

$$\hat{y}_{t+1} = \frac{\sum\limits_{i=t+1-k}^{N} w_i y_i}{\sum\limits_{i=t+1-k}^{N} w_i} \qquad (6-18)$$

Example 6-3

   Suppose the demand for a product from period 1 to 5 are 8, 12, 14, 18, 22 find the forecast for period 6 assigning the weight 0.8 for Period 5 and 0.2 for Period 4.

Solution

   From Eq.   6-17: $\hat{y}_6 = 0.8 \times 22 + 0.2 \times 18 = 21.2$

6-10 Exponential Smoothing

   Exponential smoothing which is sometimes called Exponentially weighted moving average, developed by Holt(1957), is actually a weighted MA with a fairly easy to use formula. Practically it uses very little of the past data record. Holt's primary approach did not   consider trend and seasonality; however , later he introduced trend in the model.   Winters (1960) extended the model for reasonability.

   The basic exponential smoothing uses the following formula:

$$\hat{y}_{t+1} = \hat{y}_t + \alpha\left(y_t - \hat{y}_t\right) = \alpha\, y_t + \left(1 - \alpha\right)\hat{y}_t \qquad (6\text{-}19)$$

   Where

$\hat{y}_{t+1}$	New forecast
$\hat{y}_t$	Last period's forecast
y_t	Last period's actual demand
α	A smoothing constant that lies between 0 and 1 often $0.1 < \alpha < 0.5$ and in practice is usually chosen equal to 0.1, 0.3 or 0.5(Winston, 1994, page1262)

   More details about the   above formula could be found in references such as Johnson & Montgomeri(1974)

The appropriate α for a particular case could be found by a computer code which minimizes a forecast error measure like MAD or RSME

As said before exponential smoothing is   in essence a weighted MA. As we move backward the weighting and importance of the data points decreases depending on the value of α.

To forecast the next period demand(t+!), the use of exponential smoothing , requires an initial forecast for current Period t.   A common initial forecast is the arithmetic average of the past data up to the current Period (Housyar,1985). If we have a largish record   of data the initial forecast could be replaced by $\hat{y}_{n+1} = \alpha \sum\limits_{i=0}^{i=n-1} (1-\alpha)^i \, y_{n-i}$. The reason for this will be shown soon.

Example 6-4

Using the data in the following table, find   the forecast for Period 7 with simple exponential smoothing.

t	1	2	3	4	5	6
y	30	32	30	39	33	34

Solution

$$\hat{y}_7 = 0.1\, y_6 + (1-0.1)\, \hat{y}_6 \quad \hat{y}_6 = \frac{34 + ... + 30}{6} = 33$$

$$\hat{y}_7 = 0.1\, y_6 + (1-0.1)\, \hat{y}_6 = 33.1 \blacktriangle$$

Notice that this was an exercise. The utilized method is not necessarily the best method for the case .

As the following calculations shows, in exponential smoothing the weightings assigned to the data points decreases as the data get older

$$\hat{y}_{t+1} = \alpha\, y_t + (1-\alpha)\hat{y}_t = \alpha\, y_t + (1-\alpha)[\alpha y_{t-1} + (1-\alpha)\hat{y}_{t-1}]$$
$$= \alpha y_t + \alpha(1-\alpha)y_{t-1} + (1-\alpha)^2\hat{y}_{t-1}$$

Then

$$\hat{y}_{t+1} = \alpha\, y_t + \alpha(1-\alpha)y_{t-1} + \alpha(1-\alpha)^2 y_{t-2} + \ldots + \alpha(1-\alpha)^k y_{t-k}$$
$$+ \ldots + \alpha(1-\alpha)^{t-1}y_1 + (1-\alpha)^t\hat{y}_1 \Rightarrow$$
$$\hat{y}_{t+1} = \alpha\sum_{i=0}^{i=t-1}(1-\alpha)^i\, y_{t-i} + (1-\alpha)^t\hat{y}_1$$

If the number terms is largish($t \longrightarrow \infty$), $(1-\alpha)^t\hat{y}_1 \rightleftharpoons$ zero and

then   if $t \longrightarrow \infty$, $\hat{y}_{t+1} \cong \alpha\sum_{i=0}^{i=t-1}(1-\alpha)^i\, y_{t-i}$ .The   sum   of   the

coefficients in the first part approaches one:

$$\alpha(1-\alpha)^0 + \alpha(1-\alpha)^1 + \alpha(1-\alpha)^2 + \ldots = \alpha[\frac{1}{1-(1-\alpha)}] = 1$$

   The   above   calculations   shows   if   we   proceed   further
backward as much as possible   we will notice that the   forecast
resulted from exponential smoothing is   a weighted average
from all data. The weightings decrease exponentially(Fig. 6-8).
Furthermore if $n \longrightarrow \infty$ i.e. we have a lot of information as past

data ,then $\hat{y}_{n+1} \cong \alpha\sum_{i=0}^{i=n-1}(1-\alpha)^i\, y_{n-i}$ could be used as the intitial

forecast for using Eqs. 6-18 & 6-19.
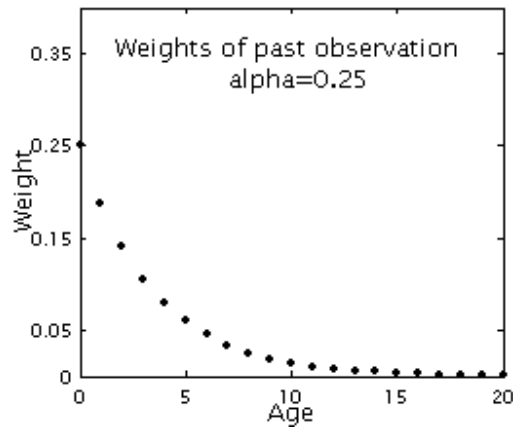
Fig. 6-8 The weightings in a simple exponential Smoothing with $\alpha = 0.25$.

The calculation in Table 6-5 shows   small α assingns greater weighting and importance to more recent data and less importance to older data.

Table 6-5 The values of some coefficient calculated for   some α

α	α	$\alpha(1-\alpha)$	$\alpha(1-\alpha)^2$	...	$\alpha(1-\alpha)^{10}$	......
$\alpha = 0.1$	0.1	0.09	0.081	...	0.035	...
$\alpha = 0.2$	0.2	0.16	0.128		0.0215	
$\alpha = 0.3$	0.3	021	0.147	...	0.009	...
$\alpha = 0.5$	0.5	0.25	0.125		0.0004	

Smaller α gives greater values to older data points than greater α does

Example 6-5

The second column from the left in Table 6-6'   shows the observed values for 24 periods. Calculate   the forecast for the periods using

Exponential   smoothing   with   smoothing   parameters $\alpha = 0.1$ and $\alpha = 0.2$. Which parameter do you prefer

Solution(Dilworth.1989 page111)

Columns 3 and 5 of Table 6-6' show the forecast using simple exponential smoothing method with $\alpha = 0.1$ and $\alpha = 0.2$ respectively.

Table 6-6' Forecasts for Example 6-5 using Exponential Smoothing with $\alpha = 0.1 \& 0.2$

t	Observed (y_t)	Forecast ($\hat{y}_t, \alpha = 0.1$)	squared error $(y_t - \hat{y}_t)^2$	Forecast ($\hat{y}_t, \alpha = 0.1$)	squared error $(y_t - \hat{y}_t)^2$
1	210	196.2000 A	B	196.2000 a	B
2	206	197.5800	B	198.9600	B
3	181	198.4220	B	200.3680	B
4	201	196.6798	B	196.4944	B
5	192	197.1118	B	197.3955	B
6	186	196.6006	112.3727	196.3164	106.4
7	190	195.5406	30.6982	194.2531	18.1
8	208	194.9865	169.3512	193.4025	213.1
9	190	196.2879	39.5377	196.3220	40
10	220	195.6591	592.4794	195.0576	622.1
11	223	198.0932	620.3487	200.0461	526.9
12	175	200.5839	654.5359	204.6369	878.3
13	205	198.0255	48.6437	198.7095	39.6
14	178	198.7229	429.4386	199.9676	482.6
15	214	196.6506	301.0017	195.5741	339.5
16	181	198.3856	302.2591	199.2593	333.4
17	187	196.6470	93.0646	195.6074	74.1
18	217	195.6823	454.4443	193.8859	534.3
19	184	197.8141	190.8294	198.5087	210.5
20	196	196.4327	0.1872	195.6070	2
21	202	196.3894	31.4788	195.6856	39.9
22	169	196.9505	781.2305	196.9485	781.1
23	223	194.1554	832.0109	191.3588	1001.2
24	190	197.0399	49.5602	197.6870	59.1
			RMSE= 17.37		RMSE=18

A Initial mean estimated prior to these calculations=196.2
 B omitted to reduce the effect of the initial mean

RMSE for $\alpha = 0.1$ is 17. 37 and RMSE for $\alpha = 0.2$ is 18.21.

Therefore $\alpha = 0.1$ is preferable for this case. ▲

6-10-1 Relation between simple moving average and simple exponential smoothing

As you have noticed, in the above methods, the user has to specify a parameter: In simple moving aver, the number of periods(N) must be set and in simple exponential smoothing , the smoothing parameter(α). In both cases the parameter determine the importance of fresh information over older information(Shemueli, et al 2010, page 352). It has been proved that the following relationship exits between N and α(Brown, 1962):

$$N = \frac{2-\alpha}{\alpha}, \qquad (6\text{-}21)$$

In other words,, an N-period   MA method gives results approximately similar to those   of a simple exponential smoothing with

$$\alpha = \frac{2}{N+1}, \qquad (6\text{-}22)$$

6-11 Double Exponential Smoothing

Simple Exponential Smoothing cannot forecast accurately when there is trend or seasonal variation in the data   Double Exponential Smoothing extends Simple Exponential Smoothing to   support analyzing data that shows a trend   by adding a second equation with a second parameter to the procedure.

If the data involves a linear trend, there would be a time lag(LT) equal to $LT = \frac{1-\alpha}{\alpha}$ between the forecast resulted from Simple Exponential Smoothing(SES) and the corresponding observed data. Double Exponential Smoothing corrects this lag by forecasting for the next period using the following formula:

$$\hat{y}'_{t+1} = A'_t + \overline{T}_t \qquad (6\text{-}23)$$

Where

$$A'_t \qquad = A_t + \frac{1-\alpha}{\alpha}\overline{T}_t$$

$A_t \qquad = \alpha\, y_t + (1-\alpha)A_{t\text{-}1}$ The initial forecast using SES i.e. $A_t = \hat{y}_{t+1}$

$\dfrac{1-\alpha}{\alpha}\overline{T}_t$ The correction value to compensate for the trend

$\overline{T}_t \qquad = \overline{T}_{t-1} + \beta\,(T_t - \overline{T}_{t-1}) \qquad = \beta\, T_t + (1-\beta)\overline{T}_{t-1}$

$T_t \qquad = A_t - A_{t-1}$

$\beta \qquad \text{o} \le \beta \le 1$

To forecast using double exponential smoothing, initial values $(A_0, \overline{T}_0)$ are needed.   A suitable value for A_0 is the average of the past data.   And a suitable value for $\overline{T}_0$ is the average of the differences between 2 successive observed values.

Note that α, β which are smoothing coefficients between 0 and 1are not necessarily equal.

If the trend continues, the forecast for k periods from now in this method is:

$$\hat{y}'_{t+k} = A'_t + k\overline{T}_t, \qquad (6\text{-}24)$$

And the sum of corrected forecasts for L period from now is:

$$\sum_{i=1}^{L}\hat{y}'_{t+i} = L \times A'_t + \frac{(L)(L+1)}{2}\overline{T}_t \quad (6\text{-}25)$$

If the trend   does not continues, the forecast for k   periods from now in this method is:

$$\hat{y}'_{t+k} = A'_t. \qquad (6\text{-}26)$$

Example 6-6

A factory uses exponential smoothing with trend adjustment . From past data we only know that . $A_0 = 50 \, \text{ton}$ $\bar{T}_0 = 1 \, \text{ton}$ If the actual demand for the current period is $y_1 = 55 \, \text{ton}$ and $\alpha = \beta = 0.1$:

What is the forecast for the next period(t=2)?

Fid the sum of forecast for the next coming 2 periods?

Solution

a)

$$\hat{y}'_{\backslash+\backslash} = A'_{\backslash} + \bar{T}_{\backslash},$$

$$\bar{T}_1 = 0.1 \, T_1 + (1-0.1)\bar{T}_0 = 0.1T_1 + 0.9\bar{T}_0$$

$$T_1 = A_1 - A_0 = A_1 - 50$$

$$A_1 = \alpha \, y_1 + (1-\alpha)A_0 = 0.1 \times 55 + 0.9 \times 50 = 50.5 ==> T_1 = 0.5 \quad \bar{T}_1 = 0.95$$

$$A'_1 = A_1 + \frac{1-\alpha}{\alpha} \bar{T}_1 = 50.5 + \frac{1-0.1}{0.1} \times .95 = 59.5$$

$$\hat{y}'_2 = A'_1 + \bar{T}_1 = 59.5 + 0.95 = 60.45$$

b)

The sum of the forecasts is :

$$\sum_{i=1}^{L} \hat{y}'_{t+i} = L \times A'_t + \frac{(L)(L+1)}{2} \bar{T}_t = 2A'_1 + \frac{(2)(2+1)}{2} \bar{T}_1$$

$$= 2 \times 59.5 + 3 \times .95 = 120.95 \quad \text{or}$$

$$\hat{y}'_2 + \hat{y}'_3 = (A'_1 + \bar{T}_1) + (A'_1 + 2\bar{T}_1) = 120.95$$

Example 6-7

A factory uses exponential smoothing corrected for trend with $\alpha = \beta = 0.15$. The actual demand for the current month is

$y_t = 40$.   Find the forecast for the next month and Month 6.
Data for estimating $A_0, \bar{T}_0$:

The monthly demands in the previous year are as follows:

month	1	2	3	4	5	6	7	8	9	10	11	12
demand	4	6	8	10	14	18	20	22	24	28	31	34

Solution

$$A_1 = \alpha \, y_1 + (1-\alpha)A_0$$

A_0 (forecast for the current period with simple exponential
smoothing) is taken the actual demand of the last month of the
previous year plus $\bar{T}_0 : A_0 = 34 + \bar{T}_0 = 36.73$

The average of the trends of all eleven "2 successive
periods" in the last year:

$$\frac{(34-31)+(31-28)+(28-24)+...-6)+(6-4)}{11} = \frac{34-4}{11} = 2.73$$

Note that for calculating $\bar{T}_0$ m in practice the actual demand of
the first and the last month was enough.

$$A_0 = 34 + 2.73 = 36.73$$

$$A_1 = 0.15 \times 40 + (1-0.15) \times 36.73 = 37.22$$

$$\bar{T}_1 = 0.15 \, T_1 + (1-0.15)\bar{T}_0 = 0.15 \, (A_1 - A_0) + (1-0.15)\bar{T}_0$$

$$= 0.15 \times (37.22-36.73) + (1-0.15) \times 2.73 = 2.39,$$

$$A'_1 = A_1 + \frac{1-0.15}{0.15} \times \bar{T}_1 = 37.22 + 5.67 \times 2.39 = 50.77,$$

$$\hat{y}'_2 = A'_1 + 1\bar{T}_1 = 53.16, \qquad \hat{y}'_7 = A'_1 + 6\bar{T}_1 = 65.11. \blacktriangle$$

Example 6-8

The demand for a product during 24 periods are given in the
second   column   of   Table 6-7. Using double exponential
smoothing calculate the forecast for Periods   1 through 24.

$\alpha = 0.1, \beta = 0.2$ $\bar{T}_0 = $ initial Trend $= 0$, $A_0 = $ the mean pf the   actual values before

period 1 $= 196.2$

Solution

The calculations are shown in Table 6-7. The calculations were done by the following MATLAB code.

```
alpha=input('Insert alpha e.g. 0.1    ');
beta=input('Insert beta e.g. 0.2     ');
A0=input('Insert A0  e.g.196.2     ');
Tbar0=input('Insert Tbar0 e.g. 0   ');
N=input('Insert number of periods e.g. 24    ');
% alpha=.1;beta=.2;A0=196.2;Tbar0=0;
A=ones(N,8);
A(:,1)=[(1:N)]';
A(:,2)=input(' Insert observed values in brackets[ ]''    ');
% A(:,2)=[210 206 181 201 192 186 190 208 190 220 223
 175 205 178 214 181 187 217 184 196 202 169 223 190]';
A(1,3)=alpha*A(1,2)+(1-alpha)*A0;
for i=2:N
   A(i,3)=alpha*A(i,2)+(1-alpha)*A(i-1,3);end
A(1,4)=A(1,3)-A0;
for i=2:N
   A(i,4)=A(i,3)-A(i-1,3);
end
A(1,5)=beta*A(1,4)+(1-beta)*Tbar0;
for i=2:N
%    A(i,5)=alpha*A(i,4)+(1-alpha)*A(i-1,5);
   A(i,5)=(1-beta)*A(i-1,5)+beta*A(i,4);
end
 A(:,6)=(1-alpha)* A(:,5)/alpha;
 A(:,7)=A(:,3)+A(:,6);
 for i=2:N
   A(i,8)=A(i-1,7)+A(i-1,5);End
```

Table 6-7 Illustration of the calculations for forecasting with double exponential smoothing.

t	y_t	$A_t = \alpha y_t$ $+(1-\alpha)A_{t-1}$	$T_t =$ $A_t - A_{t-1}$	$\overline{T}_t = \beta T_t +$ $(1-\beta)\overline{T}_{t-1}$	$\dfrac{1-\alpha}{\alpha}\overline{T}_t$	$A'_t =$ $A_t + \dfrac{1-\alpha}{\alpha}\overline{T}_t$	Forecast $\hat{y}_{t+1} =$ $A'_t + \overline{T}_t$
1	210	197.5800	1.3800	0.2760	2.4840	200.0640	-------
2	206	198.4220	0.8420	0.3892	3.5028	201.9248	200.34
3	181	196.6798	-1.7422	-0.0371	-0.3337	196.3461	202.31
4	201	197.1118	0.4320	0.0567	0.5107	197.6225	196.31
5	192	196.6006	-0.5112	-0.0568	-0.5116	196.0890	197.68
6	186	195.5406	-1.0601	-0.2575	-2.3174	193.2232	196.03
7	190	194.9865	-0.5541	-0.3168	-2.8512	192.1353	192.97
8	208	196.2879	1.3013	0.0068	0.0615	196.3493	191.82
9	190	195.6591	0.6288	-0.1203	-1.0827	194.5764	196.36
10	220	198.0932	2.4341	0.3906	3.5152	201.6084	194.46
11	223	200.5839	2.4907	0.8106	7.2954	207.8793	202.00
12	175	198.0255	-2.5584	0.1368	1.2312	199.2567	208.69
13	205	198.7229	0.6975	0.2489	2.2404	200.9633	199.39
14	178	196.6506	-2.0723	-0.2153	-1.9378	194.7128	201.21

t	y_t	$A_t = \alpha y_t$ $+(1-\alpha)A_{t-1}$	$T_t =$ $A_t - A_{t-1}$	$\overline{T}_t = \beta T_t +$ $(1-\beta)\overline{T}_{t-1}$	$\dfrac{1-\alpha}{\alpha}\overline{T}_t$	$A'_t =$ $A_t + \dfrac{1-\alpha}{\alpha}\overline{T}_t$	Forecast $\hat{y}'_{t+1} =$ $A'_t + \overline{T}_t$
15	214	198.3856	1.7349	0.1747	1.5726	199.9582	194.50
16	181	196.6470	-1.7386	-0.2079	-1.8713	194.7757	200.13
17	187	195.6823	-0.9647	-0.3593	-3.2335	192.4488	194.57
18	217	197.8141	2.1318	0.1389	1.2504	199.0645	192.09
19	184	196.4327	-1.3814	-0.1651	-1.4862	194.9465	199.20
20	196	196.3894	-0.0433	-0.1408	-1.2669	195.1225	194.78
21	202	196.9505	0.5611	-0.0004	-0.0036	196.9469	194.98
22	169	194.1554	-2.7950	-0.5593	-5.0339	189.1215	196.95
23	223	197.0399	2.8845	0.1294	1.1649	198.2047	188.56
24	190	196.3359	-0.7040	-0.0373	-0.3353	196.0006	198.33
25							195.96

End of example▲

6-12 Forecasting techniques for time series having seasonal variations

There might be 3 kind of variations in a time series:
Seasonal variations, cyclic variations and irregular (random)
variations; the first 2 kinds are forecast-able and the last kind is
systems' inherent property(Houshyar,…).

Consider a time series whose scatter plot is similar to Fig. 6-
11. As the figure shows there are seasonal or cyclic variations
in the series.
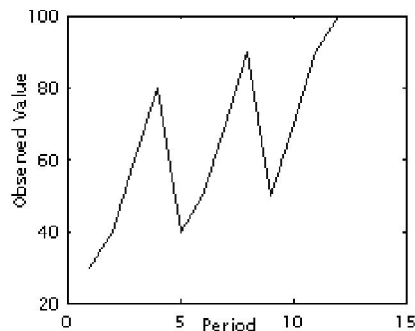
Fig. 6.11 A time series with seasonal variations

In such cases the use of the previous methods such as pure
linear regression do not answer.   Some methods have been
developed to deal with these cases e.g. ratio- trend analysis and
winter's method. The latter is described below.

6-12-1 Ratio-to-trend technique for seasonal adjustment

The steps of a Ratio-to-trend method to forecast the future based on a time series that has shown trend and seasonal variations are as follows(based on Housyar,1985):

Step 1: calculate the forecasts $(\hat{y}_t's)$ for all periods of the time series by a common method such a regression or moving average.

Step 2: calculate the ratio of the observed value (y_t) to the predicted value $(\hat{y}_t)$ for each period calculated in step 1:

$$R_t = \frac{y_t}{\hat{y}_t} \quad t = 1,..., m \times n \qquad (6\text{-}27)$$

Where

R_t The ratio of actual value to the corresponding predicted value(for period t)

m   No of cycles in a time horizon say in a year

n   No of observed values in each cycle

Step 3:There are similar(= of the same name) seasons in the time series. For each of these seasons a separate R_t has been computed using Eq. 6-27. Calculate the mean of these R_t 's calculated for similar seasons :

$$\overline{R}_j = \frac{R_j + R_{j+N} + R_{j+2N} +...+ R_{j+(m-1)N}}{m} \qquad j = 1,..., N \quad (1\text{-}27\text{-}1)$$

Where N is the number of periods in the iterative cycle e.g. 2 half-year in a year 4 seasons in a year.

call $\overline{R}_j$ the index of Season j.

Step 4 The forecast with seasonal adjustment for period t is given by:

$$\hat{y}_t^{'} = \overline{R}_j \times \hat{y}_t \qquad (6-27-2)$$

Example 6-9

A manufacture' sale during the past 3 years has been is

3.5000   4.0000   6.0000   8.0000   4.0000   5.0000

7.0000   9.0000   4.5000   7.5000   9.0000   3.5000

Apply ratio-to-trend method to forecast the sale.

Solution

As the following scatter plot shows there is seasonal variation in the sale data. Therefore the above method might be appropriate.
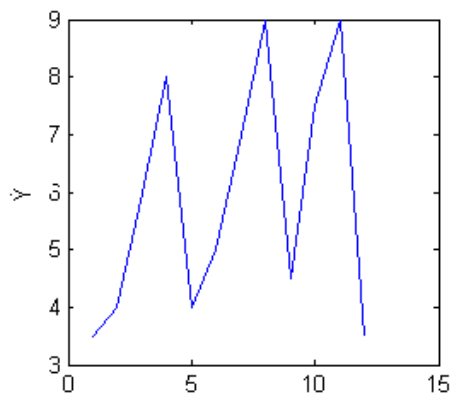
Step 1 If we use simple regression with period(t) as the dependent variable to fore cast the sale volume we would obtain:

$\hat{y}_t = 4.6667 + 0.1923\, t$. Column 5 of Table 6-8 shows the primary forecasts $(\hat{y}_t)$ for all the 12 periods using this relationship

Step 2 Column 6 shows the ratio of the observed value (y_t) to the predicted value $(\hat{y}_t)$ for each period . The scatter plot shows every 4 periods, we have an iterative cycle ; therefore $N = 4$ and 4 seasonal indices have to be calculated in order to correct y_t for seasonal adjustment:

$$\bar{R}_1 = \frac{0.7203 + 0.7107 + 0.7034}{3} = 0.7115, \bar{R}_2 = 0.9297,$$

$$\bar{R}_3 = 1.2118, \quad \bar{R}_4 = 1.1413$$

 Step 3 The corrected forecast($\hat{y}_i'$) for period i is obtained from $\hat{y}_i$ in Column 5 multiplied by the corresponding seasonal index $(\bar{R}_1, \bar{R}_2, \bar{R}_3, \bar{R}_4)$. The result is given in Column 7. The RMSE between the $\hat{y}_i'$ and y_i is rmse=sqrt(mse(y-y'))=1.6

Using MATLAB command corrcoef(y,y') gives the correlation coefficient between the $\hat{y}'_i$ and y_i is 0.62.1

The forecast for Period 13 is calculated as follows:

$\hat{y}_{13} = (4.6667 + 0.1923 \times 13) \times 0.7115 \cong 5.1$.

Table 6-8   A time series data and its forecast by Ratio-to- trend method

Year	Season	t	y_t	$\hat{y}_t$ With Regression	R_t	$\hat{y}'_t$ $(\hat{y}_t \times \bar{R}\ )$
one	Spring	1	3.5	4.8590	0.7203	3.46
	Summer	2	4.0	5.0513	0.7919	4.70
	Fall	3	6.0	5.2436	1.1443	6.35
	Winter	4	8.0	5.4359	1.4717	6.20
Two	Spring	5	4.0	5.6282	0.7107	4.00
	Summer	6	5.0	5.8205	0.8590	5.82
	Fall	7	7.0	6.0128	1.1642	7.29
	Winter	8	9.0	6.2051	1.4504	7.08
Three	Spring	9	4.5	6.3974	0.7034	4.55
	Summer	10	7.5	6.5897	1.1381	6.13
	Fall	11	9.0	6.7820	1.3270	8.22
	Winter	12	3.5	6.9743	0.5018	7.96
RMSE= 1.60 R = 0.6210						

The time series data in this problem contain both trend ad seasonal variation. A method titled Winter's method   might result in better forecasts for these kind of problems ▲

It   worth mentioning that artificial intelligence techniques such as artificial neural networks(ANNs) might be appropriate for forecasting problems.

Some artificial neural networks(ANNs) that are based on simple mathematical models of the brain could be used as forecasting methods. They allow complex nonlinear relationships between the response variable and its predictors (Hyndman &Athanasopoulos,2018,p333).The last exercise of this chapter is on ANNs.

6-13 Verifying and controlling forecasters using control charts

By: Massoud Hajghani, Hamid Bazargan,

A necessary first step after we have made a forecast is to verify that it does indeed appear to represent the data and the chance system underlying the demand for the product in question. To do a good job of forecasting requires that we continually compare the forecast against the actual demand and take action to revise the forecast when there is a statistically significant change in demand((Biegel, 1971, p51).

In this section we would like to determine the validity of the forecast values and the forecaster by appropriate statistical tools. To do this

1. we could use statistical tests,

2. we could calculate RMSE between the actual and observed values; the less this value the better the forecasting method

3. One could plot the observed values and the corresponding forecasts in an X-Y coordinate and fit a least-square-error line to them; the more the points closer to this line and this line closer to the bisector of the first quarter, the better the forecast values(See the last example of this chapter),

6-13-1 A control chart for forecast error

As said before good job of forecasting requires continual comparison of the forecasts against the actual values. If there is evidence of satisfactory forecaster , the forecaster is trusted unless the evidence no longer exist. When this happens an appropriate forecasting technique has to replace the existing one. Control chart is a graph used to study how a process changes

over time; therefore an appropriate tool for continual monitoring is plotting control chart for forecast error. Biegel(1971) introduces a control chart to monitor the forecast. Since the concept of moving range from statistics and quality control is used in this chart, the concept is reminded below:

Definition of Moving Range(MR)

Moving range denoted by MR, is dined here as follows:

$$MR = \left|\left(d'_t - d_t\right) - \left(d'_{t-1} - d_{t-1}\right)\right| \qquad (6\text{-}28)$$

where

d'_t The predicted value for Period t

d_t The actual value of Period t

d'_{t-1} The predicted value for Period t

d_{t-1} The actual value of Period t.

An application of moving range here is to estimate the standard deviation of forecast error frm the following formula:

$$\widehat{\sigma} = \frac{\overline{MR}}{d_2} \qquad (6\text{-}29)$$

where

$\overline{MR}$ The predicted value for Period t defined as:

$$\overline{MR} = \sum \frac{MR}{k-1} \qquad (6-30)$$

Note that for k period k-1

d_2 is a coefficient obtainable from statistical quality control textbooks such as Bazargan(2020). Since the moving range here is defined as the difference of consecutive errors(n=2), the value of d_2 is obtained equal to 1.128 from the books of the following MATLAB commands

```
n = ··· ;pd = makedist('normal',0,1);fun = @(x) (1-(1-cdf(pd,x)).^n-(cdf(pd,x) ) .^n); d2 = integral(fun,-inf,inf);
```

6-13-1-1 Upper and lower limits of the control chart for forecast error

The control chart used in quality control usually have a central line (CL)   and 2 limits: upper   control limit(UCL)and lower control limit (LCL).

Since forecast error(e) is sometimes negative and sometimes positive, the central line of this chart is set to zero(CL=E(e)=0). The limits are determined from the MR values calculated according to Eq.(6-28). It is advised to have at least 10 and preferably   20   MR   values   in   determining   the   control limits(Biegel,1971 p52). The upper control limit(UCL) and the lower control limit(LCL) of the control chart for forecast error are calculated from:

$$UCL=E(e)+3\sigma_e=0+3\frac{\overline{MR}}{d_2}=3\frac{\overline{MR}}{1.128}=2.66\overline{MR}$$

$$CL=0$$

$$LCL=E(e)-3\sigma_e=0-3\frac{\overline{MR}}{d_2}=-2.66\overline{MR}$$

Assuming the error is normally distributed, it is expected to have 0.27% of the points plotted on the chart to fall out of the above 3-sigma limits. In other words if we plot 10000 points on the chart, 27 points are expected to fall outside the limits; from 1000 points 3 points. Since our data are not that much if the forecasts are good no point is allowed to fall outside the limits.   Therefore if a point is out of control due to falling outside the limits or is out of control due to the criteria or   tests described later, when verifying the forecaster we have to do one of the followings(Biegel, 1971 p52):

 Discard some   data(those points from a different cause system) ;search for a new forecaster

Needles to say that if a point is outside the limits , we have to investigate the cause and try to resolve the problem.

If all points fall randomly inside the limits and form no special pattern, we could rely with certainty upon the existing forecasting method.

If points fall outside the limits we apparently do not have the correct forecasting equations and they should be revised accordingly. We can use the control chart to tell us where the change occurred and ca determine a forecasting equation from the data appropriate to the present cause system (Biegel,1971 p53)

6-13-1-2 Some criteria for out-of- control status

As wells as the case mentioned above to declare an out of control status, there are some criteria or tests based on runs of points above or below the central line of the chart

To mention the criteria, the control chart is divided into 3 regions A, B and C above and below the central line as shown in Fig. 6-12.
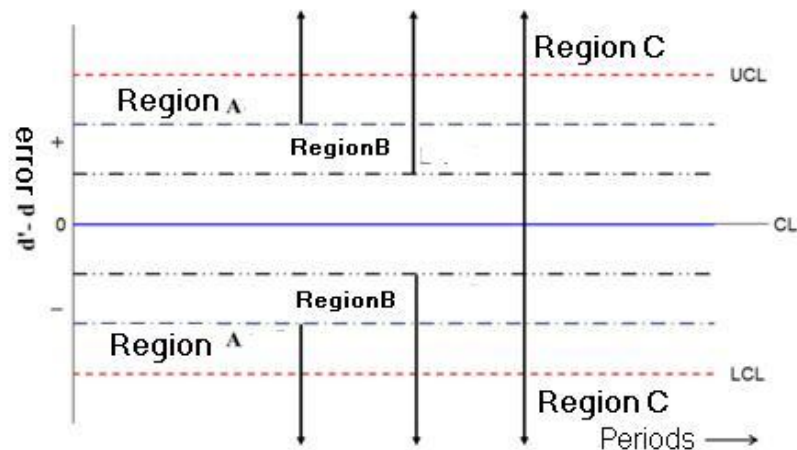
Fig. 6.12 Regions A, B and C in chart for forecast error
(based on Biegel, 1971)

Region A is within $\pm 2\sigma_e = \pm 2\dfrac{\overline{MR}}{d_2 = 1.128} = \pm 1.77\overline{MR}$ above and

below the central line.

Region B is within $\pm 1\sigma_e = \pm \dfrac{\overline{MR}}{d_2 = 1.128} = \pm 0.86\overline{MR}$ above and below
the central line.

Region C is the region above and below the central line.

Two tests that check out-of-control status in a control chart for error
are (Biegel,1971p54):
1. Of 3 successive points, at least 2 points fall on the same side
    of the central line in Region A
2. Of 5 successive points, at least 4 points fall on the same side of
    the central line in Region B.

  Grant & Levenworth(1988) suggest the following tests to detect
shifts in a universe parameter(here: forecast error) in of applications
control chart in manufacturing:

There is suspicion that the process parameter has changed if (grant
&Leavenworth ,1988 page 89):

- Whenever in 7 successive points on the control chart, all are
  on the same side of the central line(a run of 7 points all above
  or all below the central line).
- In 11 successive points on the control chart, at least 10 are on
  the same side of the central line.
- In 14 successive points on the control chart, at least 12 are on
  the same side of the central line.
- In 17 successive points on the control chart, at least 14 are on
  the same side of the central line.
- In 20 successive points on the control chart, at least 16 are on
  the same side of the central line.

    The sequences mentioned in each of these rules will occur as
a matter of chance, with no change in the universe(here: error),
more frequently   than will a point outside of 3-sigma   limits.
For this reason they provide a less reliable basis for hunting a
trouble than does the occurrence of a point outside of control
limits(Grant& Leavenworth,1988 page89).   Those interested in
the theoretical basis for the rules may refer to Chapter 6 of
Grant and Leavenworth(1988).

6-13-2 Illustrations

Below are some illustrations showing how the control chart is used for forecasts verification.   In the cases where some conditions of out-of-control appear, necessary actions have to be taken e.g. to modify the current forecast equations by removing the data points that apparently are not from the same cause system(Biegel, 1971,page55). The following symbles are used in the examples:

t period
d   demand
d'   Forecast
e   error
MR   Moving Range
CL   Central Line
UCL   Upper Control Limit
LCL   Lower Control Limit

Example 6-10   verifying constant forecasters

$$d' = \frac{\sum d_t}{12} = \frac{1191}{12} \cong 99$$ is used to forecast the demand a time series of which is given in the following table

period	1	2	3	4	5	6	7	8	9	10	11	12
demand	90	111	99	89	87	84	104	102	95	114	103	113

Use a control chart to verify the constant forecaster.

Solution
$$UCL_e = 2.66\overline{MR}, \quad CL=0, \quad LCL_e = -2.66\overline{MR}$$

$$\overline{MR} = \frac{117}{11} = 10.6, \ UCL=28.2 \quad LCL=-28.2$$

The calculations have been done in the following table and MR points have been plotted in Fig. 10-13.

period	d_t	Forecast (d')	$e = d' - d$	$MR = \left\| (d'_t - d_t) - (d'_{t-1} - d_{t-1}) \right\|$
1	90	99	9	
2	111	99	-12	21
3	99	99	0	12
4	89	99	10	10
5	87	99	12	2
6	84	99	15	3
7	104	99	-5	20
8	102	99	-3	2
9	95	99	4	7
10	114	99	-15	19
11	103	99	-4	11
12	113	99	-14	10
sum	1191	1188	-3	117

The chart indicates a stable cause(Biegel, 1971, page 55) because no point is out of the control limits and none of the tests applies. ▲
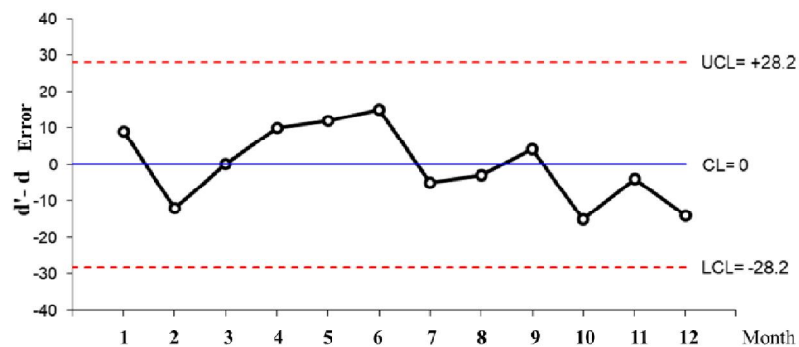
Fig. 6.13 Control chart showing forecast error of Example 6-10
(Biegel, 1971, p55)

Example 6-11 Verifying linear forecasters

The actual demand for the 12 periods of the last year have been given in the following table. Forecasting was done by

simple linear regression. The resulted function for forecasting was $193 + 3t$ given by the following commands in MATLAB environment:

   y=[199   202   199   208   212   194   214   220   219   234 219   233]';t=[1:12]'; T=[ones(size(t)) t];ab=regress(y,T)

   Calculate the moving ranges for errors, plot the control chart and comment.

Solution

   The calculations have been done in the following table and MR points have been plotted in Fig. 6-14.

t	d	d'	$d' - d$	(MR)
1	199	196	-3	
2	202	199	-3	0
3	199	202	3	6
4	208	205	-3	6
5	212	208	-4	1
6	194	211	17	21
7	214	214	0	17
8	220	217	-3	3
9	219	220	1	4
10	234	223	-11	12
11	219	226	7	18
12	233	229	-4	11
sum	2553		-3	99

$$\text{UCL}_e = 2.66\overline{\text{MR}} \ , \text{CL}=0, \text{LCL}_e = -2.66\overline{\text{MR}}$$

$$\overline{\text{MR}} = \frac{99}{11} = 9.0 \qquad \text{UCL}=23.9 \qquad \text{LCL}=-23.9$$
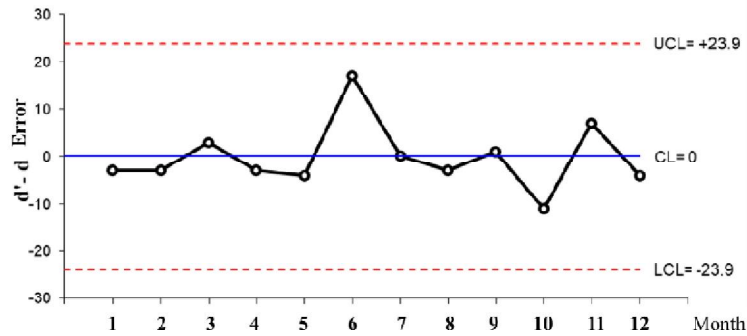
Fig 6-14 The control chart foe Example 6-11
(based on Biegel, 1971)

The chart in Fig.6-14 shows a stable cause system and a
statistically valid forecasting   function(Biegel, 1971, page 57)
because points are distributed randomly within the limits , no

point falls out of the limits and none of the tests applies. ▲

Example 6-12   Verifying a cyclic forecaster

Consider the forecasting function

$$d'_t = 495.6 + 5.7\,t - 10.8\cos\frac{\pi}{6}t + 4.9\,sin\frac{\pi}{6}t$$

proposed[1] to forecast the demand given in the following table.

period	1	2	3	4	5	6	7	8	9	10	11	12
demand	72	83	92	107	114	129	91	108	116	79	92	93

Calculate the moving ranges   for errors,   plot the control chart
and comment.

Solution
The   calculations   have been done in the following table and
MR points have been plotted in Fig. 6-15.

---

[1] To see how it has been derived one might refer to Biegel(1971) page 34.

$\text{UCL}=2.66\overline{\text{MR}}$, $\text{CL}=0$, $\text{LCL}=-2.66\overline{\text{MR}}$

$\overline{\text{MR}}=\dfrac{155}{11}=14.1$ UCL=37.5   LCL=-37.5

t	d	$d\,'$	$d'-d$	MR
1	72	82	10	
2	83	87	4	6
3	92	95	3	1
4	107	103	-4	7
5	114	110	-4	0
6	129	114	-15	11
7	91	114	23	38
8	108	109	1	22
9	116	101	-15	16
10	79	93	14	29
11	92	86	-6	20
12	93	82	-11	5
	1176		0	155
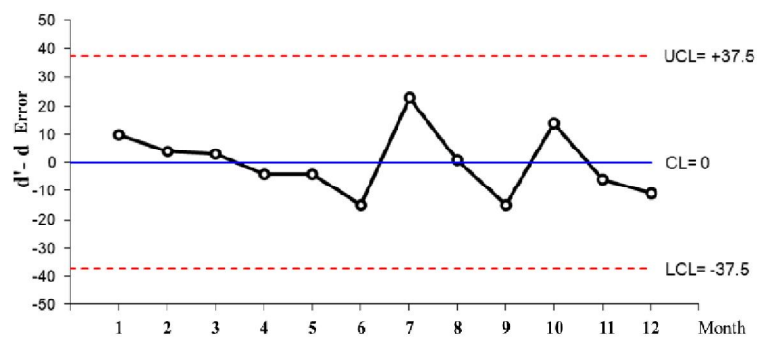
Fig. 6-15 The control chart for Example 6-12(based on Biegel, 1971)

The control chart in Fig 6.15 shows a status of in-control. Therefore   it is concluded that we have a statistically valid forecasting function.▲

Example 6-13 Verifying a linear-cyclic forecaster

Consider the forecasting function

$$d'_t = 495.6 + 5.7\,t - 10.8\cos\frac{\pi}{6}t + 4.9\,sin\frac{\pi}{6}t \quad \text{proposed[1] to}$$

forecast the demand given in the following table.

t	1	2	3	4	5	6	7	8	9	10	11	12
d	498	505	517	521	535	548	544	546	529	548	543	557

plot the control chart and comment.

Solution

   The   calculations   have been done in the following table and
MR points have been plotted in Fig. 6-15.

$$\overline{MR}=\frac{82}{11}=7.4 \qquad UCL=19.7 \qquad LCL=-19.7$$

(t)	(d')	(d)	$d'-d$	(MR)
1	494	498	-4	
2	506	505	1	5
3	518	517	1	0
4	528	521	7	6
5	536	535	1	6
6	541	548	-7	8
7	542	544	-2	5
8	542	546	-4	2
9	542	529	13	17
10	543	548	-5	18
11	546	543	3	8
12	553	557	-4	7
		6391	0	82

---

[1] To see how it has been derived one might refer to Biegel(1971) pp 36-39.
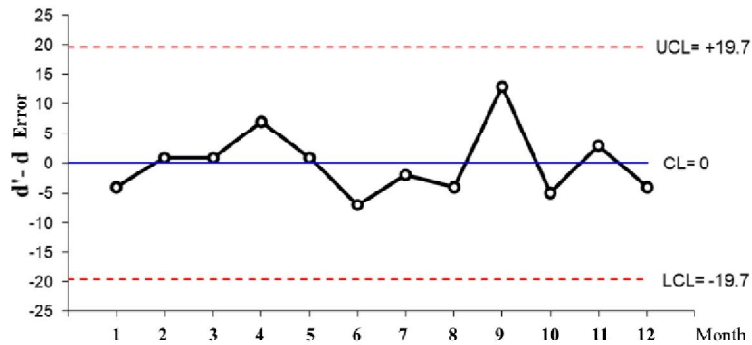
Fig. 6.16 Error control chart for Example 6-13 (Biegel, 1971p 59)

As with 3 above examples , no point is out of the the control chart in Fig 6.16 m, no special pattern has been formed and none of the tests apply. Therefore we have a state of in-control and could rely upon the forecasting function as far as no evidence of being out-of-control appears.▲

Example 6-14
Perhaps the real test of   the control chart con on this example(based on Bigel,1971 page41 and 60), since the data was from real   world.   The following table shows monthly "Revenrue Miles flown" on an international carrier.

t	1	2	3	4	5	6
Miles flown(d)	10885	10465	10143	9273	9378	9378
t	7	8	9	10	11	12
Miles flown(d)	8705	10091	10145	10995	11605	12311

The following linear cyclic function was suggested   to forecast d (for details of the computations see Biegel,1971p41):

$$d'_t = 9450 + 133\,t + 1110\cos\frac{\pi}{6}t + 329\,sin\frac{\pi}{6}t$$

Is this a reliable forecasting function for the Miles flown(d)?
Solution
To verify the function, the moving ranges are calculated and the error control chart is plotted.

$$\overline{MR} = \frac{4600}{11} = 418.2 \quad UCL = 2.66\overline{MR} = 1112 \quad LCL = -2.66\overline{MR} = -1112$$

The following table shows the computations results

t	d	d'	$d' - d$	MR
1	10885	10709	-176	
2	10465	10556	91	267
3	10143	10178	35	56
4	9273	9712	439	404
5	9768	9318	450	889
6	9378	9137	-241	209
7	8705	9254	549	790
8	10091	9672	-419	968
9	10145	10316	171	590
10	10995	11049	53	118
11	11605	11708	103	50
12	12311	12154	-156	259
sum	123764	123763	-1	4600

For example for Period 8 :

t=8;d'$_8$=9450+133*t+1110*cos(pi*t/6)+329*sin(pi*t/6)

ans = 9674.

t=7;d'$_7$=9450+133*t+1110*cos(pi*t/6)+329*sin(pi*t/6)

ans = 9255.

$$t = 8, MR_8 = \left| \left(d'_8 - d_8 \right) - \left(d'_{8-1} - d_{8-1} \right) \right| = |\, 9674 - 10093 - (9255 - 8705) \,| = 969$$

These results some how differs from those in the table; the reason could be due to rounding up the numbers.
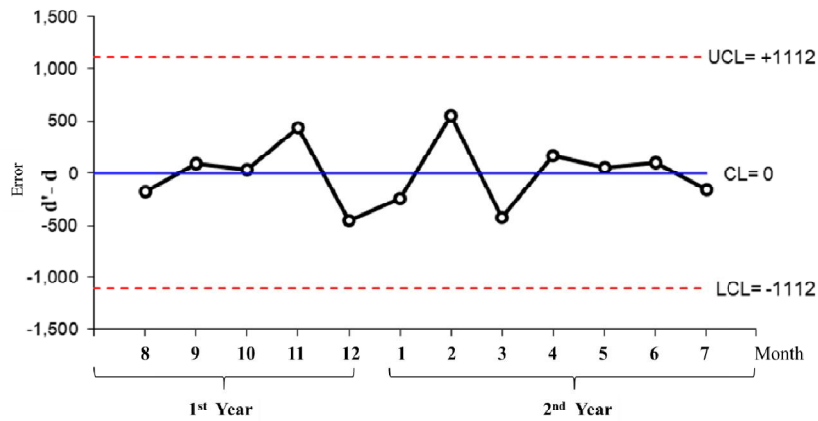
Fig 6-17 The error control chart for Example 6-14
(based on Biegel, 1971, p60)

Figure 4.6 shows the error control chart which indicates an in-control status and then a valid estimator function(Biegel,1971 p60).

The reader should bear in mind that the discussion of occurrences and actions tends to eliminate the time aspect which is present in the generation of data. For the remainder of of this chapter he should assume that the demand data become available to us , piece by piece, over a span of time.▲

Example 6-15

In Example 6.10 We have forecast the demand should average 99. Suppose the demand for the 7 month of the second year is:

month	13	14	15	16	17	18	19
d	105	89	114	109	112	107	116
$d'-d$ *	-6	10	-15	-10	-13	-8	-17

* $d' = 99$

Plot a new error control chart related to Moth 1 through 19. Comment and do the necessary actions.

Solution

Figure 6.18 shows the control chart for the new data and the data related to the year before. In this chart the point related to period 19, marked   with X, indicates an out-of control condition(4 out of 5 successive points in Region B). This means that the forecaster is underestimating the demand(Biegle,1971 p 61).
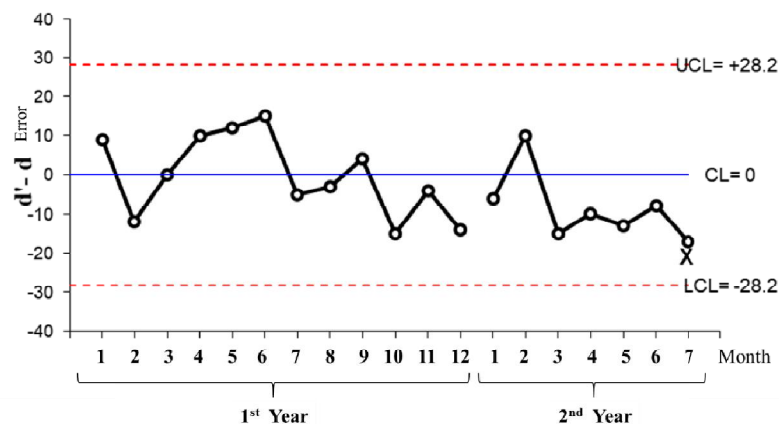
Fig 6.18 Error control chart for 19- period time horizon related
Example 6-10 (based on Biegel, 1971 p62)

The out- of -control condition   indicates a necessity to establish a new forecaster. As a first attempt the mean of all the data was examined to see if it fits. The result of the calculations are (Biegel,1971 page 63)

$$\bar{d} = \frac{1943}{19} = 102 \qquad \overline{MR} = 10.4 \qquad UCL = 27.8 \qquad LCL = -27.8$$

Standard deviation of 19 demand values is $S_d = 10.44$

A new constant forecaster is chosen :d'=102. Suppose the demands for the rest of the second year is as follows

period	20	21	22	23	24
demand(d)	105	109	93	110	116

Fig 19.6 shows the chart for the first and the second year .
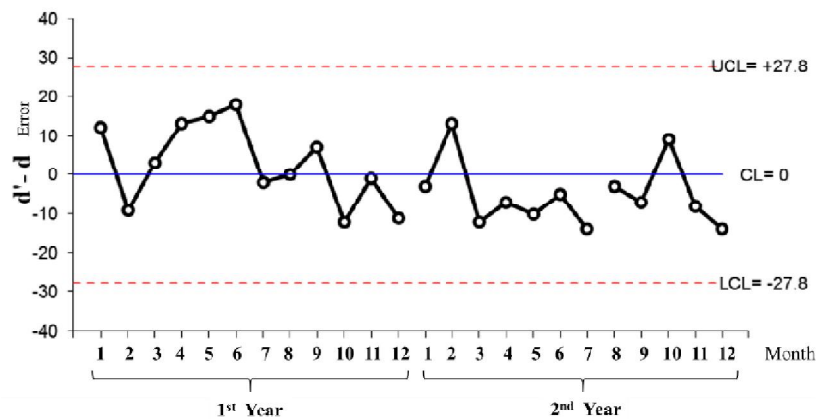
Fig. 6-19 Final control cart for 24-period data of Examples 6-10 and 6-15 (based on Biegel, 1971)

It is found the chart shows statistical control. There fore d'=102 is a better forecaster than d'=99.   This same cart should be used until we have evidence of lack of control(Biegle, 1971 p62).▲

Example 6-16

In Example 6-11, suppose the actual demand values for the 7 month of Year 2 (i.e Periods 13 through19) were 209, 226,224, 221, 250, 235, 233.

a)Use the same forecaster and error chart used in Example 11-6 to show the forecast error for Period 13 to 16 and comment.

b) If there is an indication of out-of-control status , What is your suggestion?

Solution

a)

Using $d'_t = 193 + 3t$ to forecast the demand for periods 13 through 15 and calculate error = d'-d yields:

If we plot the errors of Periods 13-16 on   the chart of Example 6-11 we would obtain the following chart. Of 5 successive points 12,13,14,.15, 16, four points fall above the central line in Region B, which indicates a state of out of control.
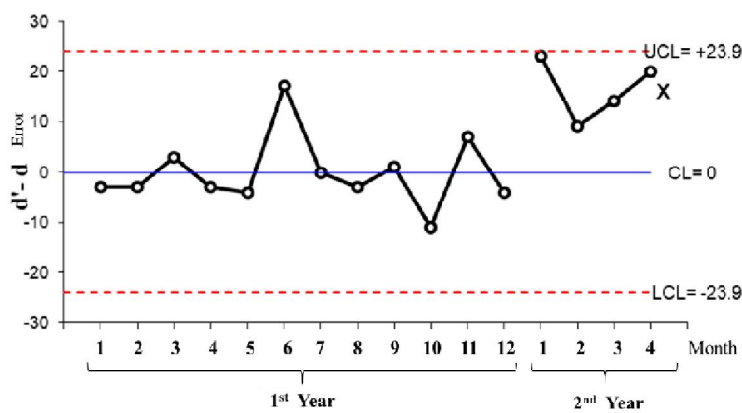
Fig 6-20 Control chart for first 16 month of Example 6-11
(Bielgel 1971 page 64)

b)

A new for regression forecaster based on the 16-month demand data is calculated using the following MATLAB commands:

y=[199   202   199   208   212   194   214   220   219   234   219   233 209   226 224   221]';t=[1:16]'; T=[ones(size(t)) t];ab=regress(y,T)

Although the answer is a=198.9000 and b=1.8426, but the forecaster is chosen as: $d' = 199 + 2t$.

The new limits (based on 16 periods) are:

$$\overline{MR} = \frac{99+27+14+5+6}{15} \cong 10.1 \qquad UCL = 26.9 \qquad LCL = -26.9$$

The new chart for 16 periods is shown in Fig 6.21. Since it shows a sate of in control, it it is concluded that the new forecasting function is satisfactory(Biegel,1971, page63).
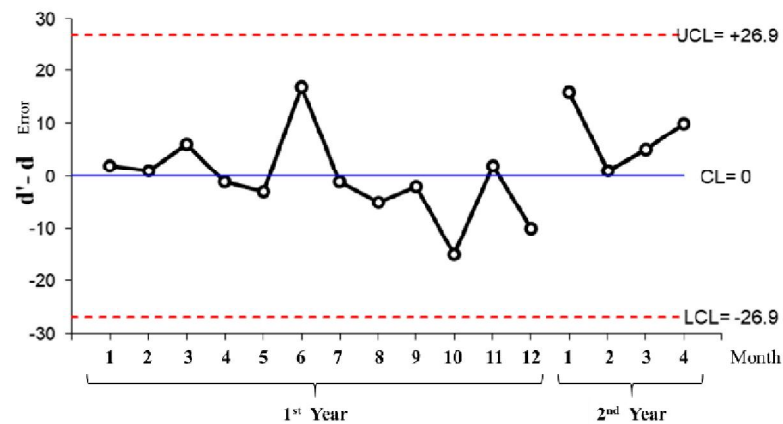
Fig 6-21 control chart for !6-period data of Examples 6-11 & 6-15
(based on Biegel, 1971)

Although the data for the rest of Year 2 is not given in this example, the control chart for both years has been redrawn in Fig. 6.22 from Fig. 4.8 on page 63 Biegel(1971).
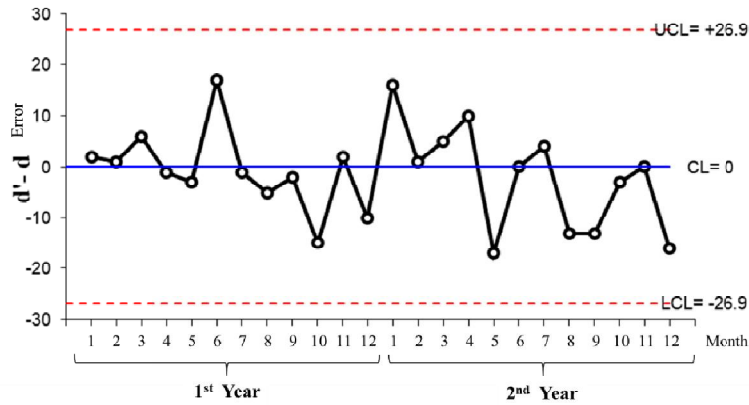
Fig 6-22 New chart for 24 periods of Examples 6-11 &6-16
(Biegel, 1971 p65)

This control chart should be used until a sign of out of control
appears.▲

Example 6-17

  Plot the actual and predicted values related to 19 periods of
Example 6-16 in an X-Y coordinates and calculate RMSE. How
do you evaluate the forecasting function $d' = 199 + 2t$?
Solution

month	1	2	3	4	5	6	7	8	9	10
demand(y)	199	202	199	208	212	194	214	220	219	234
Forecast($\hat{y}$)	196	199	202	205	208	211	214	217	220	223
month	11	12	13	14	15	16	17	18	19	
demand(y)	219	233	209	226	224	221	233	235	250	
Forecast($\hat{y}$)	226	230	238	241	244	247	250	253	256	

The above table shows the values   and the following MATLAB
commands has plotted Fig. 6.23

```
>> yhat=[196  199  202 205  208  211  214  217 220 223 226 230 238
241 244 247 250 253 256]';y=[199 202 199 208 212 194 214 220 219
234 219 233  209 226 224 221 233 235 250]';
Y=[ones(size(y)) y];ab=regress(yhat,Y);plot(y,yhat,'+');
```

xp=190:0.01:250;yp=ab(1)+ab(2)*xp;hold on;plot(xp,yp)
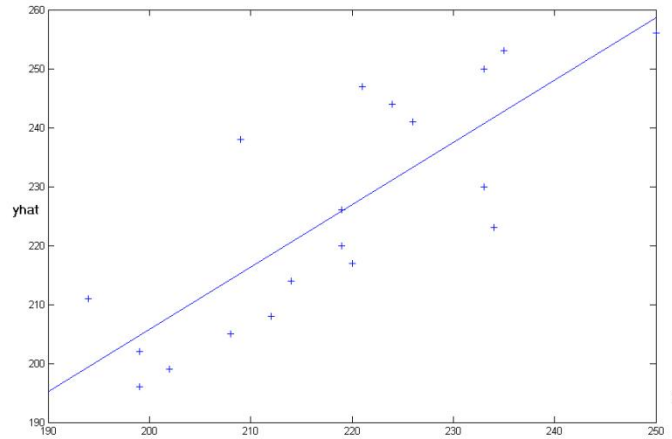
Fig 6-23 Actual demand(y) and forecast (yhat)
for 19 months of Example 6-16

RMSE=sqrt(mse(y-yhat)) in MATLAB gives RMSE= 13.2208 .

From Fig 5-23 it is evident that the points are around the line and the line is near to the bisector of the first quarter.   Then the forecasts could be acceptable. ▲

Exercises

1-Thefollowing table shows the maintenance cost per annum for a kind of vehicle versus the age of the vehicle and annual vehicle mileage.

Maintenance Cost	832	733	647	553	467	373	283	189	96
Annual Mileage(×1000)	6	7	9	11	13	15	17	18	19
Age at the beginning of the current year	8	7	6	5	4	3	2	1	0

a)Find the correlation coefficient   between the cost   and the mileage and between the cost and the age.

b) Find the simple regression forecaster for the   better correlation coefficient obtained in a.

c) Use a software such as MATLAB, Minitab, Lotus to forecast the cost form a 2-variabe(age and mileage) regression equation.

d)What would be the cost forecast from the forecasters in b and c if the age and the mileage are 3.5 years an 16000 respectively.

2. Find the regression equation for predicting cost from age in Problem 1. Forecast all the costs from the given corresponding age in the table. How much is RMSE between the actual costs and the predicted costs? How much is the correlation coefficient between the age and the predicted costs.   Use the t-test for paired data to compare the mean of the actual costs and the predicted costs($\alpha = 10\%$).

3. Suppose in the past 10 years, the increase in the price of iron compared to the price in Year 0 is as given in the following table.   Also suppose the increase in the price of a specific commodity for the same time horizon is also given in the table. Could it be concluded that the increase in the price of the commodity compared to Year 0 is proportional to the price increase of iron?

Is it better to forecast the increase in the commodity price from the increase in the price of iron or from year no. ?

Year No.	1	2	3	4	5	6	7	8	9	10
Increase In Iron price	100	102	103	110	121	124	127	130	139	145
Increase In the commodity price	100	103	106	119	126	127	128	123	140	144

4.A vendor believes that the demand for one of his goods depend on the number of houses built exactly 3 month ago in a district. Use the following table to verify his claim. Find the regression relationship to forecast the sale of the vendor from the number houses. Is it better to forecast the increase in the slae from the number houses built exactly 3 month ago   o from month no. ?

Month no.	1	2	3	4	5	6	7	8	9	10	11	12
Sale volume (× 1000)	45	60	62	30	40	45	68	75	80	45	30	25
Houses built(× 10)	26	25	32	38	50	48	32	40	35	25	10	15

5.Using the time series given in the following table, determine which N has the least RMSE for using in N-period simple moving average (with equal weighting)?

t(month)	1	2	3	4	5	6
sale	30	32	30	39	32	34

6.In Problem 5 if we want to replace moving average with simple exponential smoothing, calculate the   appropriate α . Forecast the sale volume for Month 7 if

a) the sale forecast for Month 1 is 32

b)we want to use the mean of the data as the forecast required in the exponential smoothing formula.

7. Use the following data and 2-period   weighted moving average to forecast the quantity for Periods 3 through 10. Use a weight of 0.55 for just the previous month and a weight of 0.45 for the other month.

Month	Ja	F	M	Ap	M	J	Ju	Au	S
Quantity	19.36	25.45	19.13	21.48	20.77	25.42	23.79	28.35	26.80

8. Choose ratio-to-trend algorithm to forecast the quantity for all periods of the previous problem. Calculate The RMSE and the correlation coefficient between the actual and the predicted quantities. Apply the paired data t-test. Compare this algorithm with the one used in Problem 7.

9. The following table shows the sale volume of a store of home appliances during the past 10 half-years.   Predict the sale for Year 7.

Half-year	1	2	3	4	5	6	7	8	9	10
sale	15.5	14.2	15.1	12.9	14.8	12.5	14.4	13.2	16.50	15

10. The demand for a product in January was 65 and during the previous year were as given in the table below. Forecast the demand for February using the regression method and double exponential smoothing with $\alpha = \beta = 0.1$.

t(month)	1	2	3	4	5	6	7	8	9	10	11	12
demand	52	48	36	49	65	54	60	48	51	62	66	62

11. The following data shows various   thickness of a plastic reservoir and the corresponding air pressure blown when it was being produced. Is there a linear correlation between the air pressure and the thickness?

Air pressure (kgf/cm^2)	10.0	9.5	9.0	8.5	8.0
Wall thickness(mm)	1.83	2.86	3.21	4.12	4.62
	2.02	2.53	3.05	3.88	4.50
	2.24	2.71	3.16	4.01	4.43
	1.95	2.62	3.30	3.67	4.81

12.For both methods of Problem 10, plot the control chart described in this chapter. What meth the control chart suggest to use?

13. The demand for a product during a year was as follows:

t	1	2	3	4	5	6	7	8	9	10	11	12
D	80	100	79	98	95	104	80	98	102	96	115	88

a) After determining the parameters of the following forecasts from the above data (if, necessary use least squatted error method),plot the   forecast error control chart for each of above-mentioned methods.

b) Suppose the demands for the next 12 months are 90,105,97,100,117,101,103,95,87,80,78,79 and continue one of the control charts. Is the forecaster acceptable?

14. After learning artificial neural networks of type Multilayer Perceptron (MLP), write some MATLAB commands for creating an MLP with two hidden layers and use Moore's data set in MATLAB to train it. Then simulate

$y=[-0.2218\ -0.3979\ -0.5229\ -0.0458]$

related to moore(17:20,1:5)   i.e. the rows 17 through 20 columns 1:5 of the data set.

Hint: the following commands could be used iun MATLAB[1];
Creating MLP
        net=newff(p,y,[1 11],{'tansig','logsig', 'purelin'})
instructions for training
input matrix: moore(1:16,1:5)
target vector: moore(1:16,6)
        load moore;
        p=(moore(1:16,1:5))';
        T=(moore(1:16,6))';
        net=train(shabake,p,T);
To simulate y:

---

[1] These commands were edited by the Late F. M Pourhosseini, the student of our department.

```
p2=(moore(17:20,1:5))';
yhat=sim(net,p2)
Calculation of RMSE between y and yhat:
y=[-0.2218 -0.3979 -0.5229 -0.0458];
rmse=sqrt(mse(y-yhat))
```

In general to forecast Vector y2 from Input Matrix P1 the MATLAB instructions for creating, training and simulation of an MLP with 1 hidden layer and using Matrix P as input and Vector y as target for training could written as follows:

```
P=…..;
y=….';
net=newff(P,y,[1 11],{'tansig', 'tansig', 'purelin'})
net.trainparam.epochs=100;
net=train(net,P,y);
P1=…;
yhat=sim(net,P1);
y2=[…];
rmse=sqrt(mse(y2-yhat))
```

The last instruction calculates the root mean squared error of given vector and its forecast by the MLP.

If youth but knew,
If old age but could,

Si jeunesse savait, Si vieillesse pouvait,

(French proverb)

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Tables

Table A   unit Loss Normal Integrals

$$G_U(k) = \int_k^\infty (u - k) \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

MATLAB:Guk= exp(-k^2/2)/sqrt(2*pi)-k*(1-normcdf(k))

Multiply the values by 10^{-4} e.g. Gu (0.28)=0.2745

 For values k<0 Gu (k) = Gu (-k) – k   e.g.: Gu (-2) = 0.0085+ 2 = 2.0085

k=-2;exp(-k^2/2)/sqrt(2*pi)-k*(1-normcdf(k)) → 2.0085

k	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	3989	3940	3890	384 1	3793	3744	3697	3649	3602	3556
0.1	3509	3464	3418	3373	3328	3284	3240	3197	3154	3111
0.2	3069	3027	2986	2944	2904	2863	2826	2784	2745	2706
0.3	3668	2630	2592	2555	2518	248	2445	2409	2374	2339
0.4	2304	2270	2236	2203	2169	2137	2104	2072	2040	2009
0.5	1978	1947	1917	1887	1857	1828	1800	1771	1742	1714
0.6	1687	1659	1632	1606	1580	1554	1528	1503	1478	1453
0.7	1429	1405	1381	1358	1334	1312	1289	1267	1245	1223
0.8	1202	1181	1160	1140	1120	1100	1080	1061	1042	1023
0.9	1004	0986	0968	0950	0933	0916	0899	0882	0865	0849
1.0	0833	0817	0802	0787	0772	0757	0742	0728	0714	0700
1.1	0686	0673	0660	0646	0634	0621	0609	0596	0584	0573
1.2	0561	0550	0538	0527	0577	0506	0495	0485	0475	0465
1.3	0455	0466	0436	0472	0418	0409	0400	0392	0383	0375
1.4	03 6	0359	0351	0343	0336	0328	0321	0314	0307	0300
1.5	029 3	0286	0280	0274	0267	0261	0255	0249	0244	0238
1.6	0212	0227	0222	0216	0211	0206	0201	0197	0192	0187
1.7	0183	0178	0174	0 1 70	0166	0162	0158	0154	0150	0146
1.8	0143	0139	0136	0132	0129	0126	0123	0119	0116	0113
1.9	0111	0 1 08	0105	0102	0100	0097	0094	0092	0090	0087
2.0	0085	0083	0080	0078	0076	0074	0072	0070	0068	0066
2.1	0065	0061	0061	0060	0058	0056	0055	0053	0052	0050
2.2	0049	004 8	0046	0045	0044	0042	004 1	0040	0039	0038
2.3	003 7	0036	0035	0034	0033	0032	0031	0030	0029	0028
2.4	0027	0026	0026	0025	0024	0023	0023	0022	0021	0021
2.5	0020	0019	00 19	0018	0018	0017	0017	0016	0016	0015
2.6	0015	0014	0014	0013	0013	0012	0012	0012	0011	0011
2.7	0011	0010	0010	0010	0009	0009	0009	0008	0008	0008
2.8	0008	0007	0007	0007	0007	0006	0006	0006	0006	0006
2.9	0005	0005	0005	0005	0005	0005	0004	0004	0004	0004

:Adopted from: Love, S.F. 19 79, Inventory Control   McGraw Hill

λ or np	K														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0.01	0.990	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.02	0.980	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.03	0.970	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.04	0.961	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.05	0.951	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.06	0.942	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.07	0.932	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.08	0.923	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.09	0.914	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.10	0.905	0.995	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.11	0.896	0.994	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.12	0.887	0.993	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.13	0.878	0.992	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.14	0.869	0.991	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.15	0.861	0.990	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.16	0.852	0.988	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.17	0.844	0.987	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.18	0.835	0.986	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.19	0.827	0.984	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.20	0.819	0.982	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.25	0.779	0.974	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.30	0.741	0.963	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.35	0.705	0.951	0.994	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.40	0.670	0.938	0.992	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.45	0.638	0.925	0.989	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table B Cumulative Poisson Probabilities $\Pr(X \le x)$ (Adopted from Housyar,1985)

Table B Cumulative Poisson Probabilities $\Pr(X \leq x)$ (Adopted from Housyar,1985)

λ or np	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0.50	0.607	0.910	0.986	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.55	0.577	0.894	0.982	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.60	0.549	0.878	0.977	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.65	0.522	0.861	0.972	0.996	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.70	0.497	0.844	0.966	0.994	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.75	0.472	0.827	0.959	0.993	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.80	0.449	0.809	0.953	0.991	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.85	0.427	0.791	0.945	0.989	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	0.407	0.772	0.937	0.987	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.95	0.387	0.754	0.929	0.984	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.00	0.368	0.736	0.920	0.981	0.996	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.10	0.333	0.699	0.900	0.974	0.995	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.20	0.301	0.663	0.879	0.966	0.992	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.30	0.273	0.627	0.857	0.957	0.989	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.40	0.247	0.592	0.833	0.946	0.986	0.997	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.50	0.223	0.558	0.809	0.934	0.981	0.996	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.60	0.202	0.525	0.783	0.921	0.976	0.994	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.70	0.183	0.493	0.757	0.907	0.970	0.992	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.80	0.165	0.463	0.731	0.891	0.964	0.990	0.997	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.90	0.150	0.434	0.704	0.875	0.956	0.987	0.997	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2.00	0.135	0.406	0.677	0.857	0.947	0.983	0.995	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2.10	0.122	0.380	0.650	0.839	0.938	0.980	0.994	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2.20	0.111	0.355	0.623	0.819	0.928	0.975	0.993	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2.30	0.100	0.331	0.596	0.799	0.916	0.970	0.991	0.997	0.999	1.000	1.000	1.000	1.000	1.000	1.000

λ or np	Table B Cumulative Poisson Probabilities $\Pr(X \leq x)$ (Adopted from Housyar,1985)													
K	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2.40	0.091	0.308	0.570	0.779	0.904	0.964	0.988	0.997	0.999	1.000	1.000	1.000	1.000	1.000	1.000
2.50	0.082	0.287	0.544	0.758	0.891	0.958	0.986	0.996	0.999	1.000	1.000	1.000	1.000	1.000	1.000
2.60	0.074	0.267	0.518	0.736	0.877	0.951	0.983	0.995	0.999	1.000	1.000	1.000	1.000	1.000	1.000
2.70	0.067	0.249	0.494	0.714	0.863	0.943	0.979	0.993	0.998	0.999	1.000	1.000	1.000	1.000	1.000
2.80	0.061	0.231	0.469	0.692	0.848	0.935	0.976	0.992	0.998	0.999	1.000	1.000	1.000	1.000	1.000
2.90	0.055	0.215	0.446	0.670	0.832	0.926	0.971	0.990	0.997	0.999	1.000	1.000	1.000	1.000	1.000
3.00	0.050	0.199	0.423	0.647	0.815	0.916	0.966	0.988	0.996	0.999	1.000	1.000	1.000	1.000	1.000
3.10	0.045	0.185	0.401	0.625	0.798	0.906	0.961	0.986	0.995	0.999	1.000	1.000	1.000	1.000	1.000
3.20	0.041	0.171	0.380	0.603	0.781	0.895	0.955	0.983	0.994	0.998	1.000	1.000	1.000	1.000	1.000
3.30	0.037	0.159	0.359	0.580	0.763	0.883	0.949	0.980	0.993	0.998	1.000	1.000	1.000	1.000	1.000
3.40	0.033	0.147	0.340	0.558	0.744	0.871	0.942	0.977	0.992	0.997	1.000	1.000	1.000	1.000	1.000
3.50	0.030	0.136	0.321	0.537	0.725	0.858	0.935	0.973	0.990	0.997	1.000	1.000	1.000	1.000	1.000
3.60	0.027	0.126	0.303	0.515	0.706	0.844	0.927	0.969	0.988	0.996	1.000	1.000	1.000	1.000	1.000
3.70	0.025	0.116	0.285	0.494	0.687	0.830	0.918	0.965	0.986	0.995	1.000	1.000	1.000	1.000	1.000
3.80	0.022	0.107	0.269	0.473	0.668	0.816	0.909	0.960	0.984	0.994	0.998	0.999	1.000	1.000	1.000
3.90	0.020	0.099	0.253	0.453	0.648	0.801	0.899	0.955	0.981	0.993	0.998	0.999	1.000	1.000	1.000
4.00	0.018	0.092	0.238	0.433	0.629	0.785	0.889	0.949	0.979	0.992	0.997	0.999	1.000	1.000	1.000
4.10	0.017	0.085	0.224	0.414	0.609	0.769	0.879	0.943	0.976	0.990	1.000	0.999	1.000	1.000	1.000
4.20	0.015	0.078	0.210	0.395	0.590	0.753	0.867	0.936	0.972	0.989	1.000	0.999	1.000	1.000	1.000
4.30	0.014	0.072	0.197	0.377	0.570	0.737	0.856	0.929	0.968	0.987	1.000	0.998	0.999	1.000	1.000
4.40	0.012	0.066	0.185	0.359	0.551	0.720	0.844	0.921	0.964	0.985	0.990	0.998	0.999	1.000	1.000
4.50	0.011	0.061	0.174	0.342	0.532	0.703	0.831	0.913	0.960	0.983	0.990	0.998	0.999	1.000	1.000
4.60	0.010	0.056	0.163	0.326	0.513	0.686	0.818	0.905	0.955	0.980	0.990	0.997	0.999	1.000	1.000
4.70	0.009	0.052	0.152	0.310	0.495	0.668	0.805	0.896	0.950	0.978	0.990	0.997	0.999	1.000	1.000

λ or np	K 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Table B Cumulative Poisson Probabilities $\Pr(X \le x)$ (Adopted from Housyar,1985)															
4.80	0.008	0.048	0.143	0.294	0.476	0.651	0.791	0.887	0.944	0.975	0.990	0.996	0.999	1.000	1.000
4.90	0.007	0.044	0.133	0.279	0.458	0.634	0.777	0.877	0.938	0.972	0.990	0.995	0.998	0.999	1.000
5.00	0.007	0.040	0.125	0.265	0.440	0.616	0.762	0.867	0.932	0.968	0.990	0.995	0.998	0.999	1.000
5.20	0.006	0.034	0.109	0.238	0.406	0.581	0.732	0.845	0.918	0.960	0.980	0.993	0.997	0.999	1.000
5.40	0.005	0.029	0.095	0.213	0.373	0.546	0.702	0.822	0.903	0.951	0.980	0.990	0.996	0.999	1.000
5.60	0.004	0.024	0.082	0.191	0.342	0.512	0.670	0.797	0.886	0.941	0.970	0.988	0.995	0.998	0.999
5.80	0.003	0.021	0.072	0.170	0.313	0.478	0.638	0.771	0.867	0.929	0.970	0.984	0.993	0.997	0.999
6.00	0.002	0.017	0.062	0.151	0.285	0.446	0.606	0.744	0.847	0.916	0.960	0.980	0.991	0.996	0.999
6.20	0.002	0.015	0.054	0.134	0.259	0.414	0.574	0.716	0.826	0.902	0.950	0.975	0.989	0.995	0.998
6.40	0.002	0.012	0.046	0.119	0.235	0.384	0.542	0.687	0.803	0.886	0.940	0.969	0.986	0.994	0.997
6.60	0.001	0.010	0.040	0.105	0.213	0.355	0.511	0.658	0.780	0.869	0.930	0.963	0.982	0.992	0.997
6.80	0.001	0.009	0.034	0.093	0.192	0.327	0.480	0.628	0.755	0.850	0.920	0.955	0.978	0.990	0.996
7.00	0.001	0.007	0.030	0.082	0.173	0.301	0.450	0.599	0.729	0.830	0.900	0.947	0.973	0.987	0.994
7.20	0.001	0.006	0.025	0.072	0.156	0.276	0.420	0.569	0.703	0.810	0.890	0.937	0.967	0.984	0.993
7.40	0.001	0.005	0.022	0.063	0.140	0.253	0.392	0.539	0.676	0.788	0.870	0.926	0.961	0.980	0.991
7.60	0.001	0.004	0.019	0.055	0.125	0.231	0.365	0.510	0.648	0.765	0.850	0.915	0.954	0.976	0.989
7.80	0.000	0.004	0.016	0.048	0.112	0.210	0.338	0.481	0.620	0.741	0.840	0.902	0.945	0.971	0.986
8.00	0.000	0.003	0.014	0.042	0.100	0.191	0.313	0.453	0.593	0.717	0.820	0.888	0.936	0.966	0.983
8.20	0.000	0.003	0.012	0.037	0.089	0.174	0.290	0.425	0.565	0.692	0.800	0.873	0.926	0.960	0.979
8.40	0.000	0.002	0.010	0.032	0.079	0.157	0.267	0.399	0.537	0.666	0.770	0.857	0.915	0.952	0.975
8.60	0.000	0.002	0.009	0.028	0.070	0.142	0.246	0.373	0.509	0.640	0.750	0.840	0.903	0.945	0.970
8.80	0.000	0.001	0.007	0.024	0.062	0.128	0.226	0.348	0.482	0.614	0.730	0.822	0.890	0.936	0.965
9.00	0.000	0.001	0.006	0.021	0.055	0.116	0.207	0.324	0.456	0.587	0.710	0.803	0.876	0.926	0.959

λ or np	Table B Cumulative Poisson Probabilities $\Pr(X \leq x)$ (Adopted from Housyar,1985)													
K	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
9.20	0.000	0.001	0.005	0.018	0.049	0.104	0.189	0.301	0.430	0.561	0.680	0.783	0.861	0.916	0.952
9.40	0.000	0.001	0.005	0.016	0.043	0.093	0.173	0.279	0.404	0.535	0.660	0.763	0.845	0.904	0.944
9.60	0.000	0.001	0.004	0.014	0.038	0.084	0.157	0.258	0.380	0.509	0.630	0.741	0.828	0.892	0.936
9.80	0.000	0.001	0.003	0.012	0.033	0.075	0.143	0.239	0.356	0.483	0.610	0.719	0.810	0.879	0.927
10.00	0.000	0.000	0.003	0.010	0.029	0.067	0.130	0.220	0.333	0.458	0.580	0.697	0.792	0.864	0.917
10.50	0.000	0.000	0.002	0.007	0.021	0.050	0.102	0.179	0.279	0.397	0.520	0.639	0.742	0.825	0.888
11.00	0.000	0.000	0.001	0.005	0.015	0.038	0.079	0.143	0.232	0.341	0.460	0.579	0.689	0.781	0.854
11.50	0.000	0.000	0.001	0.003	0.011	0.028	0.060	0.114	0.191	0.289	0.400	0.520	0.633	0.733	0.815
12.00	0.000	0.000	0.001	0.002	0.008	0.020	0.046	0.090	0.155	0.242	0.350	0.462	0.576	0.682	0.772
12.50	0.000	0.000	0.000	0.002	0.005	0.015	0.035	0.070	0.125	0.201	0.300	0.406	0.519	0.628	0.725
13.00	0.000	0.000	0.000	0.001	0.004	0.011	0.026	0.054	0.100	0.166	0.250	0.353	0.463	0.573	0.675
13.50	0.000	0.000	0.000	0.001	0.003	0.008	0.019	0.041	0.079	0.135	0.210	0.304	0.409	0.518	0.623
14.00	0.000	0.000	0.000	0.000	0.002	0.006	0.014	0.032	0.062	0.109	0.180	0.260	0.358	0.464	0.570
14.50	0.000	0.000	0.000	0.000	0.001	0.004	0.010	0.024	0.048	0.088	0.140	0.220	0.311	0.413	0.518
15.00	0.000	0.000	0.000	0.000	0.001	0.003	0.008	0.018	0.037	0.070	0.120	0.185	0.268	0.363	0.466
15.50	0.000	0.000	0.000	0.000	0.001	0.002	0.006	0.013	0.029	0.055	0.100	0.154	0.228	0.317	0.415
16.00	0.000	0.000	0.000	0.000	0.000	0.001	0.004	0.010	0.022	0.043	0.080	0.127	0.193	0.275	0.368
16.50	0.000	0.000	0.000	0.000	0.000	0.001	0.003	0.007	0.017	0.034	0.060	0.104	0.162	0.236	0.323
17.00	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.005	0.013	0.026	0.050	0.085	0.135	0.201	0.281
17.50	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.004	0.009	0.020	0.040	0.068	0.112	0.170	0.243
18.00	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.003	0.007	0.015	0.030	0.055	0.092	0.143	0.208
18.50	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.005	0.012	0.020	0.044	0.075	0.119	0.177
19.00	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.004	0.009	0.020	0.035	0.061	0.098	0.150
19.50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.003	0.007	0.010	0.027	0.049	0.081	0.126

λ or np	Table B Cumulative Poisson Probabilities Pr($X \leq x$) (Adopted from Housyar,1985)														
	K														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
20.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.005	0.010	0.021	0.039	0.066	0.105
20.50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.004	0.010	0.017	0.031	0.054	0.087
21.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.003	0.010	0.013	0.025	0.043	0.072
21.50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.005	0.010	0.019	0.035	0.059
22.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.004	0.008	0.015	0.028	0.048
22.50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.003	0.006	0.012	0.022	0.039
23.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.004	0.009	0.017	0.031
23.50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.003	0.007	0.014	0.025
24.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.003	0.005	0.011	0.020
24.50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.004	0.008	0.016

$z = \frac{x-\mu}{\sigma}$	Table C Area under normal curve from $-\infty$ to $z = \frac{x-\mu}{\sigma}$: Pr $(Z \le z)$.									
	Example Pr($z \le -3.00$) = 0.00135									
	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00
-3.5	0.00017	0.00017	0.00018	0.00019	0.00019	0.0002	0.00021	0.00022	0.00022	0.00023
-3.4	0.00024	0.00025	0.00026	0.00027	0.00028	0.00029	0.0003	0.00031	0.00032	0.00034
-3.3	0.00035	0.00036	0.00038	0.00039	0.0004	0.00042	0.00043	0.00045	0.00047	0.00048
-3.2	0.0005	0.00052	0.00054	0.00056	0.00058	0.0006	0.00062	0.00064	0.00066	0.00069
-3.1	0.00071	0.00074	0.00076	0.00079	0.00082	0.00084	0.00087	0.0009	0.00094	0.00097
-3	0.001	0.00104	0.00107	0.00111	0.00114	0.00118	0.00122	0.00126	0.00131	0.00135
-2.9	0.00139	0.00144	0.00149	0.00154	0.00159	0.00164	0.00169	0.00175	0.00181	0.00187
-2.8	0.00193	0.00199	0.00205	0.00212	0.00219	0.00226	0.00233	0.0024	0.00248	0.00256
-2.7	0.00264	0.00272	0.0028	0.00289	0.00298	0.00307	0.00317	0.00326	0.00336	0.00347
-2.6	0.00357	0.00368	0.00379	0.00391	0.00402	0.00415	0.00427	0.0044	0.00453	0.00466
-2.5	0.0048	0.00494	0.00508	0.00523	0.00539	0.00554	0.0057	0.00587	0.00604	0.00621
-2.4	0.00639	0.00657	0.00676	0.00695	0.00714	0.00734	0.00755	0.00776	0.00798	0.0082
-2.3	0.00842	0.00866	0.00889	0.00914	0.00939	0.00964	0.0099	0.01017	0.01044	0.01072
-2.2	0.01101	0.01130	0.0116	0.01191	0.01222	0.01255	0.01287	0.01321	0.01355	0.01390
-2.1	0.01426	0.01463	0.015	0.01539	0.01578	0.01618	0.01659	0.01700	0.01743	0.01786
-2	0.01831	0.01876	0.01923	0.0197	0.02018	0.02068	0.02118	0.02169	0.02222	0.02275
-1.9	0.0233	0.02385	0.02442	0.025	0.02559	0.02619	0.0268	0.02743	0.02807	0.02872
-1.8	0.02938	0.03005	0.03074	0.03144	0.03216	0.03288	0.03362	0.03438	0.03515	0.03593
-1.7	0.03673	0.03754	0.03836	0.0392	0.04006	0.04093	0.04182	0.04272	0.04363	0.04457

Table C (continued)   Area under normal curve : Pr $(Z \le z)$ Example $\Pr(z < -1.06) = 0.14457$

$z = \dfrac{x-\mu}{\sigma}$	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00
-1.6	0.04551	0.04648	0.04746	0.04846	0.04947	0.0505	0.05155	0.05262	0.0537	0.0548
-1.5	0.05592	0.05705	0.05821	0.05938	0.06057	0.06178	0.06301	0.06426	0.06552	0.06681
-1.4	0.06811	0.06944	0.07078	0.07215	0.07353	0.07493	0.07636	0.0778	0.07927	0.08076
-1.3	0.08226	0.08379	0.08534	0.08691	0.08851	0.09012	0.09176	0.09342	0.0951	0.0968
-1.2	0.09853	0.10027	0.10204	0.10383	0.10565	0.10749	0.10935	0.11123	0.11314	0.11507
-1.1	0.11702	0.119	0.121	0.12302	0.12507	0.12714	0.12924	0.13136	0.1335	0.13567
-1	0.13786	0.14007	0.14231	0.14457	0.14686	0.14917	0.15151	0.15386	0.15625	0.15866
-0.9	0.16109	0.16354	0.16602	0.16853	0.17106	0.17361	0.17619	0.17879	0.18141	0.18406
-0.8	0.18673	0.18943	0.19215	0.19489	0.19766	0.20045	0.20327	0.20611	0.20897	0.21186
-0.7	0.21476	0.2177	0.22065	0.22363	0.22663	0.22965	0.2327	0.23576	0.23885	0.24196
-0.6	0.2451	0.24825	0.25143	0.25463	0.25785	0.26109	0.26435	0.26763	0.27093	0.27425
-0.5	0.2776	0.28096	0.28434	0.28774	0.29116	0.2946	0.29806	0.30153	0.30503	0.30854
-0.4	0.31207	0.31561	0.31918	0.32276	0.32636	0.32997	0.3336	0.33724	0.3409	0.34458
-0.3	0.34827	0.35197	0.35569	0.35942	0.36317	0.36693	0.3707	0.37448	0.37828	0.38209
-0.2	0.38591	0.38974	0.39358	0.39743	0.40129	0.40517	0.40905	0.41294	0.41683	0.42074
-0.1	0.42465	0.42858	0.43251	0.43644	0.44038	0.44433	0.44828	0.45224	0.4562	0.46017
0	0.46414	0.46812	0.4721	0.47608	0.48006	0.48405	0.48803	0.49202	0.49601	0.5

Table C(continued) Area under Normal curve   Pr $(Z \leq z)$ Example: $\mathrm{Pr}(z < 3.44) = 0.99971$

$z = \dfrac{x-\mu}{\sigma}$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999
3.1	0.99903	0.99906	0.9991	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.9994	0.99942	0.99944	0.99946	0.99948	0.9995
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.9996	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.9997	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.9998	0.99981	0.99981	0.99982	0.99983	0.99983

$Z = \left\|\frac{x-\mu}{\sigma}\right\|$	Table C(continued) Area under normal curve Pr $(Z \leq z)$									
	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062

Table D Area under normal curve from Z_α to ∞: $Pr(Z > Z_\alpha) = \alpha$

Z_α	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5	0.49601	0.49202	0.48803	0.48405	0.48006	0.47608	0.4721	0.46812	0.46414
0.1	0.46017	0.4562	0.45224	0.44828	0.44433	0.44038	0.43644	0.43251	0.42858	0.42465
0.2	0.42074	0.41683	0.41294	0.40905	0.40517	0.40129	0.39743	0.39358	0.38974	0.38591
0.3	0.38209	0.37828	0.37448	0.3707	0.36693	0.36317	0.35942	0.35569	0.35197	0.34827
0.4	0.34458	0.3409	0.33724	0.3336	0.32997	0.32636	0.32276	0.31918	0.31561	0.31207
0.5	0.30854	0.30503	0.30153	0.29806	0.2946	0.29116	0.28774	0.28434	0.28096	0.2776
0.6	0.27425	0.27093	0.26763	0.26435	0.26109	0.25785	0.25463	0.25143	0.24825	0.2451
0.7	0.24196	0.23885	0.23576	0.2327	0.22965	0.22663	0.22363	0.22065	0.2177	0.21476
0.8	0.21186	0.20897	0.20611	0.20327	0.20045	0.19766	0.19489	0.19215	0.18943	0.18673
0.9	0.18406	0.18141	0.17879	0.17619	0.17361	0.17106	0.16853	0.16602	0.16354	0.16109
1	0.15866	0.15625	0.15386	0.15151	0.14917	0.14686	0.14457	0.14231	0.14007	0.13786
1.1	0.13567	0.1335	0.13136	0.12924	0.12714	0.12507	0.12302	0.121	0.119	0.11702
1.2	0.11507	0.11314	0.11123	0.10935	0.10749	0.10565	0.10383	0.10204	0.10027	0.09853
1.3	0.0968	0.0951	0.09342	0.09176	0.09012	0.08851	0.08691	0.08534	0.08379	0.08226
1.4	0.08076	0.07927	0.0778	0.07636	0.07493	0.07353	0.07215	0.07078	0.06944	0.06811
1.5	0.06681	0.06552	0.06426	0.06301	0.06178	0.06057	0.05938	0.05821	0.05705	0.05592
1.6	0.0548	0.0537	0.05262	0.05155	0.0505	0.04947	0.04846	0.04746	0.04648	0.04551
1.7	0.04457	0.04363	0.04272	0.04182	0.04093	0.04006	0.0392	0.03836	0.03754	0.03673

Table D (continued) some values of $\boldsymbol{Z_\alpha}$ $\alpha = 0.05$ $Z_{\frac{\alpha}{2}} = 1.96$									
Z_α	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.8	0.03593	0.03515	0.03438	0.03362	0.03288	0.03216	0.03144	0.03074	0.03005	0.02938
1.9	0.02872	0.02807	0.02743	0.0268	0.02619	0.02559	0.025	0.02442	0.02385	0.0233
2	0.02275	0.02222	0.02169	0.02118	0.02068	0.02018	0.0197	0.01923	0.01876	0.01831
2.1	0.01786	0.01743	0.017	0.01659	0.01618	0.01578	0.01539	0.015	0.01463	0.01426
2.2	0.0139	0.01355	0.01321	0.01287	0.01255	0.01222	0.01191	0.0116	0.0113	0.01101
2.3	0.01072	0.01044	0.01017	0.0099	0.00964	0.00939	0.00914	0.00889	0.00866	0.00842
2.4	0.0082	0.00798	0.00776	0.00755	0.00734	0.00714	0.00695	0.00676	0.00657	0.00639
2.5	0.00621	0.00604	0.00587	0.0057	0.00554	0.00539	0.00523	0.00508	0.00494	0.0048
2.6	0.00466	0.00453	0.0044	0.00427	0.00415	0.00402	0.00391	0.00379	0.00368	0.00357
2.7	0.00347	0.00336	0.00326	0.00317	0.00307	0.00298	0.00289	0.0028	0.00272	0.00264
2.8	0.00256	0.00248	0.0024	0.00233	0.00226	0.00219	0.00212	0.00205	0.00199	0.00193
2.9	0.00187	0.00181	0.00175	0.00169	0.00164	0.00159	0.00154	0.00149	0.00144	0.00139
3	0.00135	0.00131	0.00126	0.00122	0.00118	0.00114	0.00111	0.00107	0.00104	0.001

Table E   MATLAB commands related to some distributions

Distribution	Parameter Estimator	Random number Generator	Inverse of cum. dist. Func.	Cum. Dist Func.	Prob. Dist. Fun/ Prob. Func
F		frnd(V1,V2,m,n	finv(P, V1,V2)	fcdf(x, V1,V2)	fpdf(x, V1,V2)
GEV	gevfit(X)	gevrnd(C,B,A)	gevinv	gevcdf(x,C,B,A)	gevpdf(C,B,A)
GPD	gpfit	gprnd	gpinv	gpcdf	gppdf
Rayleigh	raylfit(X)	raylrnd(B,m,n)	raylinv(P,B)	raylcdf(x,B)	raylpdf(x,B)
T		trnd(V,m,n)	tinv(P,V)	tcdf(x,V)	tpdf(x,V)
beta	betafit(X)	betarnd(A,B,m,	betainv(P,A,B)	betacdf(x,A,B)	betapdf(x,A,B)
Poisson	poissfit(X	poissrnd(λ	poissinv(P, λ	poisscdf(x, λ)	poisspdf(x, λ)
Binomial	binofit(X,	binornd(N,P,m,	binoinv(Y,N,P)	binocdf(x,N,P)	binopdf(x,N,P)
Negat. Bin.	nbinfit(X)	nbinrnd(R,P,m,	nbininv(Y,R,P)	nbincdf(x,R,P)	nbinpdf(x,R,P)
Hyper Geo		hygernd(M,K,N,	hygeinv(P,M,K	hygecdf(x,M,K,	hygepdf(x,M,K,
Gamma	gamfit(X)	gamrnd(A,B,m,	gaminv(P,A,B)	gamcdf(x,A,B)	gampdf(x,A,B)
Lognormal	lognfit(X)	lognrnd(μ,	logninv(P,μ, σ)	logncdf(x,μ, σ)	lognpdf(x,μ, σ)
Chi Squ.		chi2rnd(V,m,n)	chi2inv(P,V)	chi2cdf(x,V)	chi2pdf(x,V)
Normal	normfit(X	normrnd(μ,	norminv(P,μ,	normcdf(x μ,	normpdf(x, μ,
Exponential	expfit(X)	exprnd(mu,m,n	expinv(P,mu)	expcdf(x, mu)	exppdf(x, mu)
Geometric		geornd(P,m,n)	geoinv(Y,P)	geocdf(x,P)	geopdf(x,P)
Weibul	wblfit(X)	wblrnd(B,C,m,n	wblinv(P, B,C)	wblcdf(x, B,C)	wblpdf(x, B,C)
Uniform	unifit(X)	unifrnd(A,B,m,	unifinv(P,A,B)	unifcdf(x,A,B)	unifpdf(x,A,B)

Table F Some characteristics of 6 distributions

Distribution	Moment Gen Func $\phi(t)$	Variance	mean	Density /probability Function
Unifotm on [a b]	$\dfrac{e^{tb}-e^{ta}}{t(b-a)}$	$\dfrac{(b-a)^2}{12}$	$\dfrac{(a+b)}{2}$	$\dfrac{1}{b-a},a<x<b$
Exponential with $\lambda>0\ or\ \theta>0$	$\dfrac{\lambda}{\lambda-t}$	$\dfrac{1}{\lambda^2}$	$\theta=\dfrac{1}{\lambda}$	$\lambda e^{-\lambda x}\ \ or\ \ \dfrac{1}{\theta}e^{-\theta x}$
Normal with parametrs (μ,σ)	$\exp\left\{\mu t+\dfrac{\sigma^2 t^2}{2}\right\}$	σ^2	μ	$\dfrac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}\quad -\infty\langle x\langle\infty$
Binomial with parametrs n &0≤p≤1	$[pe^t+(1-p)]^n$	$np(1-p)$	np	$\binom{n}{x}p^x(1-p)^{n-x}\ \ x=0,1,\ldots,n$
Poisson with parameter $\lambda>0$	$\exp[\lambda(e^t-1)]$			$e^{-\lambda}\dfrac{\lambda^x}{x!}\ \ x=0,1,2,\ldots$
Weibul		$\dfrac{\pi^2 B^2}{6}$	$A+\gamma B,\gamma=$ 0.057720	$\dfrac{C}{B}(\dfrac{x-A}{B})^{C-1}e^{-(\frac{x-A}{B})^C}$ $x\geq A$

Table G Some useful formulas for Inventory Models

Model			TVC^*	r (ROP)	T or Q
Classic Econimic order(EOQ)	$\bar{I} = \dfrac{Q_W}{2}$	$Imax = Q_W$	$TC_W = \sqrt{2DC_OC_h} = C_hQ_W$	$\begin{cases} DT_L \\ DT_L - KQ^* \end{cases}$	$Q_{W=}\sqrt{\dfrac{2DC_O}{C_h}}$
EOQ- Discrete					$\begin{array}{l} Q^*(Q^* - n) \leq Q_W^2 \\ \leq Q^*(Q^* + n) \end{array}$
Multiple-item EOQ No constraint			$TVC^* = \displaystyle\sum_{i=1}^{n}\sqrt{2D_iC_{O_i}C_{h_i}}$		$Q_i^* = \sqrt{\dfrac{2D_iC_{O_i}}{C_{h_i}}}$
Multiple-item EOQ No constraint The same T One Co for ordering all together			$TVC^* = \dfrac{C_O}{T} + \displaystyle\sum_{j=1}^{n}C_{h_j}\left(\dfrac{D_jT}{2}\right)$		$Q_j^* = D_jT^*$ $T^* = \sqrt{\dfrac{2C_O}{\sum C_{h_j}D_j}}$
Multiple-item EOQ No constraint The same T separate Co for each item			$TVC^* = \displaystyle\sum_{j=1}^{n}C_{O_j}\left(\dfrac{1}{T}\right) + \displaystyle\sum_{j=1}^{n}C_{h_j}D_j\dfrac{T}{2}$		$Q_j^* = D_jT^*$ $T^* = \sqrt{\dfrac{2\sum C_{O_j}}{\sum C_{h_j}D_j}}$

Table G Some useful formulas for Inventory Models

Model			TVC^*	r (ROP)	T or Q
Economic order interval (Single-item EOI)		$Imax = DT^* + DT_L$			$Q^* = DT^* \quad T^* = \sqrt{\dfrac{2C_O}{DC_h}}$
Back-ordered EOQ($\pi \neq 0$)		$Imax = S^* = Q^* - b^*$		$\begin{cases} DT_L - b^* \\ DT_L - b^* - kQ^* \end{cases}$	$Q^* = \dfrac{\pi D}{C_h} + \left(1 + \dfrac{\hat{\pi}}{C_h}\right)b^*$ $b^* = \dfrac{1}{\hat{\pi} + C_h}(C_h Q^* - \pi D)$
Back-ordered EOQ($\hat{\pi} \neq 0$ $\pi = 0$)		$Imax = S^*$ $= Q^* \dfrac{\hat{\pi}}{\hat{\pi} + C_h}$	$TVC^* = TC_w \sqrt{\dfrac{\hat{\pi}}{\hat{\pi} + C_h}}$	$r = DT_L - b^*$	$Q^* = Q_w \sqrt{\dfrac{\hat{\pi} + C_h}{\hat{\pi}}}$ $b^* = Q^*\left(\dfrac{C_h}{\hat{\pi} + C_h}\right)$
Lost-sale EOQ					$\begin{cases} Q = 0 \quad T_2^* = \infty \quad\quad \pi D < TC_w \\ Q^* = \dfrac{\pi D}{C_h} \text{ هر مقدار } T_2^* \quad \pi D = TC_w \\ Q = Q_w \quad T_2^* = 0 \quad\quad \pi D < TC_w \end{cases}$
Temporary reduction in proce			$G^* = \dfrac{C_O(P-d)}{P}\left(\dfrac{Q^*}{Q_w} - 1\right)^2 = 0$		$Q^* = \dfrac{dD}{I(P-d)} + \dfrac{PQ_w}{P-d} - q$
EOQ-increase of price(inflation)			A)$G^* = C_O\left[\left(\dfrac{Q^*}{Q_w}\right)^2 - 1\right]$ B)$G^* = C_O\left(\dfrac{Q^*}{Q_w} - 1\right)^2$ q=ROP		$Q'^* = Q_a^* + \dfrac{a}{P}(IQ_a^* + D) - q$ $+ D\,T_L\, Q_a^*$ $= \sqrt{\dfrac{2DC_O}{I(P+a)}}$

Table G Some useful formulas for Inventory Models

Model			TVC^*	r (ROP)	T or Q
EPQ-single item	$\bar{I} = \dfrac{Q^*}{2} \times$ $\left(1 - \dfrac{D}{R}\right)$		$TVC^* = \sqrt{2DC_OC_h\left(1-\dfrac{D}{R}\right)}$ $= C_hQ_w\sqrt{1-\dfrac{D}{R}}$	$\begin{cases} DT_L - KQ \\ \quad T_L(D-R) + \\ \quad (K+1)\left(\dfrac{R}{D}-1\right)Q \end{cases}$	$EPQ = Q^* = \sqrt{\dfrac{2DC_O}{IP\left(1-\dfrac{D}{R}\right)}}$
EPQ-single item: Back ordered		$Imax = Q^*\left(1-\dfrac{D}{R}\right)$ $- b^*$			Q^* $= \sqrt{\dfrac{2DC_O}{C_h\left(1-\dfrac{D}{R}\right)} - \dfrac{\hat{\pi}^2D^2}{C_h(C_h+\hat{\pi})}}\sqrt{\dfrac{\hat{\pi}+C_h}{\hat{\pi}}}$ $b^* = \dfrac{[C_hQ^* - \pi D](1-\dfrac{D}{R})}{\hat{\pi}+C_h}$
EPQ-single item: Back ordered ($\pi = 0$)			$C_hQ^*\left(1-\dfrac{D}{R}\right)\sqrt{\dfrac{\hat{\pi}}{\hat{\pi}+C_h}}$		$Q^* = \sqrt{\dfrac{2DC_O}{C_h\left(1-\dfrac{D}{R}\right)}}\sqrt{\dfrac{\hat{\pi}+C_h}{\hat{\pi}}}$
Multiple-item EPQ No constraint	$\bar{I}_i = \dfrac{Q_i}{2} \times$ $\left(1 - \dfrac{D_i}{R_i}\right)$		$TVC^* =$ $\displaystyle\sum_{i=1}^{n}\sqrt{2D_iC_{O_i}C_{h_i}\left(1-\dfrac{D_i}{R_i}\right)}$		$Q_i^* = \sqrt{\dfrac{2D_iC_{O_i}}{C_{h_i}\left(1-\dfrac{D_i}{R_i}\right)}}$

Table G Some useful formulas for Inventory Models

Model			TVC^*	r (ROP)	T or Q
Multiple-item EPQ One station & The same T for all $\sum_{i=1}^{n}\frac{D_i}{R_i} < 1$	$\overline{I}_i = \frac{D_i}{2m} \times \left(1 - \frac{D_i}{R_i}\right)$		برای حالت $T^* = T_0^*$ $TVC^* = 2\sum_{i=1}^{n}\frac{(C_O)_i}{T^*}$ $= 2m^* \sum_{i=1}^{n}(C_O)_i$		$Q_i^* = D_i T^*$ $T^* = Max\{T_0^*, T_{min}\}$ $T_0^* = \sqrt{\frac{2\sum_{i=1}^{n}(C_O)_i}{\sum_{i=1}^{n}(C_h)_i D_i\left(1-\frac{D_i}{R_i}\right)}}$ $T_{min} = \frac{\sum S_i}{1 - \sum_{i=1}^{n}\frac{D_i}{R}}$
(r, Q)=FOS	$\overline{I} = \frac{Q}{2} + S.S$ $SS = r^* - E(DL)$ اگر D_L:نرمال $SS = Z_{1-p}\sigma_{DL}$ $\overline{b}(r) = \sigma_{D_L} G_U(k)$ $k = \frac{r-\mu_{D_L}}{\sigma_{D_L}}$	$Var(D_L)$ $= \mu_D^2\sigma_L^2 + \mu_L\sigma_D^2$		1)$p = Pr(D_L) \le r$ 2)$r =$ $= max(D) \times E(TL)$ 3)$r = max(TL) \times E(D)$ توزیع D_L:نرمال $r = E(D_L) + Z_{1-p}\sigma_{DL}$	$Q^* = \sqrt{\frac{2 C_o E(D)}{C_h}}$
(r Q) Back ordered	$SS = r - E(DL)$			A)$F_{D_L}(r^*)=1-\frac{C_h Q^*}{\pi D}$ B) $f_{D_L}(r^*) = \frac{C_h Q}{gD}$ $p_{D_L}(r^*) = \frac{C_h Q}{gD}$	

Table G Some useful formulas for Inventory Models

Model			TVC^*	r (ROP)	T or Q
(r Q) Lost-Sale	$SS =$ $r^* - E(D_L)$ $+\overline{b(r)}$			A) $\Pr(D_L > r^*) = \dfrac{C_h Q^*}{C_h Q^* + \pi E(D)}$ B) $\dfrac{f_{D_L}(r^*)}{F_{D_L}(r^*)} = \dfrac{C_h Q}{gD}$	$Q^* = \sqrt{\dfrac{2C_o E(D)}{C_h}}$
(R T) =FOI	$SS = R - E(D_{T+L})$ D_{T+L} نرمال : $SS = Z_{1-p}\,\sigma_{D_{T+L}}$ $\overline{b(R)} =$ $\sigma_{D_{T+L}} G_U(k)$ $k = \frac{R - \mu_{D_{L+T}}}{\sigma_{D_{L+T}}}$	$\sigma_{D_{T+L}} =$ $\sqrt{\mu_{T+L}\,\mathrm{Var}(D) + \mu_D^2\,\mathrm{Var}(T+L)}$		$Continuous\ D_{T+L}$ $F_{D_{T+L}}(R) = p$ $Discrete\ \ D_{T+L}$ $F_{D_{T+L}}(R) \ge p$ Normal $R = \mu_{D_{T+L}} + Z_{1-p}\,\sigma_{D_{T+L}}$	$T^* = \sqrt{\dfrac{2C_0}{C_h \mu_D}}$

Table G Some useful formulas for Inventory Models					
Model			TVC^*	r (ROP)	T or Q
(R T): back ordered				A) $\Pr(D_{L+T} > R^*) = \dfrac{C_h T^*}{\pi}$ B) $f_{D_{L+T}}(R^*) = \dfrac{C_h T}{g}$	
(R T) Los sale				A) $\Pr(D_{L+T} > R) = \dfrac{C_h T}{\pi + C_h T}$ B) $\dfrac{f_{D_{L+T}}(R^*)}{F_{D_{L+T}}(R^*)} = \dfrac{C_h T}{g}$	

College of Engineering of Shahid Bahonar University of Kerman , Iran.

دانشکده فنی دانشگاه شهید باهنر کرمان